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TWO MODEL THEORETIC PROOFS OF RÜCKERT'S NULLSTELLENSATZ

BY

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ABSTRACT. Rückert's Nullstellensatz for germs of analytic functions and its analogue for germs of real analytic functions are proved by a combination of nonstandard analysis with a model theoretic transfer principle. It is also shown that Rückert's Nullstellensatz is constructive essentially relative to the Weierstrass preparation theorem.

Introduction. In [4] A. Robinson introduced nonstandard (n.s.) analytic function germs and used them to give an elegant formulation and proof of Rückert's Nullstellensatz. He showed moreover that the nonstandard version of Rückert's Nullstellensatz can be generalized to arbitrary ideals of n.s. cylindrical analytic function germs at the origin. In this context he raised the question whether the model theoretic proof of Hilbert's Nullstellensatz for polynomials, which uses the algebraic construction of a generic point for a prime ideal and the model completeness of the theory of algebraically closed fields, could be adapted to prove Rückert's Nullstellensatz. This is done in the present paper. We also consider the corresponding problem for n.s. real analytic function germs.

In §1 we set up a first order theory A and a slightly weaker theory A' which sum up the essential algebraic features of n.s. analytic function germs at the origin. The language L of A and A' contains in particular a unary predicate symbol U for the monad $\mu(0 \cdots 0)$ of the origin. The models of A and A' are characterized and certain "algebraic" models of A' are constructed by purely algebraic means.

In §2 we prove the following transfer principle: Let M' be a model of A', M a model of A, and φ an existential sentence in L with quantifiers ranging over U. Then φ holds in M, if it holds in M'.

In §3 the construction of an "algebraic" model of A' containing a generic

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point for a given prime ideal in a ring of convergent power series is combined with the transfer principle of §2 to a model theoretic proof of Rückert's Nullstellensatz.

§4 investigates the constructive aspects of Rückert's theorem. A. Seidenberg's constructive proof of Hilbert's Nullstellensatz [7] is extended to a proof of Rückert's Nullstellensatz which is constructive essentially relative to the Weierstrass preparation theorem and certain linear substitutions in a ring of convergent power series over **C**.

In §5 we consider the corresponding problem for n.s. germs of real analytic functions at the origin. It turns out that the argument used in §§1–3 to prove Rückert's theorem can be carried over with slight modifications to prove a Nullstellensatz for n.s. real analytic function germs. Our theorem is essentially the n.s. version of a theorem of J.-J. Risler [3]; it relates to Dubois' Nullstellensatz [1] for polynomials over ordered fields as Rückert's Nullstellensatz relates to Hilbert's Nullstellensatz.

1. **The theories** A' **and** A. Let **C** be the field of complex numbers, $\mathbf{C}[[X_1 \cdots X_n]]$ the ring of formal power series in n variables over **C**. The convergent power series in $\mathbf{C}[[X_1 \cdots X_n]]$ form a subring R_n of $\mathbf{C}[[X_1 \cdots X_n]]$. We need the following facts about R_n (see [9, Chapter VII, §1]).

1.1. $f \in R_n$ is invertible iff $\operatorname{ord}(f) = 0$, where $\operatorname{ord}(f)$ denotes the order of f. The elements $f \in R_n$ of order $\geqslant 1$ form a maximal ideal R_n^0 in R_n.

1.2. For fixed elements $g_1 \cdots g_n$ of R_m^0 the substitution map s: $R_n \longrightarrow R_m$, defined by

$$s(f(X_1 \cdots X_n)) = f[g_1(X_1 \cdots X_m), \ldots, g_n(X_1 \cdots X_m)],$$

is a homomorphism and s maps R_n^0 into R_m^0.

1.3. For finitely many elements $f_1 \cdots f_k \in R_n$ there exist $c_1 \cdots c_{n-1} \in \mathbf{C}$, such that $f_i[X_1 + c_1X_n, \ldots, X_{n-1} + c_{n-1}X_n, X_n]$ are regular of order $\operatorname{ord}(f_i)$ in X_n for $i = 1, \ldots, k$.

1.4 (*Taylor expansion*). For every $f \in R_n$ there is a $h \in R_n$, such that, in R_{n+1}

$$f(X_1 \cdots X_{n-1}, X) - f(X_1 \cdots X_{n-1}, Y) = h(X_1 \cdots X_{n-1}, (X - Y)) \cdot (X - Y).$$

1.5 (*Weierstrass' preparation theorem*). For every $f \in R_n^0$, which is regular of order $k > 0$ in X_n, there exist uniquely determined elements $v \in R_n$, $u_0 \cdots u_{k-1} \in R_{n-1}^0$, such that

$$\operatorname{ord}(v) = 0 \quad \text{and} \quad f = v \cdot \left(\sum_{i=0}^{k-1} u_i X_n^i + X_n^k \right), \quad n \geqslant 1.$$

We consider R_m in a natural way as subring of R_n for $m < n$, and identify R_0 with **C**. $\bigcup_{n<\omega} R_n$ is denoted by R_ω and $\bigcup_{n<\omega} R_n^0$ by R_ω^0. The ring structure of R_ω will now be used in order to describe the theories A' and A.

