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ASYMPTOTIC VALUES OF MODULUS 1 OF FUNCTIONS IN THE UNIT BALL OF H^∞

BY

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ABSTRACT. The main purpose of this paper is to prove a theorem concerning a necessary and sufficient condition for an inner function to have a limiting value of modulus 1 along an arc inside the unit disc, terminating at a point of the unit circle.

1. **Introduction.** A function $h(z)$ defined and analytic in the unit disc $U = \{z: |z| < 1\}$ is said to be in the unit ball $\mathcal{B}$ of H^∞ if $|h(z)| \leq 1$ for all $z \in U$. A sequence $\{a_n\}$ of complex numbers is called a Blaschke sequence if $\Sigma_{n=1}^\infty (1 - |a_n|)$ converges and $0 \leq |a_n| < 1$. If $\{a_n\}$ is a Blaschke sequence of nonzero numbers the associated Blaschke product is defined by the formula

$$B(z, \{a_n\}) = \prod_{n=1}^\infty \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \bar{a}_n z}. \tag{1.1}$$

We assume that the function $h(z)$ in $\mathcal{B}$ has no zeros at the origin. It is well known that the zeros $\{a_n\}$ of $h(z)$ form a Blaschke sequence, and $h(z)$ admits the following factorization:

$$h(z) = cB(z, \{a_n\})\, \phi(z) \tag{1.2}$$

where c is a constant of modulus one. The function $\phi(z)$ is defined such that

$$\phi(z) = \exp\left\{-\int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\mu(\theta)\right\} \tag{1.3}$$

where μ is a positive finite Borel measure defined on the unit circle $\partial U = \{z: |z| = 1\}$.

A function $h_1(z)$ in $\mathcal{B}$ is said to be a divisor of the function $h(z)$ in $\mathcal{B}$ provided that

$$h_1(z) = c_1 B(z, \{b_n\})\, \phi_E(z) \tag{1.4}$$

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where c_1 is a constant of modulus one, $\{b_n\}$ is a subset of the set $\{a_n\}$ of zeros of $h(z)$ and

$$\phi_E(z) = \exp\left\{-\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} d\mu_E(\theta)\right\} \tag{1.5}$$

where μ_E is the restriction of μ to a measurable subset E of ∂U.

Ahern and Clark [1] have proved the following result.

THEOREM A. *A necessary and sufficient condition for each divisor of a function* $h(z)$ *in* $\mathcal{B}$ *to have a radial limit of modulus one at the point* 1 *is* $\mu(\{0\}) = 0$ *and*

$$\sum_{n=1}^{\infty} \frac{1-|a_n|}{|1-a_n|} + \int_{-\pi}^{\pi} \frac{d\mu(\theta)}{|1-e^{i\theta}|} < \infty. \tag{1.6}$$

It should be noted that Theorem A has been proved by Frostman [5] in the case $h(z) = B(z, \{a_n\})$.

In [6] we obtained the following result for the case $h(z) = B(z, \{a_n\})$.

THEOREM B. *Let* r *denote a function which is decreasing on an interval* $(0, \theta_0)$, *let* $\lim_{\theta\to 0} r(\theta) = 1$, *and let* r *satisfy a Lipschitz condition*

$$|r(\theta) - r(\theta')| \leq Q|\theta - \theta'| \tag{1.7}$$

for some constant Q, *let* $\{r_n e^{i\theta_n}\}$ *be a Blaschke sequence, and let* $\Gamma = \{r(\theta)e^{i\theta}, \theta \in (0, \theta_0)\}$. *Then* $\lim_{z\to 1;z\in\Gamma} B(z, \{b_n\})$ *exists, and has modulus one for each subsequence* $\{b_n\}$ *of* $\{r_n e^{i\theta_n}\}$ *if and only if both of the conditions*

$$\sum_{n=1}^{\infty} \frac{1-r_n}{|1-r_n e^{i\theta_n}|} < \infty \tag{1.8}$$

and

$$\lim_{t\to 0+} \sum_{t/2<\theta_n<2t} \frac{1-r_n}{1-r(t)+|t-\theta_n|} = 0 \tag{1.9}$$

hold.

The main purpose of this paper is to extend Theorem B to the general inner functions.

