

ON ENTIRE FUNCTIONS OF FAST GROWTH

BY

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ABSTRACT. Let

$$(*) \quad f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$$

be a transcendental entire function. Set

$$M(r) = \max_{|z|=r} |f(z)|, \quad m(r) = \max_{n \geq 0} \{a_n |r^{\lambda_n}\}$$

and

$$N(r) = \max_{n \geq 0} \{\lambda_n | m(r) = a_n |r^{\lambda_n}\}.$$

Sato introduced the notion of growth constants, referred in the present paper as S_q -order λ and S_q -type T , which are generalizations of concepts of classical order and type by defining

$$(**) \quad \lambda = \limsup_{r \rightarrow \infty} (\log^{[q]} M(r) / \log r)$$

and if $0 < \lambda < \infty$, then

$$(***) \quad T = \limsup_{r \rightarrow \infty} (\log^{[q-1]} M(r) / r^\lambda)$$

for $q = 2, 3, 4, \dots$ where $\log^{[0]} x = x$ and $\log^{[q]} x = \log(\log^{[q-1]} x)$. Sato has also obtained the coefficient equivalents of (**) and (***) for the entire function (*) when $\lambda_n = n$. It is noted that Sato's coefficient equivalents of λ and T also hold true for (*) if n 's are replaced by λ_n 's in his coefficient equivalents. Analogous to (**) and (***) lower S_q -order ν and lower S_q -type t for entire function $f(z)$ are introduced here by defining

$$\nu = \liminf_{r \rightarrow \infty} (\log^{[q]} M(r) / \log r)$$

and if $0 < \lambda < \infty$ then

$$t = \liminf_{r \rightarrow \infty} (\log^{[q-1]} M(r) / r^\lambda), \quad q = 2, 3, 4, \dots$$

For the case $q = 2$, these notions are due to Whittaker and Shah respectively.

For the constant ν , two complete coefficient characterizations have been found

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which generalize the earlier known results. For t coefficient characterization only for those entire functions for which the consecutive principal indices are asymptotic is obtained. Determination of a complete coefficient characterization of t remains an open problem. Further S_q -growth and lower S_q -growth numbers for entire function $f(z)$ we defined

$$\delta = \lim_{\mu \rightarrow \infty} \sup \inf (\log^{[q-1]} N(r) |r^\lambda),$$

for $q = 2, 3, 4, \dots$ and $0 < \lambda < \infty$. Earlier results of Juneja giving the coefficients characterization of δ and μ are extended and generalized. A new decomposition theorem for entire functions of S_q -regular growth but not of perfectly S_q -regular growth has been found.

1. Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be a transcendental entire function. Set the following:

$$M(r) \equiv M(r, f) = \max_{|z|=r} |f(z)|, \quad m(r) \equiv m(r, f) = \max_{n \geq 0} \{|a_n| r^{\lambda_n}\},$$

and

$$N(r) \equiv N(r, f) = \max_{n \geq 0} \{\lambda_n | m(r) = |a_n| r^{\lambda_n}\}.$$

$M(r)$, $m(r)$ and $N(r)$ are known as the maximum modulus, maximum term and the rank of the maximum term, respectively, for $f(z)$. It is well known that the functions $\log M(r)$ and $\log m(r)$ are increasing convex functions of $\log r$. Further, it is known that $N(r)$ is an unbounded, nondecreasing step function of r with a left-hand discontinuity at jump points. The values attained by $N(r)$ at these points are called principal indices. We denote by $\{\rho(n_s)\}$ the jump points of $N(r)$ and its range by $\{\lambda_{n_s}\}$. Following Sato [9], we define

$$(1.1) \quad \lambda = \lim_{r \rightarrow \infty} \sup \frac{\log^{[q]} M(r)}{\log r}$$

and

$$(1.2) \quad T = \lim_{r \rightarrow \infty} \sup \frac{\log^{[q-1]} M(r)}{r^\lambda}, \quad q = 2, 3, 4, \dots,$$

where $\log^{[q]} x$ stands for $\log \log \dots (q \text{ times}) x$ and (1.2) has a meaning only when $0 < \lambda < \infty$. Sato [9] obtained the following coefficient characterization for λ and T .