The language L of A' and A contains the following nonlogical symbols. (We count variables and the symbol "=" for equality among the logical symbols.)

1.6. (i) The binary function symbols $+, -, \cdot$, and for every $f \in R_n, n \geqslant 1$, a n-ary function symbol **f**.

(ii) For every $c \in \mathbf{C}$ a constant symbol **c**.

(iii) A unary predicate symbol U.

The set T^0 of special terms and the set T of terms are defined inductively as follows:

1.7. (i) $\mathbf{0} \in T^0$ and $x \in T^0$ for every variable x.

(ii) If $t_1, t_2 \in T^0$, then $t_1 + t_2, t_1 - t_2, t_1 \cdot t_2 \in T^0$. If $t_1 \cdots t_n \in T^0$ and $f \in R_n^0$ for some $n \geqslant 1$, then $f(t_1 \cdots t_n) \in T^0$. If $t \in T$ and $t' \in T^0$, then $t \cdot t', t' \cdot t \in T^0$.

(iii) $\mathbf{c} \in T$ for every $c \in \mathbf{C}$. If $t \in T^0$, then $t \in T$. If $t_1 \cdots t_n \in T^0$ and $f \in R_n$ for some $n \geqslant 1$, then $f(t_1 \cdots t_n) \in T$.

(iv) If $t_1, t_2 \in T$, then $t_1 + t_2, t_1 - t_2, t_1 \cdot t_2 \in T$. Every term $t \in T$ determines a unique element $f_t \in R_\omega$, which is obtained from t by copying the construction of t in R_ω. Thus $f_t \in R_\omega^0$ for $t \in T^0$. We call terms of the form

$$\sum_{i=0}^{k} t_i \cdot x^i,$$

where $t_0 \cdots t_k$ are terms not containing the variable x, *pseudo-polynomials* in x of degree k, and terms of the form

$$\sum_{i=0}^{k-1} \mathbf{u}_i \cdot x_n^i + x_n^k,$$

where $u_i \in R_{n-1}^0$, *Weierstrass polynomials* in x_n of degree k. Atomic formulas are expressions of form $t_1 = t_2$ for $t_1, t_2 \in T$, or $U(t)$ for $t \in T$. Formulas are built up from atomic formulas as usual in a first order theory, using the connectives $\wedge, \vee, \neg, \longrightarrow, \longleftrightarrow$, and quantification. A formula containing no free variable is called a sentence. We abbreviate formulas of form $\exists x(U(x) \wedge \varphi)$, $\forall x(U(x) \longrightarrow \varphi)$, where φ is a formula, by $(\exists x)_U(\varphi)$, respectively $(\forall x)_U(\varphi)$, and refer to the expressions $(\exists x)_U, (\forall x)_U$ as *restricted quantifiers.* A formula of form $(\exists x_1)_U \cdots (\exists x_n)_U(\varphi)$, where φ is quantifier-free, is called a *restricted existential formula.*

A' has the following axioms:

1.8. Axioms for algebraically closed fields, formulated as usual as a set of sentences in L.

1.9. Axioms about U:

(i) $U(\mathbf{0}), \neg U(\mathbf{c})$ for every $c \in \mathbf{C}, c \neq 0, (\exists x)_U(\neg x = \mathbf{0})$.

(ii) For every term $t(x_1 \cdots x_n) \in T^0, n \geqslant 0$, the axiom

$$(\forall x_1)_U \cdots (\forall x_n)_U(U(t(x_1 \cdots x_n))).$$

1.10. (i) For every pair $t_1(x_1 \cdots x_n), t_2(x_1 \cdots x_n)$ of terms, $n \geq 0$, such that $f_{t_1} = f_{t_2}$ holds in R_n, the axiom

$$(\forall x_1)_U \cdots (\forall x_n)_U(t_1(x_1 \cdots x_n) = t_2(x_1 \cdots x_n)).$$

(ii) For every pair t_1, t_2 of variable-free terms, such that $f_{t_1} \neq f_{t_2}$ holds in $R_0 = \mathbf{C}$, the axiom $t_1 \neq t_2$.

A has all the axioms of A' together with the following set of axioms:

1.11. For every $n \geq 1$:

$$(\forall y_0)_U \cdots (\forall y_{n-1})_U(\forall x)\left(\sum_{i=0}^{n-1} y_i x^i + x^n = 0 \longrightarrow U(x)\right).$$

The contents of these axioms are best understood in terms of the following characterization of the models of A' and A.