2. We first prove the following lemma.

LEMMA 1. *Let* μ *be a positive Borel measure defined on* ∂U *and assume that* $\mu(\{0\}) = 0$, *and*

$$\int_{-\pi}^{\pi} \frac{d\mu(\theta)}{|\theta|} < \infty. \tag{2.1}$$

Let ϕ be the function in $\mathcal{B}$ associated with μ. Let Γ be a curve as defined in Theorem B. Then if $0 < \epsilon < \sqrt{2}/4\pi$, there exists a positive number δ such that

$$|\phi_E(z) - \phi^*(1)| < 52\epsilon \tag{2.2}$$

for each $E = (-\pi, \pi)\backslash(t/2, 2t)$, $z = r(t)e^{it}$, $0 < t < \delta/2$, and $\phi^*(1)$ is the radial limit of ϕ at 1.

PROOF. We first have, for each number f,

$$|1 - e^f| = \left|1 - \sum_{n=0}^{\infty} \frac{f^n}{n!}\right| \leq \sum_{n=1}^{\infty} \frac{|f|^n}{n!} = |f| \sum_{n=0}^{\infty} \frac{|f|^n}{(n+1)!} \leq |f|e^{|f|}. \tag{2.3}$$

From (2.1) we can find a positive number $\delta_1 < \pi/2$ such that $r(\delta_1) > ½$ and

$$\int_{-\Phi}^{\Phi} d\mu(\theta)/|\theta| < \epsilon \tag{2.4}$$

for all $\Phi \in (0, \delta_1)$.

We denote the set $[-\Phi, \Phi]$ by E_1. Let t be a positive number less than $\Phi/2$. Then

$$\begin{aligned} |\phi_E(z) - \phi^*(1)| &\leq |\phi_E(z) - \phi_{\partial U\backslash E_1}(z)| + |\phi_{\partial U\backslash E_1}(z) - \phi^*_{\partial U\backslash E_1}(1)| \\ &\quad + |\phi^*_{\partial U\backslash E_1}(1) - \phi^*(1)|. \end{aligned} \tag{2.5}$$

Now

$$\begin{aligned} |\phi^*_{\partial U\backslash E_1}(1) - \phi^*(1)| &= \left|\exp\left\{-\int_{\partial U\backslash E_1} \frac{e^{i\theta}+1}{e^{i\theta}-1}\, d\mu(\theta)\right\} - \exp\left\{-\int_{-\pi}^{\pi} \frac{e^{i\theta}+1}{e^{i\theta}-1}\, d\mu(\theta)\right\}\right| \\ &= \left|1 - \exp\left\{-\int_{E_1} \frac{e^{i\theta}+1}{e^{i\theta}-1}\, d\mu(\theta)\right\}\right|. \end{aligned}$$

But

$$\int_{E_1} \left|\frac{e^{i\theta}+1}{e^{i\theta}-1}\right| d\mu(\theta) < 2\pi \int_{E_1} \frac{e^{i\theta}+1}{e^{i\theta}-1}\, d\mu(\theta) < 2\pi\epsilon < 1.$$

We therefore have

$$|\phi^*_{\partial U\backslash E_1}(1) - \phi^*(1)| < 21\epsilon. \tag{2.6}$$

Since the function $\phi_{\partial U\backslash E_1}$ is analytic at 1, there exists a positive number δ_2 such that

$$|\phi_{\partial U\backslash E_1}(z) - \phi^*_{\partial U\backslash E_1}(1)| < \epsilon \tag{2.7}$$

for all $z = r(t)e^{it}$, $0 < t \leq \delta_2$.

$$|\phi_E(z) - \phi_{\partial U\setminus E_1}(z)| = \left|\exp\left\{-\int_E \frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\mu(\theta)\right\} - \exp\left\{-\int_{\partial U\setminus E_1} \frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\mu(\theta)\right\}\right|$$

$$< \left|1 - \exp\left\{-\int_{E\cap E_1} \frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\mu(\theta)\right\}\right|.$$

