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PL INVOLUTIONS OF $S^1 \times S^1 \times S^1$

BY

KYUNG WHAN KWUN[1] AND JEFFREY L. TOLLEFSON[2]

ABSTRACT. We prove that the 3-dimensional torus $S^1 \times S^1 \times S^1$ admits exactly nine nonequivalent PL involutions. With the exception of the four fixed point free ones, the involutions may be distinguished by their fixed point sets: (1) eight points, (2) two simple closed curves, (3) four simple closed curves, (4) one torus, (5) two tori.

1. Introduction. Recently, Hempel [3] classified all free cyclic actions on $S^1 \times S^1 \times S^1$ and in particular showed there exist exactly four nonequivalent free involutions of $S^1 \times S^1 \times S^1$. Also, Showers has shown [8] that there exist exactly two nonequivalent PL involutions of $S^1 \times S^1 \times S^1$ with 2-dimensional fixed point set. Employing the general principle initiated in [9] (also see [4] and [6]), we will classify *all* PL involutions of $S^1 \times S^1 \times S^1$. More precisely, we prove

THEOREM A. *The 3-dimensional torus $S^1 \times S^1 \times S^1$ admits exactly nine nonequivalent PL involutions. With the exception of the four fixed point free ones, the involutions may be distinguished by their fixed point sets*: (1) *eight points,* (2) *two simple closed curves,* (3) *four simple closed curves*, (4) *one torus, and* (5) *two tori.*

A main tool of the proof of Theorem A is the following:

THEOREM B. *Let M be an orientable 3-manifold and T a torus PL-embedded in M such that the homomorphism $H_1(T; Z) \to H_1(M; Z)$ induced by inclusion is injective. Suppose h is a PL involution of M and U a neighborhood of $T \cup h(T)$. Then there exists a torus T' PL-embedded in U such that the homomorphism $H_1(T'; Z) \to H_1(M; Z)$ is injective, either $h(T') = T'$ or $T' \cap h(T') = \emptyset$, and T' is in general position relative to the fixed point set of h.*

Throughout the paper, we will work in the PL category and the fixed point set of an involution h will be denoted by Fix h.

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2. PL involutions of $S^1 \times S^1$. There are exactly five nonequivalent PL involutions of $S^1 \times S^1$. With the exception of the two obvious free ones, they are distinguished by the fixed point sets: (1) four points, (2) one simple closed curve, and (3) two simple closed curves. This must be well known. It follows anyhow from a simple version of our proof for the case of $S^1 \times S^1 \times S^1$.

3. Involutions of $S^1 \times S^1 \times S^1$. We regard S^1 as the set of complex numbers of norm 1. We list nine involutions h_i of $S^1 \times S^1 \times S^1$ with fixed point sets F_i $(i = 1, 2, \cdots, 9)$ here, and we later show that every involution is equivalent to one of these h_i. For the sake of convenience, an involution which is obviously equivalent to h_i will be denoted by h_i even if it is not exactly as listed below.

(1) $h_1(x, y, z) = (x, y, -z)$, $F_1 = \emptyset$;
(2) $h_2(x, y, z) = (\bar{x}, \bar{y}, -z)$, $F_2 = \emptyset$;
(3) $h_3(x, y, z) = (x, \bar{y}, -z)$, $F_3 = \emptyset$;
(4) $h_4(x, y, z) = (xy, \bar{y}, -z)$, $F_4 = \emptyset$;
(5) $h_5(x, y, z) = (\bar{x}, \bar{y}, \bar{z})$, $F_5 =$ eight points;
(6) $h_6(x, y, z) = (\bar{x}, y, \bar{z})$, $F_6 =$ four simple closed curves;
(7) $h_7(x, y, z) = (y, x, \bar{z})$, $F_7 =$ two simple closed curves;
(8) $h_8(x, y, z) = (x, y, \bar{z})$, $F_8 =$ two tori, and
(9) $h_9(x, y, z) = (y, x, z)$, $F_9 =$ one torus.