THEOREM S. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function with the constants λ and T as defined in (1.1) and (1.2), then

$$(1.3) \quad \lambda = \lim_{n \rightarrow \infty} \sup \frac{n \log^{[q-1]} n}{\log |a_n|^{-1}}$$

and

$$(1.4) \quad T = \limsup_{n \rightarrow \infty} |a_n|^{\lambda/n} \log^{[q-2]} \frac{n}{e\lambda}$$

for $q = 2, 3, 4, \dots$.

Note that, if instead of $f(z) = \sum_{n=0}^{\infty} a_n z^n$, we choose the entire function $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, then Theorem S holds true if we replace n by λ_n in (1.3) and (1.4). The constants λ and T are, in fact, the generalizations of the notions of order and type introduced in Valiron [13, p. 34]. We call these constants Sato q th-iterate order and Sato q th-iterate type and, for simplicity, S_q -order λ and S_q -type T . Similar to S_q -order λ and S_q -type T we introduce the following:

$$(1.5) \quad \nu = \liminf_{r \rightarrow \infty} \frac{\log^{[q]} M(r)}{\log r}$$

and

$$(1.6) \quad t = \liminf_{r \rightarrow \infty} \frac{\log^{[q-1]} M(r)}{r^\lambda}.$$

We call these constants lower S_q -order ν and lower S_q -type t . In fact, these constants are generalizations of the concepts of lower order and lower type introduced respectively by Whittakar [14] and Shah [11]. Whenever $\lambda = \nu$, $f(z)$ is said to be of S_q -regular growth and if, in addition $T = t$, it is said to be of perfectly S_q -regular growth. In case $\nu < \lambda$, $f(z)$ is of S_q -irregular growth. We, further, introduce the S_q -growth number δ and lower S_q -growth number μ for $f(z)$ by the following:

$$(1.7) \quad \begin{aligned} \delta &= \limsup_{r \rightarrow \infty} \frac{\log^{[q-2]} N(r)}{r^\lambda} \\ \mu &= \liminf_{r \rightarrow \infty} \frac{\log^{[q-2]} N(r)}{r^\lambda} \end{aligned}$$

The purpose of the present paper is to investigate the coefficient characterizations of the constants defined above. It will be seen that the results which we obtain generalize and improve considerably the results contained in [1], [5], [10], [11], etc. To avoid unnecessary repetition, we shall denote throughout the paper,

$$\psi(n) = (\lambda_{n+1} - \lambda_n)^{-1} \log \left| \frac{a_n}{a_{n+1}} \right|, \quad \psi(n_k) = (\lambda_{n_{k+1}} - \lambda_{n_k})^{-1} \log \left| \frac{a_{n_k}}{a_{n_{k+1}}} \right|$$

for the entire function $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$.

2. We shall need the following lemmas to prove our theorems in the following sections. Lemmas 1 and 3 generalize the results contained in Valiron [13, p. 33], and Whittakar [14]. Lemmas 4–7 generalize the results contained in [7], [10], [11].

LEMMA 1. Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be an entire function of S_q -order λ and S_q -type T ; then

$$(2.1) \quad \limsup_{r \rightarrow \infty} \frac{\log^{[q]} m(r)}{\log r} = \lambda = \limsup_{r \rightarrow \infty} \frac{\log^{[q-1]} N(r)}{\log r}$$

and if $0 < \lambda < \infty$, then

$$(2.2) \quad T = \limsup_{r \rightarrow \infty} \frac{\log^{[q-1]} m(r)}{r^\lambda}.$$

PROOF. (2.1) and (2.2) follow in a straightforward manner by using the following well-known results [13, pp. 31–32].

$$(2.3) \quad m(r) < M(r) < m(r)[2N(r) + r/N(r) + 1]$$

and

$$(2.4) \quad \log m(r) = \log m(r_0) + \int_{r_0}^r \frac{N(x)}{x} dx.$$

LEMMA 2. Let (i) $\log^{[q-1]} \phi(x)$ be a positive increasing function of x ;

$$(ii) \quad \liminf_{x \rightarrow \infty} \frac{\log^{[q-1]} \phi(x)}{\log x} = \alpha \quad (0 \leq \alpha < \infty).$$

Then, for each pair of positive numbers β, γ satisfying the inequalities $\alpha < \beta$; $\alpha/\beta < \gamma < 1$, there is a sequence $x_1, x_2, \dots, x_n \rightarrow \infty$ such that

$$(2.5) \quad \log^{[q-2]} \phi(x) < x^\beta \quad (x_n^\gamma \leq x \leq x_n).$$

This lemma is due to Whittakar [14] and so we omit the proof.