1.12. A structure $M = \langle M, U_M, +_M, -_M, \cdot_M, \langle f_M : f \in R_\omega \rangle, \langle c_M : c \in \mathbf{C} \rangle\rangle$ for L is a model of A' if and only if

(i) $\langle M, +_M, -_M, \cdot_M \rangle$ is an algebraically closed field.

(ii) Let ${}^{(U_M^n)}M$, $n \geq 0$, denote the ring of functions from U_M^n into M with pointwise operations. The map $R_\omega \longrightarrow \bigcup_{n<\omega} {}^{(U_M^n)}M$, given by $f \longmapsto f_M \restriction U_M^n$ for $f \in R_n$, $n \geq 0$, is a ring homomorphism and its restriction to $R_0 = \mathbf{C}$ is a monomorphism, i.e. the set $\mathbf{C}_M = \{c_M : c \in \mathbf{C}\}$ with the restrictions of $+_M, -_M, \cdot_M$ to $\mathbf{C}_M^2$ is an isomorphic copy of the field $\mathbf{C}$.

(iii) U_M is a subring of $\langle M, +_M, -_M, \cdot_M \rangle$ under the restrictions of $+_M, -_M, \cdot_M$ to U_M^2, $U_M \neq \{0_M\}$, $U_M \cap \mathbf{C}_M = \{0_M\}$. U_M is closed under the functions f_M for $f \in R_\omega^0$ and hence in particular a vector space over $\mathbf{C}_M$.

Moreover M is a model of A iff in addition:

(iv) The ring $\langle U_M, +_M \restriction U_M^2, -_M \restriction U_M^2, \cdot_M \restriction U_M^2 \rangle$ is integrally closed in $\langle M, +_M, -_M, \cdot_M \rangle$.

We shall now describe two different ways to obtain models of A'. First, let C be the higher order structure over the set $\mathbf{C}$ and ${}^*C$ an enlargement of C (for the basic concepts of nonstandard analysis see [5]). Every $f \in R_n$, $n \geq 0$, determines a unique n.s. analytic function germ $f_{{}^*C}$ at the origin, which is defined as follows: Let $\bar{f}$ be the function analytic at the origin determined by f, and ${}^*\bar{f}$ the extension of $\bar{f}$ in ${}^*C$. Then $f_{{}^*C}$ is the restriction of ${}^*\bar{f}$ to the monad of the origin in ${}^*C^n$ (cf. [4]). Using 1.12, it is easy to verify that the structure $M({}^*C)$ given by $M = {}^*\mathbf{C}$, $U_M = \mu(0)$, $+_M, -_M, \cdot_M$ the field operations in ${}^*\mathbf{C}$, f_M an arbitrary extension of $f_{{}^*C}$ to a function on $({}^*\mathbf{C})^n$ for $f \in R_n$, $n \geq 0$, and $\mathbf{c}_M = c$ for $c \in \mathbf{C}$ is a model of A'. M is even a model of A, since $\mu(0)$ is apparently an integrally closed subring of the field $^*\mathbf{C}$. We refer to $M(^*C)$ as a "standard" model of A.

Next, we construct certain models of A' by purely algebraic means. For fixed $n \geqslant 0$, consider the map

$$\sim : R_\omega \longrightarrow \bigcup_{m<\omega} {}^{((R_n^0)^m)}R_n$$

given by $\tilde{f}(g_1 \cdots g_m) = f[g_1 \cdots g_m]$ for $f \in R_m$, $g_1 \cdots g_m \in R_n^0$, $m \geqslant 0$, where $f[g_1 \cdots g_m]$ denotes the substitution of $g_1 \cdots g_m$ for $X_1 \cdots X_m$ in f. Using 1.2 we see that $\sim$ is a ring homomorphism mapping R_ω^0 into $\bigcup_{m<\omega} {}^{((R_n^0)^m)}R_n^0$. We denote the image of R_ω under $\sim$ by F. Let J be an ideal in R_n, $J \subsetneqq R_n^0$, $h: R_n \longrightarrow S_n = R_n/J$ the canonical homomorphism, and $h[R_n^0] = S_n^0$. By 1.4, $g_1 \cdots g_m, k_1 \cdots k_m \in R_n^0$, $\tilde{f}(x_1 \cdots x_n) \in F$, and $g_i - k_i \in J$ for $i = 1, \ldots, m$ implies $\tilde{f}(g_1 \cdots g_m) - \tilde{f}(k_1 \cdots k_m) \in J$. Thus the map

$$- : R_\omega \longrightarrow \bigcup_{m<\omega} {}^{((S_n^0)^m)}S_n,$$

given by $\bar{f}(h(g_1) \cdots h(g_m)) = h(\tilde{f}(g_1 \cdots g_m))$ for $f \in R_m$, $g_1 \cdots g_m \in R_n^0$ is well defined and a ring homomorphism. Besides, in case J is a prime ideal, S_n can be extended to an algebraically closed field K. Then the structure $M(J, K)$ given by $M = K$, $U_M = S_n^0$, $+_M, -_M, \cdot_M$ the field operations in K, f_M an arbitrary extension of $\bar{f}$ to K^m for $f \in R_m$, $m \geqslant 0$, and $c_M = h(c)$ for $c \in \mathbf{C}$ is a model of A' by 1.12. We refer to $M(J, K)$ as an "algebraic" model of A'.