But

$$\int_{E\cap E_1}\left|\frac{e^{i\theta}+z}{e^{i\theta}-z}\right| d\mu(\theta) \leqslant \int_{E\cap E_1} \frac{2}{|e^{i\theta}-z|}\,d\mu(\theta)$$

$$= \int_{E\cap E_1} \frac{2d\mu(\theta)}{((1-r(t))^2 + 4r(t)\sin^2(\theta - t)/2)^{1/2}}$$

$$\leqslant \int_{E\cap E_1} \frac{2d\mu(\theta)}{((1-r(t))^2 + 2(2/\pi)^2(\theta - t)^2/4)^{1/2}}$$

$$\leqslant \int_{E\cap E_1} \frac{2\pi}{\sqrt{2}\,|\theta - t|}\,d\mu(\theta)$$

$$= \int_{-\Phi}^{t/2} \frac{2\pi}{\sqrt{2}\,|\theta - t|}\,d\mu(\theta) + \int_{2t}^{\Phi} \frac{2\pi}{\sqrt{2}\,|\theta - t|}\,d\mu(\theta)$$

$$\leqslant \int_{-\Phi}^{t/2} \frac{2\pi}{\sqrt{2}\,|\theta|}\,d\mu(\theta) \quad + \int_{2t}^{\Phi} \frac{4\pi}{\sqrt{2}\,|\theta|}\,d\mu(\theta)$$

$$\leqslant \frac{4\pi}{\sqrt{2}} \int_{E_1} \frac{d\mu(\theta)}{|\theta|} \leqslant \frac{4\pi}{\sqrt{2}}\,\epsilon < 1$$

for $z = r(t)e^{it}$, $\pi/2 > t > 0$, $r(t) > 1/2$.

Again applying (2.3) we have

$$|\phi_E(z) - \phi_{\partial U\setminus E_1}(z)| < 2^{-1/2}\,4\pi\epsilon\, e^{4\pi\epsilon/2^{1/2}} < 30\epsilon. \tag{2.8}$$

Combining (2.6), (2.7),.(2.8) and taking $\delta = \min(\delta_1, \delta_2)$, we have

$$|\phi_E(z) - \phi^*(1)| < 52\epsilon. \tag{2.2}$$

This completes the proof of Lemma 1.

The following theorem is an extension of Theorem B.

THEOREM 1. *Let* h *be a function in* B *with factorization* (1.2), *let* $a_n = r_n e^{i\theta_n}$, $n = 1, 2, \cdots$, *and let* $\mu(\{0\}) = 0$. *Let* Γ *be a curve as defined in Theorem* B. *Then, for the limit*

$$\lim_{z\to 1;\, z\in\Gamma} h_1(z) \tag{2.9}$$

to exist and have modulus 1 *for each divisor* h_1 *of* h, *it is necessary and sufficient that both of the conditions*

$$\sum_{n=1}^{\infty} \frac{1-r_n}{|1-r_n e^{i\theta_n}|} + \int_{-\pi}^{\pi} \frac{d\mu(\theta)}{|1-e^{i\theta}|} < \infty \tag{2.10}$$

and

$$\lim_{t\to 0+}\left(\sum_{t/2<\theta_n<2t} \frac{1-r_n}{1-r(t)+|t-\theta_n|} + \int_{t/2}^{2t} \frac{d\mu(\theta)}{1-r(t)+|t-\theta|}\right) = 0 \tag{2.11}$$

hold.

PROOF. Theorem B is the special case of Theorem 1 where $h(z) = B(z, \{a_n\})$.

We now prove Theorem 1 for the case that $h(z) = \phi(z)$.

Let $z = r(t)e^{it}$, $t > 0$, $r(t) > ½$. Since $|(e^{i\theta} + z)/(e^{i\theta} - z)| < c/(1 - r(t) + |t - \theta|)$ for some positive constant c, we have