LEMMA 3. Let $f(z)$ be an entire function of lower S_q -order ν and lower S_q -type t ; then

$$(2.6) \quad \nu = \liminf_{r \rightarrow \infty} \frac{\log^{[q-1]} N(r)}{\log r} = \liminf_{r \rightarrow \infty} \frac{\log^{[q]} m(r)}{\log r}$$

and, whenever $0 < \lambda < \infty$,

$$(2.7) \quad t = \liminf_{r \rightarrow \infty} \frac{\log^{[q-1]} m(r)}{r^\lambda}$$

where λ denotes S_q -order of $f(z)$.

PROOF. Let

$$\alpha = \liminf_{r \rightarrow \infty} \frac{\log^{[q-1]} N(r)}{\log r}.$$

Then, from (2.3) and (2.4) we have $\alpha \leq \nu$. Now, let $\alpha < \infty$. By Lemma 2, if $\alpha < \beta$, $\alpha/\beta < \gamma < 1$, then there exists a sequence $R_1, R_2, \dots, R_n \rightarrow \infty$ for which

$$(2.8) \quad \log^{[q-1]} N(r) < \beta \log r \quad \text{for } R_n^\gamma \leq r \leq R_n.$$

Take positive numbers δ, ϵ such that $\gamma < \delta < 1$, $\gamma/\delta < \epsilon < 1$ and write $S_n = R_n^\delta$ so that $R_n^\gamma < S_n^\epsilon < S_n < \frac{1}{2}R_n$ ($n \geq n_1$). From (2.4) we have for these values of S_n ,

$$(2.9) \quad \log m(S_n) < \epsilon N(S_n) \log S_n$$

and

$$(2.10) \quad \log m(S_n) \geq \log m(S_n^\epsilon) + (1 - \epsilon)N(S_n^\epsilon) \log S_n > \frac{1}{\epsilon} \log m(S_n^\epsilon).$$

Now by using (2.8), we have for these values of S_n ,

$$(2.11) \quad (1 - \epsilon) \log m(S_n) < \int_{S_n^\epsilon}^{S_n} \frac{N(x)}{x} dx < (1 - \epsilon) \log S_n \exp^{[q-2]} S_n^\beta.$$

Thus, from (2.8)–(2.11) we get

$$\log M(S_n) < \log m(S_n) + \exp^{[q-3]} (2S_n)^\beta + o(1).$$

This implies that

$$(2.12) \quad \log M(S_n) \sim \log m(S_n) \quad \text{if } \nu > 0.$$

In the case when $\nu = 0$, (2.6) and (2.7) are obvious. Further, from (2.10) we have

$$(2.13) \quad \log^{[q]} m(S_n) < (1 + o(1)) \log^{[q-1]} N(S_n).$$

From (2.12) and (2.13) the result of the lemma follows.

LEMMA 4. Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be an entire function of lower S_q -order ν ; then for an arbitrary sequence of integers $\{\lambda_{n_m}\}$ from $\{\lambda_n\}$ and the corresponding sequence $\{a_{n_m}\}$ from $\{a_n\}$ we have

$$(2.14) \quad \nu \geq \beta \equiv \liminf_{m \rightarrow \infty} \lambda_{n_m} \log^{[q-1]} \lambda_{n_{m-1}} / |\log |a_{n_m}||^{-1}.$$

PROOF. If $\beta = 0$, then (2.14) is obvious. Hence, let $0 < \beta < \infty$. Then, for given $\epsilon > 0$, $\beta > \epsilon$ and for all $m \geq m_0(\epsilon)$, we get

$$(2.15) \quad |a_{n_m}| > [\log^{[q-2]} \lambda_{n_{m-1}}]^{-\lambda_{n_m}/(\beta-\epsilon)}$$

Define

$$(2.16) \quad \log r_{nm} = \frac{1}{\beta - \epsilon} \log^{[q-1]} \lambda_{nm-1} + p, \quad p > 1, \quad m = 1, 2, \dots$$

Then, for $r_{nm-1} \leq r \leq r_{nm}$, from (2.15) and (2.16) we get

$$(2.17) \quad \begin{aligned} \log^{[q]} M(r) &\geq \log^{[q-1]} \{ \lambda_{nm} \log r + \log |a_{nm}| \} \\ &\geq \log^{[q-1]} \{ p \exp^{[q-2]} (r_{nm+1}^{\beta-\epsilon} e^{-p(\beta-\epsilon)}) \} \\ &\geq (\beta + o(1)) \log r. \end{aligned}$$

Thus, by taking the limit inferior in (2.17), (2.14) follows whenever $0 \leq \beta < \infty$.