2. **A transfer principle for** A **and** A'. The aim of this section is to prove the following transfer principle for the theories A and A':

2.1. Let M' be a model of A', M be a model of A. Then every restricted existential sentence in L which holds in M' holds in M, too. 2.1 is an immediate consequence of the following two lemmas:

2.2. For every quantifier-free sentence φ in L either $A' \vdash \varphi$ or $A' \vdash \neg\varphi$.

2.3. For every restricted existential sentence φ in L there is a quantifier-free sentence ψ in L, such that $A' \vdash \varphi \longrightarrow \psi$ and $A \vdash \varphi \longleftrightarrow \psi$.

To prove 2.2 it suffices to consider the case that φ is an atomic sentence. If φ is of form $t_1 = t_2$, where t_1, t_2 are variable-free terms, we have by 1.10 $A' \vdash \varphi$, if $f_{t_1} = f_{t_2}$ in R_0, and $A' \vdash \neg\varphi$, if $f_{t_1} \neq f_{t_2}$ in R_0. If φ is of form $U(t_1)$, we can find $c \in \mathbf{C}$, such that $f_{t_1} = c$ holds in R_0. Then by 1.10(i) $A' \vdash t_1 = c$, and so by (1.9)(i) $A' \vdash U(t_1)$, if $c = 0$, and $A' \vdash \neg U(t_1)$, if $c \neq 0$.

2.3 is proved by induction on the number n of restricted quantifiers in

$\varphi = (\exists x_1)_U \cdots (\exists x_n)_U(\psi(x_1 \cdots x_n))$. The case $n = 0$ is trivial. For $n > 0$ we may assume that $\psi(x_1 \cdots x_n)$ is a conjunction of atomic formulas of form

2.4. $t(x_1 \cdots x_n) = 0$, or

2.5. $U(t(x_1 \cdots x_n))$, with $t(x_1 \cdots x_n) \in T$, and their negations. From 1.9 and 1.10 we get easily

$$A' \vdash (\forall x_1)_U \cdots (\forall x_n)_U(U(t'(x_1 \cdots x_n) \wedge \neg U(t(x_1 \cdots x_n) \wedge t(x_1 \cdots x_n) \neq 0)))$$

for $t'(x_1 \cdots x_n) \in T^0$, $t(x_1 \cdots x_n) \in T$ with $f_t \in R_n - R_n^0$. Hence we may assume without restriction that all the atomic subfoımulas of φ are of form 2.4 with $t \in T^0$. Let $t_1 \cdots t_n$ be all such terms occurring in φ. By 1.3 there are $c_1 \cdots c_{n-1} \in \mathbf{C}$, such that $g_i(X_1 \cdots X_n) = f_{t_i}[X_1 + c_1X_n, \ldots, X_{n-1} + c_{n-1}X_n, X_n]$ are regular in X_n of order $k_i > 0$ for $i = 1, \ldots, m$. Using now 1.9 and 1.10, we see that

$$\varphi = (\exists x_1)_U \cdots (\exists x_n)_U\left(\bigwedge_{i=1}^{m} t_i(x_1 \cdots x_n) \mathrel{\overset{=}{\neq}} 0\right)$$

is in A' equivalent to

$$\varphi_1 = (\exists x_1)_U \cdots (\exists x_n)_U\left(\bigwedge_{i=1}^{m} g_i(x_1 \cdots x_n) \mathrel{\overset{=}{\neq}} 0\right).$$

Next, we apply the Weierstrass preparation theorem 1.5 to $g_1 \cdots g_m$, and get $v_i \in R_n - R_n^0$, $u_{ij} \in R_{n-1}^0$, $j = 1, \cdots, k_i$, $i = 1, \ldots, m$, such that

2.6. $g_i = v_i \cdot p_i$, where $p_i(X_1 \cdots X_n)$ are the Weierstrass polynomials $\Sigma_{j=0}^{k_i-1} u_{ij}(X_1 \cdots X_{n-1})X_n^j + X_n^{k_i}$.

2.6 combined with 1.9 and 1.10 entails that φ_1 is in A' equivalent to

$$(\exists x_1)_U \cdots (\exists x_n)_U\left(\bigwedge_{i=1}^{m} \mathrm{p}_i(x_1 \cdots x_n) \mathrel{\overset{=}{\neq}} 0\right), \tag{2.7}$$

where $\mathrm{p}_i(x_1 \cdots x_n) = \Sigma_{j=0}^{k_i-1} \mathrm{u}_{ij}(x_1 \cdots x_{n-1})x_n^j + x_n^{k_i}$ for $i = 1, \ldots, m$. Recall that so far all equivalences were in A'. Now, let 2.8 be the sentence resulting from 2.7 by replacing the restricted quantifier $(\exists x_n)_U$ by the unrestricted quantifier $(\exists x_n)$. Then, obviously $A' \vdash 2.7 \longrightarrow 2.8$, and by 1.11, 1.9 $A \vdash 2.7 \longleftrightarrow 2.8$.