$$\begin{aligned}\int_{-\pi}^{\pi} \left|\frac{e^{i\theta}+z}{e^{i\theta}-z}\right| d\mu(\theta) &\leq \int_{-\pi}^{\pi} \frac{c}{1-r(t)+|t-\theta|}\, d\mu(\theta) \\ &\leq c\int_{2t}^{\pi} \frac{d\mu(\theta)}{|t-\theta|} + c\int_{-\pi}^{t/2} \frac{d\mu(\theta)}{|t-\theta|} + c\int_{t/2}^{2t} \frac{d\mu(\theta)}{1-r(t)+|t-\theta|} \\ &\leq 2c\int_{2t}^{\pi} \frac{d\mu(\theta)}{|\theta|} + c\int_{-\pi}^{t/2} \frac{d\mu(\theta)}{|\theta|} + c\int_{t/2}^{2t} \frac{d\mu(\theta)}{1-r(t)+|t-\theta|} \\ &\leq 3c\int_{-\pi}^{\pi} \frac{d\mu(\theta)}{|\theta|} + c\int_{t/2}^{2t} \frac{d\mu(\theta)}{1-r(t)+|t-\theta|}\, .\end{aligned}$$

Now (2.10) and (2.11) imply the uniform convergence [8] of the integral $\int_{-\pi}^{\pi} (e^{i\theta} + z)/(e^{i\theta} - z)\,d\mu(\theta)$ when $z = r(t)e^{it}$ for all $t \in (0, C_1)$ for some $C_1 > 0$. Therefore the limit

$$\lim_{z\to 1;\, z\in\Gamma} \phi(z) \tag{2.12}$$

exists and has modulus one. The same arguments are also valid for the divisors of $\phi(z)$.

To see the converse, we assume that each divisor of $\phi(z)$ has a limit of modulus one as $z \longrightarrow 1$, along the curve Γ. A theorem of Lindelöf [4, p. 10] implies that each divisor of $\phi(z)$ has a radial limit of modulus one at 1. By Theorem A, we have $\mu(\{0\}) = 0$ and

$$\int_{-\pi}^{\pi} \frac{d\mu(\theta)}{|\theta|} < \infty. \tag{2.13}$$

We want to show that

$$\lim_{t\to 0+}\int_{t/2}^{2t}\frac{d\mu(\theta)}{1-r(t)+|t-\theta|}=0. \tag{2.14}$$

Suppose, on the contrary,

$$\overline{\lim_{t\to 0+}}\int_{t/2}^{2t}\frac{d\mu(\theta)}{1-r(t)+|t-\theta|}>0 \tag{2.15}$$

then

$$\overline{\lim_{t\to 0+}}\int_{t/2}^{2t}\frac{1-r(t)+|t-\theta|}{(1-r(t)+|t-\theta|)^2}d\mu(\theta)>0. \tag{2.16}$$

Hence one of the following two cases must arise

$$\text{(I)}\quad \overline{\lim_{t\to 0+}}\int_{t/2}^{2t}\frac{1-r(t)}{(1-r(t)+|t-\theta|)^2}d\mu(\theta)>0, \tag{2.17}$$

$$\text{(II)}\quad \overline{\lim_{t\to 0+}}\int_{t/2}^{2t}\frac{|t-\theta|}{(1-r(t)+|t-\theta|)^2}d\mu(\theta)>0. \tag{2.18}$$

In the case (I) there is a positive number $p > \infty$ such that

$$\int_{t_n/2}^{2t_n}\frac{1-r(t_n)}{(1-r(t_n)+|t_n-\theta|)^2}\,d\mu(\theta)>p \tag{2.19}$$

for each $t_n \in F$, where F is a set of positive numbers with 0 as an accumulation point. But we have for $z_n = r(t_n)e^{it_n}$

$$\begin{aligned}
|\phi(z_n)| &= \exp\left\{-\mathrm{Re}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\mu(\theta)\right\}\\
&= \exp\left\{-\int_{-\pi}^{\pi}\frac{1-r(t_n)^2}{[1-r(t_n)]^2+4r(t_n)\sin^2(t_n-\theta)/2}\,d\mu(\theta)\right\}\\
&\leqslant \exp\left\{-\int_{-\pi}^{\pi}\frac{1-r(t_n)}{[1-r(t_n)]^2+(t_n-\theta)^2}\,d\mu(\theta)\right\}\\
&\leqslant \exp\left\{-\int_{-\pi}^{\pi}\frac{1-r(t_n)}{(1-r(t_n)+|t_n-\theta|)^2}\,d\mu(\theta)\right\}\\
&\leqslant \exp\left\{-\int_{t_n/2}^{2t_n}\frac{1-r(t_n)}{(1-r(t_n)+|t_n-\theta|)^2}\,d\mu(\theta)\right\}\\
&< \exp(-p) < 1.
\end{aligned}$$