In the case when $\beta = \infty$, choose an arbitrarily large number γ instead of $(\beta - \epsilon)$, and then following exactly as above, we get $\nu \geq \gamma$. Since γ is arbitrary, it follows $\nu = \infty$. This completes the proof of Lemma 4.

LEMMA 5. Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be an entire function of lower S_q -order ν ($0 \leq \nu \leq \infty$) such that $\{\psi(n)\}$ forms a nondecreasing function of n for $n > n_0$; then $\nu \leq \alpha$ where

$$(2.18) \quad \alpha \equiv \liminf_{n \rightarrow \infty} \frac{\lambda_n \log^{[q-1]} \lambda_{n-1}}{\log |\lambda_n|^{-1}}.$$

PROOF. Since $\{\psi(n)\}$ forms a nondecreasing function of n and $f(z)$ is entire, it follows that $\psi(n) > \psi(n-1)$ for infinitely many values of n . Choose $n > n_0$ and $\psi(n-1) \leq \log r < \psi(n)$; then $m(r) = \lambda_n r^{\lambda_n}$ and $N(r) = \lambda_n$. It follows from Lemma 3, that

$$(2.19) \quad N(r) > \exp^{[q-1]} ((\nu - \epsilon) \log r)$$

for all $r \geq r_0(\epsilon)$ and $0 < \epsilon < \nu < \infty$. Let $|a_{m_1}| z^{\lambda_{m_1}}$ and $|a_{m_2}| z^{\lambda_{m_2}}$ ($m_1, m_2 > n_0$; $\psi(m_1 - 1) > r_0(\epsilon)$) be any two consecutive maximum terms, then $m_1 \leq m_2 - 1$ and let $m_1 < n \leq m_2$. Since $|a_{m_1}| z^{\lambda_{m_1}}$ is a maximum term, we have

$$N(r) = \lambda_{m_1} \quad \text{for } \psi(m_1 - 1) \leq \log r < \psi(m_1).$$

Then, from (2.19), we have for every r in this interval

$$\lambda_{m_1} > \exp^{[q-1]} ((\nu - \epsilon) \log r).$$

In particular,

$$N(r) = \lambda_{m_1} > \exp^{[q-1]} [(\psi(m_1) - C)(\nu - \epsilon)].$$

where $C = \min [1, \frac{1}{2}(\psi(m_1) - \psi(m_1 - 1))]$. Further, we have $\psi(m_1) = \psi(m_1 + 1) = \dots = \psi(n - 1)$ and so

$$\begin{aligned}
 \log |a_{n_0}| + \log |a_n|^{-1} &= \sum_{m=n_0}^n (\lambda_m - \lambda_{m-1}) \psi(m-1) \\
 (2.20) \quad &\leq (\lambda_n - \lambda_{n_0-1}) \psi(n-1) \leq (\lambda_n - \lambda_{n_0-1}) \left(\frac{\log^{[q-1]} \lambda_{n-1}}{\nu - \epsilon} + C \right).
 \end{aligned}$$

This gives, on proceeding to limits, $\nu \leq \alpha$, whenever $0 < \nu < \infty$. For $\nu = 0$ it is obvious and for $\nu = \infty$, it follows exactly on the same lines as above with the same reasonings as in Lemma 4.

By combining Lemmas 3 and 4 we have the following which includes a theorem of Shah [10] and Juneja and Singh [7] for $q = 2$.

LEMMA 6. Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be an entire function of lower S_q -order ν such that $\{\psi(n)\}$ forms a nondecreasing function of n for $n > n_0$, then $\nu = \alpha$.