Since the terms $\mathrm{p}_1 \cdots \mathrm{p}_m$ are pseudo-polynomials in x_n, we can apply the usual quantifier-elimination procedure for the theory of algebraically closed fields to the quantifier $(\exists x_n)$ in 2.8 and thus arrive at a sentence.

$$(\exists x_1)_U \cdots (\exists x_{n-1})_U(\vartheta(x_1 \cdots x_{n-1})) \tag{2.9}$$

in L, such that $\vartheta(x_1 \cdots x_{n-1})$ is quantifier-free and $A' \vdash 2.8 \leftrightarrow 2.9$. Finally, the induction assumption is applied to 2.9.

3. **Rückert's Nullstellensatz.** Let ${}^*C$ be an enlargement of C and $\Gamma_n({}^*C)$ the ring of n.s. analytic function germs of n variables at the origin. For $\Phi_1 \cdots \Phi_m \in \Gamma_n$ the n.s. variety germs of $\Phi_1 \cdots \Phi_m$ is defined by $v(\Phi_1 \cdots \Phi_m) = \{\langle a_1 \cdots a_n\rangle \in \mu(0 \cdots 0) : \Phi_i(a_1 \cdots a_n) = 0$ for $i = 1, \ldots, m\}$. Rückert's Nullstellensatz in its nonstandard version (see [4]) asserts that

3.1. For all $\Phi_1 \cdots \Phi_m$, $\Psi \in \Gamma_n$, $m \geq 1$, $n \geq 0$, such that $v(\Psi) \supseteq v(\Phi_1 \cdots \Phi_m)$, some power Ψ^k, $k \geq 1$, of Ψ belongs to the ideal generated by $\Phi_1 \cdots \Phi_m$ in Γ_n.

We prove 3.1 in spirit of A. Robinson's model theoretic proof of Hilbert's Nullstellensatz (cf. [4, p. 141]). First, observe that the map $R_n \longrightarrow \Gamma_n$ given by $f \longmapsto f_{{}^*C}$, as defined in §1, is actually an isomorphism for every $n \geq 0$. So 3.1 can be reformulated in terms of the model $M({}^*C)$ of A as follows:

3.2. For all $f_1 \cdots f_n$, $g \in R_n$, $m \geq 1$, $n \geq 0$, such that

$$(\forall x_1)_U \cdots (\forall x_n)_U \left(\bigwedge_{i=1}^{m} \mathrm{f}_i(x_1 \cdots x_n) = 0 \longrightarrow \mathrm{g}(x_1 \cdots x_n) = 0 \right)$$

holds in $M({}^*C)$, there is a natural number $k \geq 1$ and $h_1 \cdots h_m \in R_n$, such that $g^k = \Sigma_{i=1}^m h_i f_i$.

Let I be the ideal generated by $f_1 \cdots f_m$ in R_n, and suppose $g^k \notin I$ for all $k \geq 1$. Then $I \subseteq R_n^0$, since R_n^0 is the maximal ideal of R_n. We want to show that

$$(3.3) \quad (\exists x_1)_U \cdots (\exists x_n)_U \left(\bigwedge_{i=1}^{m} \mathrm{f}_i(x_1 \cdots x_n) = 0 \wedge \mathrm{g}(x_1 \cdots x_n) \neq 0 \right)$$

holds in $M({}^*C)$. If $g \in R_n - R_n^0$, $g_{{}^*C}(0 \cdots 0) \neq 0$ but $f_{i^*C}(0 \cdots 0) = 0$ for $i = 1, \ldots, m$, and so 3.3 holds in $M({}^*C)$. If $g \in R_n^0$, we extend I by a familiar algebraic argument to a prime ideal $J \subsetneqq R_n^0$ excluding all the powers of g. Taking for K an algebraically closed field which contains R_n/J, we form the model $M(J, K)$ of A', as described in §1. With the notation used there, we find that $\langle h(X_1) \cdots h(X_n)\rangle$ is a generic point for J, i.e., for all $f \in R_n$, $f_{M(J,K)}(h(X_1) \cdots h(X_n)) = 0_M$ iff $f \in J$. In particular, 3.3 holds in $M(J, K)$, and so, by the transfer principle 2.1, 3.3 holds in $M({}^*C)$.