4. Proof of Theorem A. We assume Theorem B which will be proved in § 6. In this section, we will assume that T' of Theorem B misses the fixed point set of the involution. In the next section, we will show that this can always be assumed. Let h be the involution and $F = \operatorname{Fix} h$. Since the case for $\dim F = -1$ or 2 is known [3], [8], we only consider the case $\dim F = 0$ or 1.

The incompressible torus T' as above will be simply denoted by T. Cut $M = S^1 \times S^1 \times S^1$ along T to obtain a space homeomorphic to $T \times [0, 1]$. This follows from [1], as the inclusion of T into this space induces an isomorphism of fundamental groups (as can be easily seen). Before we proceed, we state the following proposition which is a special case of Theorem B of [4].

PROPOSITION 4.1. *Let* $\alpha\colon T \times [0, 1] \longrightarrow T \times [0, 1]$ *be an involution. Then there exists an involution* β *of* T *and a product structure of* $T \times [0, 1]$ *such that* $\alpha(x, t) = (\beta(x), t)$ *or* $\alpha(x, t) = (\beta(x), 1 - t)$ *for* $x \in T$ *and* $t \in [0, 1]$.

Now $h(T) = T$ or $h(T) \cap T = \emptyset$. Suppose $h(t) = T$. If near T, the two sides are not switched under h. Then by the above proposition and the assumption that $F \cap T = \emptyset$, F would be empty. But $\dim F = 0$ or 1.

If the two sides are switched, we may choose T so that $T \cap h(t) = \varnothing$. In any case, we may assume $T \cap h(T) = \varnothing$. In fact, h and M may be viewed as follows. M is obtained from $T \times [0, 1]$ by a suitable identification in which $T = T \times 0$ and $h(T) = T \times 1/2$. $h|T \times [0, 1/2]$ is given by $h(x, t) = (g(x), 1/2 - t)$ and $h|T \times [1/2, 1]$ is given by $h(x, t) = (\bar{g}(x), 3/2 - t)$, where g and $\bar{g}$ are involutions of T and $(x, 0)$ is identified with $(\bar{g}g^{-1}(x), 1)$ for each $x \in T$. That is, M is viewed as $T \times [0, 1]/\bar{g}g^{-1}$.

Case 1. $\dim F = 0$. We may assume that the fixed point set of $h|T \times [0, 1/2]$ is 0-dimensional. In this case we may assume g is given by $g(x, y) = (\bar{x}, \bar{y})$ (see §2). Then by Conner [2], $h|T \times [1/2, 1]$ also has four points as its fixed point set. Hence by §2, $\bar{g} = fgf^{-1}$ for some homeomorphism f of T. We assert that f is isotopic to a homeomorphism $\bar{f}$ that commutes with g. To this end, recall that the group if isotopy classes of T is isomorphic to the multiplicative group of unimodular 2×2 matrices and is generated by the isotopy classes of the three homeomorphism of T defined as follows:

$$\psi_1(x, y) = (\bar{y}, x), \quad \psi_2(x, y) = (\bar{y}, xy), \quad \psi_3(x, y) = (\bar{x}, y).$$

Observe that g commutes with each ψ_i. As f is isotopic to a product of the ψ_i's, it follows that the desired homeomorphism $\bar{f}$ does exist.

Let $H: T \times [0, 1] \longrightarrow T$ be an isotopy from the identity to $f\bar{f}^{-1}$. We define a homeomorphism θ from $T \times [0, 1]/1_T$ to $T \times [0, 1]/\bar{g}g^{-1}$ in the following manner:

$$(*) \qquad \theta(x, t) = \begin{cases} [x, t] & 0 \leqslant t \leqslant 1/2, \\ [H(x, 4t - 2), t], & 1/2 < t < 3/4, \\ [\bar{g}H(g^{-1}(x)), 4 - 4t], & 3/4 \leqslant t \leqslant 1. \end{cases}$$

It is easily checked that θ is a well-defined homeomorphism. Moreover, θ is an equivalence between the involutions h and h_5; that is $h = \theta h_5 \theta^{-1}$.