Hence in the case (I) if the limit (2.12) exists, it cannot have modulus one. Therefore

(2.20) $$\lim_{t\to 0+}\int_{t/2}^{2t}\frac{1-r(t)}{(1-r(t)+|t-\theta|)^2}\,d\mu(\theta)=0.$$

It remains to consider the case (II). In this case we construct a divisor of $\phi(z)$ which does not have the appropriate limit

(2.18) $$\overline{\lim_{t\to 0+}}\int_{t/2}^{2t}\frac{|t-\theta|}{(1-r(t)+|t-\theta|)^2}\,d\mu(\theta)>0;$$

then either

(2.21) $$\overline{\lim_{t\to 0+}}\int_{t}^{2t}\frac{\theta-t}{(1-r(t)+\theta-t)^2}\,d\mu(\theta)>0$$

or

(2.22) $$\overline{\lim_{t\to 0+}}\int_{t/2}^{t}\frac{t-\theta}{(1-r(t)+t-\theta)^2}\,d\mu(\theta)>0.$$

We suppose without loss of generality that it is the case that (2.21) holds. Then there exist a positive constant q and a set $\{t_n\}\subset(0,\pi)$, $t_n\longrightarrow 0$ as $n\longrightarrow\infty$, $4t_{n+1}<t_n$ for $n=1,2,\cdots$, such that for some $\xi_n\in[t_n,2t_n]$

$$q<\int_{t_n}^{\xi_n}\frac{\theta-t_n}{(1-r(t_n)+\theta-t_n)^2}\,d\mu(\theta)<2q<\tfrac{1}{2}\pi.$$

Let $F_n=[t_n,\xi_n]$ and let $F=\bigcup_{n=1}^{\infty}F_n$; condition (2.13) implies that if $\phi_F(z)$ has a limit as $z\longrightarrow 1$ along Γ, then this limit is $\phi_F^*(1)$. We can show that $\phi_F(z)$ does not have this limit by verifying that the argument of $\phi_{F_n}(z_n)$ ($z_n=r(t_n)e^{it_n}$) does not have 0 as a limit (modulo 2π) as $n\longrightarrow\infty$.

Since

$$\frac{e^{i\theta}+r(t)e^{it}}{e^{i\theta}-r(t)e^{it}}=\frac{1-r(t)^2+i[2r(t)\sin(t-\theta)]}{|e^{i\theta}-r(t)e^{it}|^2},$$

by our choice of the sequence $\{t_n\}$ we have

$$\begin{aligned}\arg\phi_{F_n}(z_n)&=\int_{-\pi}^{\pi}\frac{2r(t_n)\sin(\theta-t_n)}{|e^{i\theta}-r(t_n)e^{it_n}|^2}\,d\mu_{F_n}(\theta)\\&=\int_{t_n}^{\xi_n}\frac{2r(t_n)\sin(\theta-t_n)}{|e^{i\theta}-r(t_n)e^{it_n}|^2}\,d\mu(\theta)\\&=[2-\delta(n)]\int_{t_n}^{\xi_n}\frac{\sin(\theta-t_n)}{|1-r(t_n)+i(\theta-t_n)|^2}\,d\mu(\theta)\end{aligned}$$

where $\delta(n) > 0$ and $\lim_{n\to\infty}\delta(n) = 0$. Since $2^{-1/2}[1 - r(t_n) + (\theta - t_n)] < |1 - r(t_n) + i(\theta - t_n)| < 1 - r(t_n) + (\theta - t_n)$ for $\theta \in F_n$, we see that

$$q < \arg\phi_{F_n}(z_n) < 2q \tag{2.23}$$

for sufficiently large n. Apply Lemma 1 with μ replaced by μ_F; we have $\mu_E = \mu_{F\setminus F_n}$. So the lemma implies $\phi_{F\setminus F_n}(z_n) \longrightarrow \phi_F^*(1)$, and this gives a contradiction to (2.23).

This completes the proof of Theorem 1 for the case $h(z) = \phi(z)$.