Note, that we cannot prove in this way the existence of a generic point $\langle a_1 \cdots a_n\rangle \in \mu(0 \cdots 0)$ for every proper prime ideal J in Γ_n, since the notion of a generic point is not expressible by a formula of finite length in L. However, for every prime ideal $J \subsetneqq R_n^0$ and every pair of finite sets $A \subseteq J$, $B \subseteq R_n - J$, a transfer argument similar to the above yields an "approximatively generic" point, i.e. a point $\langle a_1 \cdots a_n \rangle \in \mu(0 \cdots 0)$, such that $f_{*_C}(a_1 \cdots a_n) = 0$ for $f \in A$, and $g_{*_C}(a_1 \cdots a_n) \neq 0$ for $g \in B$. (The ideal R_n^0 has obviously the generic point $(0 \cdots 0)$ in $M({}^*C)$.) From this the existence of a generic point in $\mu(0 \cdots 0)$ for every proper prime ideal in Γ_n, as well as the generalization of Rückert's Nullstellensatz for n.s. cylindrical analytic function germs, can be derived as in [4, §5].

4. **Constructive aspects of Rückert's theorem.** In [7], A. Seidenberg observed that the quantifier-elimination procedure for the theory of algebraically closed fields gives rise to a constructive version of Hilbert's Nullstellensatz in the following sense:

4.1. Let F be an algebraically closed field, $f_1 \cdots f_m, g \in F[X_1 \cdots X_n]$, and F_0 the subfield of F generated by the coefficients of $f_1 \cdots f_m, g$. Suppose

$$(\forall x_1) \cdots (\forall x_n)\left(\bigwedge_{j=1}^{m} f_i(x_1 \cdots x_n) = 0 \longrightarrow g(x_1 \cdots x_n) = 0\right)$$

holds in F. Then a natural number $k \geqslant 1$ and polynomials $h_1 \cdots h_m \in F_0[X_1 \cdots X_n]$ can be constructed, such that $g^k = \Sigma_{i=1}^m h_i \cdot f_i$.

More precisely, the construction of k and $h_1 \cdots h_m$ involves the following operations and decisions:

4.2. (i) Addition, subtraction, multiplication in $F_0[X_1 \cdots X_n]$, and division in F_0.

(ii) Assigning to every $f \in F_0[X_1 \cdots X_n]$ the highest number m, such that X_m occurs in f, the formal degree and the coefficients of f considered as a polynomial in X_m.

(iii) Deciding, whether some polynomial relation $g(c_1 \cdots c_h) = 0$ between elements $c_1 \cdots c_h$ of F_0 holds in F_0 or not.

It is natural to ask whether Seidenberg's construction can be extended to yield a proof of Rückert's Nullstellensatz, which is in some sense "constructive". This is indeed possible, if we allow certain operations and decisions in the ring R_n to enter our constructions in place of those mentioned in 4.2. Analyzing the quantifier-elimination procedure in A for restricted existential sentences in L, we find that the following operations were used:

4.3. (i) Addition, subtraction, multiplication in R_n.

(ii) Assigning to every $f \in R_n$ its order.

(iii) Assigning to finitely many $f_1 \cdots f_m \in R_k^0$, $k \leqslant n$, elements $c_1 \cdots c_{k-1} \in \mathbf{C}$, such that $f_i[X_1 + c_1X_k, \ldots, X_{k-1} + c_{k-1}X_k, X_k]$ is regular in X_k for $i = 1, \ldots, m$.

(iv) Performing the linear substitution

$$f(X_1 \cdots X_k) \mapsto f[X_1 + c_1 X_k, \ldots, X_{k-1} + c_{k-1} X_k, X_k]$$

for given $f \in R_k$, $c_1 \cdots c_{k-1} \in \mathbf{C}$, $k \leq n$.

(v) Assigning to every $f \in R_k$, $k \leq n$, which is regular in X_k, the element $v \in R_k - R_k^0$ and the Weierstrass polynomial $p \in R_k$ associated with f by the Weierstrass preparation theorem, such that $f = v \cdot p$.

For the constructive version of Rückert's Nullstellensatz we include in addition

(vi) Division in $R_n - R_n^0$.

Interpreting "constructive" with respect to these operations, Rückert's theorem, as formulated in 3.2, is constructive. This is proved by induction on n: For $n = 0$ the construction is obvious. For $n > 0$, we assume without restriction $f_1 \cdots f_m$, $g \in R_n^0$. Similar to the proof of 2.3, we "construct" $c_1 \cdots c_{n-1} \in \mathbf{C}$, $v_1 \cdots v_m$, $v \in R_n - R_n^0$, and Weierstrass polynomials in X_n, $p_1 \cdots p_m$, $p \in R_n$, such that

$$f_i[X_1 + c_1 X_n, \ldots, X_{n-1} + c_{n-1} X_n, X_n] = v_i(X_1 \cdots X_n) \cdot p_i(X_1 \cdots X_n) \quad \text{for } i = 1, \ldots, m,$$

$$g[X_1 + c_1 X_n, \ldots, X_{n-1} + c_{n-1} X_n, X_n] = v(X_1 \cdots X_n) \cdot p(X_1, \ldots, X_n), \tag{4.4}$$

$$(\forall x_1)_U \cdots (\forall x_{n-1})_U (\forall x_n) \cdot \left(\bigwedge_{i=1}^{m} \mathrm{p}_i(x_1 \cdots x_n) = 0 \rightarrow \mathrm{p}(x_1 \cdots x_n) = 0 \right)$$

hold in $M({}^*C)$.