Case 2. $\dim F = 1$ and g is conjugate to $\bar{g}$. We may assume g is now given by $g(x, y) = (x, \bar{y})$. Let f be a homeomorphism such that $\bar{g} = fgf^{-1}$. As $T \times [0, 1]/\bar{g}g^{-1}$ is a product, we must have $\bar{g}g^{-1} = fgf^{-1}g^{-1}$ isotopic to the identity. Consequently, the isotopy class of f commutes with that of g. Consider the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ corresponding to g. It is easily checked that, in order to commute with $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, the matrix corresponding to f must be one of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, or $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. It follows that f is isotopic to one of four obvious homeomorphisms, corresponding to these four matrices, which commutes with g. Denote it by $\bar{f}$. Now, as in Case 1, let H be an isotopy from the identity to $f\bar{f}^{-1}$ and define a homeomorphism $\theta: T \times S^1 \longrightarrow T \times [0, 1]/\bar{g}g^{-1}$ as shown in $(*)$. θ has the property that $h = \theta h_6 \theta^{-1}$.

Case 3. $\dim F = 1$ and g is not conjugate to $\bar{g}$. We may assume that $g(x, y) = (x, \bar{y})$ and $T \times [\frac{1}{2}, 1]$ is disjoint from F. Then $\bar{g} = frgf^{-1}$ for some homeomorphism f of T, where r is the rotation defined by $r(x, y) = (-x, y)$. Exactly as in Case 2, it follows that f is isotopic to a homeomorphism $\bar{f}$ commuting with g. Let H be an isotopy from the identity to $f\bar{f}^{-1}$. Define the homeomorphism

$$\theta: T \times [0, 1]/r \longrightarrow T \times [0, 1]/\bar{g}g^{-1}$$

again by the formula (*). This time we have $h = \theta h_7 \theta^{-1}$.

5. **Proof of Theorem A continued.** Let h be an involution on $M = S^1 \times S^1 \times S^1$ with $\dim F = 1$. Suppose we have an incompressible torus T in M that is invariant under h, meets F and is in general position with respect to F. The goal of this section is to show that there is also an incompressible torus T' in M disjoint from F such that either $hT' \cap T' = \emptyset$ or $hT' = T'$. Once this is accomplished, it follows that the classification of §4 includes all involutions of M with $\dim F = 0$ or 1.

As before, if we cut M along T we obtain a space homeomorphic to $T \times [0, 1]$. Let us view M as $T \times [0, 1]/\varphi$ with $T = T \times 0$. It follows from Proposition 4.1 that we may assume h defines an involution $g \times 1$ on $T \times [0, 1]$, where $g(x, y) = (\bar{x}, \bar{y})$, $(x, y) \in T$. Observe that $T = T \times 0$ meets every component of F. Thus, if we succeed in finding an incompressible torus in M that misses at least one component of F, then an application of Theorem B will give us our desired torus disjoint from F.

We first consider the case when F has four components and then proceed to the remaining cases. We require the following observation.

PROPOSITION 5.1. *Let α, β be two isotopic, nonseparating, simple closed curves in $S^1 \times S^1$ and let p be any point in $S^1 \times S^1 - (\alpha \cup \beta)$. Then α is isotopic to β in $S^1 \times S^1 - \{p\}$.*

PROOF. Isotop α into general position with respect to β. We induct on the number of components in $\alpha \cap \beta$. If $\alpha \cap \beta = \emptyset$ then the conclusion is obvious. Suppose $\alpha \cap \beta$ consists of n points. Let D be an innermost disk in $S^1 \times S^1 - (\alpha \cup \beta)$ bounded by the union of an arc a in α and an arc b in β such that $a \cap \beta = b \cap \alpha = \partial a = \partial b$. If $p \notin D$ then use D to isotop α in $S^1 \times S^1 - \{p\}$ to reduce the number of points in $\alpha \cap \beta$ by two. If $p \in D$ then consider a simple closed curve α' obtained in the following manner. Let $U(a)$ be a regular neighborhood of a in α such that $\overline{U(a) \cap \beta} = \partial a$. Construct α' from $\overline{\alpha - U(a)}$ by attaching an arc to $\partial(\overline{\alpha - U(a)})$ that is parallel to

b and outside D such that α' is isotopic to α and $\alpha' \cap \beta$ contains $n-2$ points. Now α' is isotopic to β in $S^1 \times S^1 - \{p\}$ by the induction hypothesis and α is clearly isotopic to α' in $S^1 \times S^1 - \{p\}$.