Combining the proof for the cases $h(z) = B(z, \{a_n\})$, and $h(z) = \phi(z)$, the theorem follows without difficulty.

3. Upon generalizing theorems of Ahern and Clark [1] and Leung and Linden [6], we obtain the following.

THEOREM 2. *Let* h *be a function in* $\mathcal{B}$ *with factorization* (1.2) *and* $\gamma \geqslant 1$. *A necessary and sufficient condition for* h *and all its divisors to have* T_γ *limits of modulus* 1 *at the point* 1 *is that*

$$\sum_{n=1}^{\infty} \frac{1 - |a_n|}{|1 - a_n|} + \int_{-\pi}^{\pi} \frac{d\mu(\theta)}{|1 - e^{i\theta}|} < \infty \tag{3.1}$$

and

$$\lim_{t\to 0+}\left\{\sum_{t/2<|\theta_n|<2t} \frac{1 - r_n}{t^\gamma + |t - |\theta_n||} + \left(\int_{-2t}^{-t/2} + \int_{t/2}^{2t}\right) \frac{d\mu(\theta)}{t^\gamma + |t - |\theta||}\right\} = 0 \tag{3.2}$$

hold.

THEOREM 3. *Let* $\gamma \geqslant 1$, *and* k *be a nonnegative integer. Then the* k*th derivatives of the function* $h(z)$ *in* $\mathcal{B}$ *and all its divisors have* T_γ *limits at* 1 *if*

$$\sum_{n=1}^{\infty} \frac{1 - |a_n|}{|1 - a_n|^{k+1}} + \int_{-\pi}^{\pi} \frac{d\mu(\theta)}{|1 - e^{i\theta}|^{k+1}} < \infty \tag{3.3}$$

and

$$\lim_{t\to 0+}\left\{\sum_{t/2<|\theta_n|<2t} \frac{1 - r_n}{(t^\gamma + |t - |\theta_n||)^{k+1}} + \left(\int_{-2t}^{-t/2} + \int_{t/2}^{2t}\right) \frac{d\mu(\theta)}{(t^\gamma + |t - |\theta||)^{k+1}}\right\} = 0 \tag{3.4}$$

hold.

4. The method we used in the previous sections also applies to the class N of holomorphic functions of bounded characteristic. A function f holomorphic in the unit disc is said to be in the class N if and only if

$$\sup_{0<r<1-} \int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})| d\theta < \infty.$$

It is well known that if $f \in N$, then it admits the factorization that

$$(4.1)\qquad f(z) = cz^m\, B(z, \{a_n\})\, Q(z),$$

where m is the order of zeros of $f(z)$ at the origin, $B(z, \{a_n\})$ is the Blaschke product associated with the zeros $\{a_n\}$ of $f(z)$ with $a_n \neq 0$. Hence $Q(z)$ is defined by the relation

$$Q(z) = \exp\left\{\int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\sigma(\theta)\right\},\quad d\sigma(\theta) = -d\mu(\theta) + (2\pi)^{-1} \log|f^*(e^{i\theta})|d\theta,$$

where f^* is the boundary function of f, μ is a positive singular Borel measure defined on ∂U, and c is a constant.

Again we assume without loss of generality that $m = 0$.

We obtain the following result.

THEOREM 4. *Let $f \in N$ with factorization* (4.1), *let $\gamma \geqslant 1$, and let k be a nonnegative integer. If*

$$(4.2)\qquad \sum_{n=1}^{\infty} \frac{1 - |a_n|}{|1 - a_n|^{k+1}} + \int_{-\pi}^{\pi} \frac{d|\sigma(\theta)|}{|1 - e^{i\theta}|^{k+1}} < \infty$$

and

$$(4.3)\qquad \lim_{t\to 0+}\left\{\sum_{t/2<|\theta_n|<2t} \frac{1 - r_n}{(t^\gamma + |t - |\theta_n||)^{k+1}} + \left(\int_{-2t}^{-t/2} + \int_{t/2}^{2t}\right) \frac{d|\sigma(\theta)|}{(t^\gamma + |t - |\theta||)^{k+1}}\right\} = 0$$

then the T_γ limit of $f^{(k)}$ exists.

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