Now we combine the inductive step in Seidenberg's construction [7] with our induction assumption, and obtain a natural number $k \geq 1$ and pseudo-polynomials $h_1 \cdots h_m$ in X_n, such that

$$p^k = \sum_{i=1}^{m} h_i \cdot p_i. \tag{4.5}$$

This together with 4.4 yields a representation $g^k = \Sigma_{i=1}^m \tilde{h}_i \cdot f_i$, where $\tilde{h}_1 \cdots \tilde{h}_m \in R_n$ are constructed from $f_1 \cdots f_m, g$ using only the operations listed in 4.3.

5. **A Nullstellensatz for nonstandard real analytic function germs.** Hilbert's Nullstellensatz has an analogue for ordered fields which is due to D. W. Dubois [1]. A slightly improved version of Dubois' theorem reads as follows:

5.1. Let F be an ordered field, $f_1, \ldots, f_m, g \in F[X_1, \ldots, X_n]$, $\bar{F}$ the real closure of F and $V(f_1, \ldots, f_m)$, $V(g)$ the variety of $\{f_1, \ldots,$

$f_m\}$ and g, respectively, in $\overline{F}$. Then $V(g) \supseteq V(f_1, \ldots, f_m)$ if and only if the following condition holds:

5.1.1. There exist $k \geqslant 1$, $r \geqslant 0$, $0 < p_i \in F$, $j_i \in F[X_1, \ldots, X_n]$, $1 \leqslant i \leqslant r$, such that $g^{2k} + p_1 j_1^2 + \cdots + p_r j_r^2$ is in the ideal generated by $f_1, \ldots, f_m$.

Remark that the theorem holds for $p_1 = \cdots = p_r = 1$, if F is a real closed field.

5.1 can be proved in complete analogy to the model theoretic proof of Hilbert's Nullstellensatz. We give a sketch of the proof: Clearly 5.1.1 is a sufficient condition for $V(g) \supseteq V(f_1, \ldots, f_m)$. For the converse we need the following lemma:

5.2. Let R be a commutative ring with identity containing an ordered field F, I an ideal in R, $g \in R$,

$$S = \{g^{2k} + p_1 j_1^2 + \cdots + p_r j_r^2 \colon k \geqslant 1, r \geqslant 0, 0 < p_i \in F, j_i \in R\}$$

and assume $I \cap S = \emptyset$. Then I can be extended to an ideal J in R such that $J \cap S = \emptyset$ and R/J can be extended to a real closed field F'. Lemma 5.2 is proved just like the lemma on p. 113 in [1].

The proof of 5.1 is then completed as follows: Let I be the ideal generated by $f_1, \ldots, f_m$ in $R = F[X_1, \ldots, X_n]$, S, J, F' as in the lemma and $a_i \in F'$ the image of X_i under the canonical homomorphism from R onto R/J. Then $f_i(a_1, \ldots, a_n) = 0$ for $1 \leqslant i \leqslant m$ but $g(a_1, \ldots, a_n) \neq 0$. We may assume that $F \subset \overline{F} \subset F'$. So by the model completeness of the theory of real closed fields (see [6, p. 105]) there exist $b_1, \ldots, b_n \in \overline{F}$ such that $f_i(b_1, \ldots, b_n) = 0$ for $1 \leqslant i \leqslant m$ and $g(b_1, \ldots, b_n) \neq 0$.

We will now formulate a counterpart of Dubois' Nullstellensatz for nonstandard real analytic function germs. Let R be the higher order structure over the set $\mathbf{R}$ of real numbers, ${}^*R$ an enlargement of R, and $\overline{\Gamma}_n = \overline{\Gamma}_n({}^*R)$ the ring of nonstandard germs of real functions in n variables analytic at the origin. As before we define the nonstandard variety germ of $\Phi_1, \ldots, \Phi_m \in \overline{\Gamma}_n$ by

$$v(\Phi_1, \ldots, \Phi_m) = \{\langle a_1, \ldots, a_n\rangle \in \mu(0, \ldots, 0) : \Phi_i(a_1, \ldots, a_n) = 0 \text{ for } 1 \leqslant i \leqslant m\}.$$

Then we have the following theorem:

5.3. *Suppose* $\Phi_1, \ldots, \Phi_m, \Psi \in \overline{\Gamma}_n$. *Then* $v(\Psi) \supseteq v(\Phi_1, \ldots, \Phi_m)$ *if and only if there exist* $k \geqslant 1$, $\Theta_1, \ldots, \Theta_r \in \overline{\Gamma}_n$ *such that* $\Psi^{2k} + \Theta_1^2 + \cdots + \Theta_n^2$ *belongs to the ideal generated by* $\Phi_1, \ldots, \Phi_m$ *in* $\overline{\Gamma}_n$.