Case 1. F consists of four circles, i.e. $\varphi|F \cap T$ is the identity. Let p denote one of the four points $F \cap T$ in T. Choose a nonseparating simple closed curve α in T disjoint from p. Since φ is isotopic to the identity, α and $\beta = \varphi(\alpha)$ are simple closed curves isotopic in T, and thus in $T - \{p\}$. Let $H: S^1 \times [0, 1] \longrightarrow T - \{p\}$ be an isotopy from α to β. Define an incompressible embedding γ of $S^1 \times S^1$ in $M = T \times [0, 1]/\varphi$ by $\gamma(x, t) = [H(x, t), t]$. Let T' be the image of γ in M. Notice that T' does not meet the component $\{p\} \times [0, 1]/\varphi|\{p\}$ of F. Therefore, there exists an incompressible torus T'' in M disjoint from F such that either $hT'' \cap T'' = \emptyset$ or $hT'' = T''$.

Case 2. $\varphi|F \cap T$ has period two. $M = T \times [0, 1]/\varphi$ is double-covered by $\widetilde{M} = T \times [0, 1]/\varphi^2$ in a manner where the involution h on M is covered by the involution $\widetilde{h}$ on $\widetilde{M}$ defined by $\widetilde{h}([x, t]) = [g(x), t]$ (this is a consequence of Case 1 and §4). Observe that F consists of two circles that are covered by the four circles of $\widetilde{F} = \mathrm{Fix}(\widetilde{h})$. As we have seen, there is an incompressible torus $\widetilde{T}$ in $\widetilde{M}$ disjoint from $\widetilde{F}$. Let τ be the covering transformation of the two-sheeted covering $\pi: \widetilde{M} \longrightarrow M$. Apply Theorem B to τ and $\widetilde{T}$ to get an incompressible torus $\widetilde{T}_1$ in $\widetilde{M}$ disjoint from $\widetilde{F}$ such that either $\tau(\widetilde{T}_1) \cap \widetilde{T}_1 = \emptyset$ or $\tau(\widetilde{T}_1) = \widetilde{T}_1$. In either case, $\pi(\widetilde{T}_1)$ is an incompressible torus or Klein bottle in M disjoint from F. If the latter situation occurs, we can find the desired incompressible torus as the boundary of a regular neighborhood of the Klein bottle. Now, as in Case 1, another application of Theorem B yields the sought after incompressible torus in M disjoint from F.

Case 3. $\varphi|F \cap T$ has period four. In this case, F is connected. We proceed in the same manner as we did in Case 2. More specifically, M is double-covered by $\widetilde{M} = T \times [0, 1]/\varphi^2$, where $\varphi^2|T \cap \widetilde{F}$ has period 2. The existence of an incompressible torus in $\widetilde{M}$ disjoint from $\widetilde{F}$ leads to the existence of an incompressible torus in M disjoint from F. Of course, it now follows from §4 that this case does not occur.

Case 4. $\varphi|F \cap T$ has period three. We will show that this case does not occur either, which then completes the proof of Theorem A. In order to rule out the present situation, we require the following proposition.

PROPOSITION 5.2. *Each component of F represents an indivisible element of $\pi_1(M)$.*

PROOF. The argument here is not unlike the one given in the proof of Proposition 2.2 of [5]. Let C be a component of F and choose a base point $x_0 \in C$ for M. Suppose $[C] = \alpha = \beta^n$ in $\pi_1(M, x_0)$, where $n > 0$. Let $p\colon (R^3, \widetilde{x}_0) \longrightarrow (M, x_0)$ be the universal covering of M. Lift h to an involution $\widetilde{h}$ of R^3 with $\widetilde{h}(\widetilde{x}_0) = \widetilde{x}_0$. $\mathrm{Fix}(\widetilde{h}) = \widetilde{C}$, where $\widetilde{C}$ is the component of $p^{-1}(C)$ containing $\widetilde{x}_0$. Let λ be a loop based at x_0 representing β in $\pi_1(M, x_0)$. Lift λ to a path $\widetilde{\lambda}$ beginning at $\widetilde{x}_0$ with endpoint y. Since $n > 0$, it follows that $y \notin \widetilde{C}$. Because $h_\#(\beta^n) = \beta^n$ and $\pi_1(T)$ is a torsion free abelian group, we see that $h_\#(\beta) = \beta$. Thus the loop λ and $h(\lambda)$ both lift to paths beginning at $\widetilde{x}_0$ and ending at y. However, $\widetilde{h}(\widetilde{\lambda})$ is also a lifting of $h(\lambda)$ beginning at $\widetilde{x}_0$. Hence $\widetilde{h}(y) = y$, in contradiction to $\mathrm{Fix}(\widetilde{h}) = \widetilde{C}$. Therefore, $[C]$ is indivisible in $\pi_1(M)$.

Returning to Case 4, we have given the period of $\varphi|F \cap T$ is three. Thus, we may choose generators α, β, γ of $\pi_1(M) \cong Z \oplus Z \oplus Z$ such that the generator α is represented by one component of F and $\alpha^3\beta^k$ is represented by the other component (for some k). Since $\alpha^3\beta^k$ is indivisible, $k \not\equiv 0 \pmod 3$. Furthermore, $h_\#(\alpha) = \alpha$ and $h_\#(\alpha^3\beta^k) = \alpha^3\beta^k$. It follows that $h_\#(\beta) = \beta$ also. Hence $h_\#$ corresponds to a matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{pmatrix}.$$

Since h is an orientation-preserving involution, we have $a = b = 0$ and $c = 1$, that is, $h_\#$ is the identity. But $h_\#$ clearly cannot be the identity. Consequently, it is not possible for two components of F to be related in $\pi_1(M)$ as α and $\alpha^3\beta^k$, where $k \not\equiv 0 \pmod 3$.

6. **Proof of Theorem B.** Let h be an involution on the 3-manifold M. A closed surface T embedded in M is in *h-general position modulo* $\mathrm{Fix}(h)$ if (i) T is in general position with respect to $\mathrm{Fix}(h)$, (ii) $T - \mathrm{Fix}(h)$ and $h(T) - \mathrm{Fix}(h)$ are in general position, and (iii) all cuts among T, $h(T)$, and $\mathrm{Fix}(h)$ are locally piercing except perhaps between T and $h(T)$ at vertices of $F = \mathrm{Fix}(h)$.

Suppose we have a torus T in M such that $H_1(T; Z) \longrightarrow H_1(M; Z)$ is an injection. We follow the idea of equivariant surgery as described in [4] and [9] to obtain the torus T' whose existence is asserted in Theorem B. Throughout, observe that all constructions may take place inside of any neighborhood of $T \cup h(T)$.

First, isotop T into h-general position modulo F. Now assign a complexity $c(T) = (a, b)$ to T. Let a denote the number of components of

$[T \cap h(T)] - F$ and let b denote the number of components of $F \cap T$. We order such complexities in lexigraphical fashion.

Let Σ denote the set of all tori T in M with the properties (1) $H_1(T; Z) \longrightarrow H_1(M; Z)$ is injective, and (2) either $h(T) = T$ and T is in general position with respect to F, or T is in h-general position modulo F. We assign the complexity $(0, 0)$ to an invariant torus in Σ.

Let $T \in \Sigma$. If $C(T) > (0, 0)$, then by equivariant surgery on T (the description of which follows) we produce a new torus $T' \in \Sigma$ such that $C(T') < C(T)$. Thus, if we choose a member T of Σ of minimal complexity, it follows that $C(T) = (0, 0)$ (Σ is clearly not empty under our hypothesis). Such a torus T will satisfy the conclusion of the theorem.

Suppose that $T \in \Sigma$ and $C(T) > (0, 0)$. We divide the surgery into three cases.