This theorem is essentially the nonstandard version of a theorem of J.-J.

Risler's (see [3, Theorem 1]); a part of this theorem was also obtained independently by G. Efroymson in [2]).

To prove 5.3 we need only make a few modifications in the proof of Theorem 3.1. We replace **C** by **R**, R_n by the ring $\bar{R}_n$ of convergent power series in $\mathbf{R}[[X_1, \ldots, X_n]]$, R_n^0 by the maximal ideal $\bar{R}_n^0$ of $\bar{R}_n$, R_ω by $\bar{R}_\omega = \bigcup_{n<\omega} \bar{R}_n$, and R_ω^0 by $\bar{R}_\omega^0 = \bigcup_{n<\omega} \bar{R}_n^0$. The validity of 1.1–1.5 is not affected by this change. L is replaced by the language $\bar{L}$ containing

(i) the function-symbols $+, -, \cdot$, and for every $f \in \bar{R}_n$, $n \geq 1$, a n-ary function-symbol **f**,

(ii) for every $c \in \mathbf{R}$ a constant-symbol **c**,

(iii) a unary predicate-symbol U and a binary relation-symbol $<$. The sets $\bar{T}, \bar{T}^0$ of terms and special terms in $\bar{L}$ are defined analogous to T, T^0. Atomic formulas in $\bar{L}$ are expressions of form $t_1 = t_2$, $t_1 < t_2$, or $U(t_1)$ for $t_1, t_2 \in \bar{T}$. We replace A', A by theories $\bar{A}' \subseteq \bar{A}$ in $\bar{L}$ with the following axioms: Axioms for the theory of real closed fields formulated in $\bar{L}$; the axioms $U(0)$, $\neg U(\mathbf{c})$ for $0 \neq \mathbf{c} \in \mathbf{R}$, $(\exists x)_U(x \neq 0)$; for every $\underline{t}(x_1, \ldots, x_n) \in \bar{T}^0$ the axiom $(\forall x_1)_U \cdots (\forall x_n)_U(U(t(x_1, \ldots, x_n)))$; the analogues of the axioms 1.10(i); for every pair t_1, t_2 of variable-free terms in $\bar{T}$ such that $f_{t_1} < f_{t_2}$ holds in $\bar{R}_0 = \mathbf{R}$ the axiom $t_1 < t_2$. These are all the axioms of $\bar{A}'$; $\bar{A}$ has in addition the axioms 1.11 and the axiom $\forall x \forall y (0 < x < y \wedge U(y) \rightarrow U(x))$.

The models of $\bar{A}'$ and $\bar{A}$ can be characterized in a way similar to 1.12. We remark in particular that a model M of $\bar{A}'$ is a model of $\bar{A}$ if and only if U_M is an isolated subring of M and integrally closed in M. Any enlargement $^*R$ of R gives rise to a model $M(^*R)$ of $\bar{A}$ defined similar to $M(^*R)$. Likewise any prime ideal $\bar{J}$ in $\bar{R}_n$, such that $\bar{J} \subsetneq \bar{R}_n^0$ and $\bar{R}_n/\bar{J}$ can be embedded into a real closed field, $\bar{K}$ induces a model $M(\bar{J}, \bar{K})$ of $\bar{A}'$ defined similar to $M(J, K)$.

The transfer principle 2.1 carries over without difficulty to the language $\bar{L}$ and the theories $\bar{A}', \bar{A}$ instead of L, A', A; it suffices to replace the quantifier elimination procedure for the theory of algebraically closed fields in the proof of 2.3 by the corresponding procedure for real closed fields (see [8]).

The proof of theorem 5.3 is then completed as in §3 by constructing a generic point $a_1, \ldots, a_n$ for suitable ideal $\bar{J} \subset R_n$ (obtained this time by lemma 5.2) in a model $M(\bar{J}, \bar{K})$ of $\bar{A}'$ and applying the transfer principle for $\bar{A}'$ and $\bar{A}$.

The author knows of no analogue of Seidenberg's constructive proof for Hilbert's Nullstellensatz that could be used to show that theorem 5.3 is constructive relative to the Weierstrass preparation theorem (in the sense of §4). It is, however, easy to see that Dubois' Nullstellensatz 5.1 is constructive, if the ground field F is the field of rational numbers.

As remarked at the end of §3, we can combine theorem 5.3 with A. Robinson's argument in [4, §5], to get an analogous theorem for n.s. real cylindrical function germs.

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