Case 1. $T \cap h(T)$ consists of a single, invariant simple closed curve J. Let U be a small invariant regular neighborhood of J such that $U \cap (T \cup h(T))$ is composed of two annuli $A \subset T$, $A' = h(A)$ crossing along the curve J. The boundary of U, ∂U, is separated by $\partial A \cup \partial A'$ into two pairs of nonadjacent annuli $\{B_1, C_1\}$, $\{B_2, C_2\}$. Observe that $h(B_i \cup C_i) = B_i \cup C_i$ $(i = 1, 2)$. Let

$$T_i = [(T \cup h(T)) - (A \cup A')] \cup [B_i \cup C_i].$$

Both T_1 and T_2 are invariant under h and at least one has the property that $H_1(T_i) \longrightarrow H_1(M)$ is injective.

Suppose that $T \cap h(T)$ is neither empty nor a simple closed curve. We shall refer to the closure of a component of $h(T) - T$ as being *innermost* in $h(T)$ if its interior is disjoint from T. Then there exists either an innermost disk or annulus E in $h(T)$.

Case 2. E is a disk in $h(T)$. Let $J = \partial E$. $J \cap F$ may consist of any one of the following: $\varnothing$, J, one point or arc, or more than one component (each an arc or a point). Since T is incompressible in M, J must separate T into two components E_1 and E_2, where E_2 is a disk bounded by J. Choose small regular neighborhoods U, V of E, T, respectively, such that $U \cap V$ is a regular neighborhood of J. Let E' be a disk inside U parallel to E such that (i) $E' \cap (T \cup h(T)) = E' \cap T = \partial E'$, (ii) $E' \cap E = J \cap F$, (iii) $\partial E' \cup J$ bounds A in E_1, where A is homeomorphic to the quotient space obtained from $J \times [0, 1]$ by identifying $\{x\} \times [0, 1]$ to a point for each $x \in J \cap F$, and (iv) the interior of the region in U bounded by $E \cup E' \cup A$ is disjoint from $T \cup h(T)$. Now define $T' = E' \cup (E_1 - A)$. If T' is not transverse to F along $J \cap F$, then pull T' and $h(T')$ apart slightly along the components of $F \cap J$ where this occurs. Then $T' \in \Sigma$ and $C(T') < C(T)$.

Case 3. E is an innermost annulus (possibly pinched at a point in $h(T)$) and Case 2 does not occur. ∂E separates T into two annuli, say E_1 and E_2. Let $T_i = E \cup E_i$ $(i = 1, 2)$. Suppose that $i_*: H_1(T_1) \longrightarrow H_1(M)$ is not injective. If α is an element of $H_1(M)$ represented by one of the components of ∂E, then we can find another element β so that the image of $H_1(T)$ in $H_1(M)$ is generated by α and β. (For simplicity let us assume $H_1(M)$ is torsion free. In the case where it is not, simply project onto $H_1(M)/\mathrm{Tor}(H_1(M))$.) Then $i_*(H_1(T_1))$ is generated by α. Hence, the image of $H_1(T_2)$ is generated by α and $\beta\alpha^n$ for some integer n. So $H_1(T_2) \longrightarrow H_1(M)$ is injective if i_* fails to be.

Let $T' = E \cup E_i$, where i is chosen so that $H_1(T') \longrightarrow H_1(M)$ is injective. If T' is invariant under h, then (if required) we shift T' into general position with respect to F with T' remaining invariant under h. If T' is not invariant, then we proceed in the following manner. (The index i, once chosen, is fixed in what follows.)

Let U and V be small regular neighborhoods in M of E and T, respectively, such that $U \cap V$ is a regular neighborhood of ∂E. First we treat the easiest situation, which is when $E_i \cap U$ lies on only one side of E.

Subcase (a). $E_i \cap (U - h(T))$ is contained in a single component of $U - h(T)$ and either $h(\partial E) = \partial E$ or $h(\partial E) \cap \partial E = \emptyset$. Find an annulus E' in U parallel to E such that (i) $E' \cap (T \cup h(T)) = E' \cap T = \partial E'$, (ii) $E' \cap E = \partial E \cap F$, (iii) $\partial E' \cup \partial E$ bounds A in E_i, where A is homeomorphic to the quotient space obtained from $\partial E \times [0, 1]$ by identifying $\{x\} \times [0, 1]$ to a point for each $x \in F \cap \partial E$, and (iv) the interior of the region in U bounded by $E \cup E' \cup A$ is disjoint from $T \cup h(T)$. Define $D'' = E' \cup [E_i - A]$. If required, we pull T'' and $h(T'')$ apart along the components of $\partial E \cap F$ where T'' is not transverse to F. Then $T'' \in \Sigma$ and $C(T'') < C(T)$.

Subcase (b). $E_i \cap (U - h(T))$ meets both components of $U - h(T)$ and either $h(\partial E) = \partial E$ or $h(\partial E) \cap \partial E = \emptyset$. Find an annulus E' in U parallel to E such that E' fulfills conditions (i), (iii), and (iv) of Subcase (a) and also the following: (ii′) $E' \cap E = (\partial E \cap F) \cup J$, where J is a simple closed curve in $E - \partial E$ that is parallel to a component of ∂E and E' crosses E at J. Now finish exactly as in Subcase (a).

Subcase (c). $h(\partial E) \cap \partial E = J_1$, where J_1 is one component of ∂E. Let J_2 denote the other component of ∂E. Now find an annulus E' in U parallel to E such that (i) $E' \cap (T \cap h(T)) = E' \cap T = \partial E'$, (ii) $E \cap E' = J_1 \cup (J_2 \cap F)$, (iii) $(\partial E' - J_1) \cup J_2$ bounds A in E_i, where A is homeomorphic to the quotient space of $J_2 \times [0, 1]$ obtained by identifying $\{x\} \times [0, 1]$ to a point for each $x \in J_2 \cap F$, and (iv) the interior of the region in U bounded by $E \cup E' \cup A$ is disjoint from $T \cup h(T)$. Define $T'' = E' \cup (E_i - A)$ and finish as above to make T'' transverse to F. This completes the proof of Theorem B.

7. **An example.** We give an example to show that Theorem B cannot be improved significantly. A class Σ of surfaces embedded in 3-manifolds is called *admissible* (for equivariant surgery) if, given any 3-manifold M with a PL involution h and a surface F in M belonging to Σ, one can perform equivariant surgery on F to obtain a new surface F' in M, also belonging to Σ, such that either $h(F') = F'$ or $h(F') \cap F' = \emptyset$.

According to Theorem B, the class of "first homology injective" tori (in orientable M) is admissible. We exhibit with the following example that the larger class of incompressibly embedded tori is not admissible.

EXAMPLE. Let M be the orientable Seifert fiber space with invariants $\{0_0; 0|0; (4, 1), (4, 1), (4, 1)\}$. $\pi_1(M)$ has the presentation $(q_1, q_2, q_3, h | q_1q_2q_3, q_1^4h, q_2^4h, q_3^4h)$. It is easily checked that $H_1(M) \cong Z_{12} \oplus Z_4$. In [10] it is shown that M does not contain any orientable, incompressible surfaces. Consider the two-sheeted covering $p: \widetilde{M} \longrightarrow M$ determined by the kernel of the homomorphism $\lambda: \pi_1(M) \longrightarrow Z_2$ defined by $\lambda(q_1) = \lambda(q_2) = 1$ and $\lambda(q_3) = \lambda(h) = 0$. Denote by h the free involution on $\widetilde{M}$ serving as the nontrivial covering transformation for p.

Observe that $\widetilde{M}$ is a Seifert fiber space with invariants $\{0_0; 0|0; (2, 1), (2, 1), (4, 1), (4, 1)\}$ (for example, see [7]). $\widetilde{M}$ contains an incompressible torus separating $\widetilde{M}$ into two components, each containing two exceptional fibers. Suppose that the class of incompressibly embedded tori were admissible. Then there would exist an incompressible torus T in $\widetilde{M}$ such that $h(T) = T$ or $h(T) \cap T = \emptyset$. In the first case, h interchanges the sides of T. In either event, there would always be an incompressible torus T in $\widetilde{M}$ with $h(T) \cap T = \emptyset$. But this T would then cover an incompressible torus $p(T)$ in M.

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DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MICHIGAN 48823 (Current address of K. W. Kwun)

DEPARTMENT OF MATHEMATICS, TEXAS A & M UNIVERSITY, COLLEGE STATION, TEXAS 77843

Current address (J. L. Tollefson): Department of Mathematics, University of Connecticut, Storrs, Connecticut 06268