LEMMA 7. Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be an entire function of lower S_q -order ν , then

$$(2.21) \quad \nu \geq \gamma = \liminf_{m \rightarrow \infty} \frac{(\lambda_{n_{m+1}} - \lambda_{n_m}) \log^{[q-1]} \lambda_{n_m}}{\log |a_{n_m}/a_{n_{m+1}}|}.$$

for any arbitrary sequence of integers $\{\lambda_{n_m}\}_{m=1}^{\infty}$ from $\{\lambda_n\}$ and for the corresponding coefficients $\{a_{n_m}\}$ from $\{a_n\}$ of $f(z)$.

PROOF. For $\gamma = 0$ or negative (2.21) is obvious, hence, as usual, first let $0 < \gamma < \infty$. Then for all large $m \geq m_0(\epsilon)$ and $0 < \epsilon < \gamma$, we have

$$(2.22) \quad (\lambda_{n_{m+1}} - \lambda_{n_m}) \log^{[q-1]} \lambda_{n_m} > (\gamma - \epsilon) \log |a_{n_m}/a_{n_{m+1}}|.$$

Summing the inequalities obtained from (2.22) by replacing m by $n_0, n_0 + 1, \dots, k$, we get

$$\begin{aligned}
 (\gamma - \epsilon) \log \left| \frac{a_{n_0}}{a_{n_{k+1}}} \right| &< \sum_{m=n_0}^k (\lambda_{n_{m+1}} - \lambda_{n_m}) \log^{[q-1]} \lambda_{n_m} \\
 (2.23) \quad &< (\lambda_{n_{k+1}} - \lambda_{n_0}) \log^{[q-1]} \lambda_{n_k}.
 \end{aligned}$$

Hence, from Lemma 4 and (2.23), we get $\gamma \leq \nu$. The case when $\gamma = \infty$ follows on the same reasonings as those of Lemma 4. This completes the proof of the lemma.

3. In this section we shall prove the following theorems:

THEOREM 1. Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be an entire function of lower S_q -order ν , then

$$(3.1) \quad \nu = \max_{\{n_m\}} \alpha(\{n_m\})$$

and

$$(3.2) \quad \nu = \max_{\{n_m\}} \beta(\{n_m\})$$

where

$$(3.3) \quad \alpha(\{n_m\}) = \liminf_{m \rightarrow \infty} \frac{\lambda_{n_m} \log^{[q-1]} \lambda_{n_{m-1}}}{\log |a_{n_m}|^{-1}}$$

and

$$(3.4) \quad \beta(\{n_m\}) = \liminf_{m \rightarrow \infty} \frac{(\lambda_{n_{m+1}} - \lambda_{n_m}) \log^{[q-1]} \lambda_{n_m}}{\log |a_{n_m}/a_{n_{m+1}}|}.$$

For $q = 2$, the result (3.1) is due to Juneja [6] but the method of proof given here is a different and more elegant one. It is pointed out here that the characterizations (3.1) and (3.2)⁽¹⁾ are complete in the essence of the definition of lower S_q -order given in the present paper.

PROOF. From Lemmas 4 and 7, it is evident that

$$(3.5) \quad \nu \geq \max_{\{n_m\}} \alpha(\{n_m\})$$

and

$$(3.6) \quad \nu \geq \max_{\{n_m\}} \beta(\{n_m\}).$$

Now, consider the function $g(z) = \sum_{s=1}^{\infty} a_{n_s} z^{\lambda_{n_s}}$ where $\{\lambda_{n_s}\}$ denotes the sequence of principal indices of $f(z)$. It is easily seen that $g(z)$ is an entire function and that $f(z)$ and $g(z)$ have the same maximum term and rank for every value of r . So, by Lemmas 1 and 3, S_q -order and lower S_q -order of $g(z)$ are the same as those of $f(z)$. Hence, $g(z)$ is of lower S_q -order ν . Further, since $\psi(n_s)$ forms a strictly increasing function of s , Lemma 6 applied to $g(z)$ gives

$$(3.7) \quad \nu = \alpha(\{n_s\}).$$

From (3.5) and (3.7), we get (3.1).

To prove (3.2), we observe that (2.6) gives

$$(3.8) \quad \lambda_{n_s} > \exp^{[q-1]}[(\nu - \epsilon)(\psi(n_s) - C)]$$

⁽¹⁾ For $q = 2$, see [8]. However the result was also known to the first author for $q = 2$.

where

$$C = \min[1, \frac{1}{2}(\psi(n_s) - \psi(n_{s-1}))].$$

Now, by substituting the value of $\psi(n_s)$ in (3.8), we get

$$(3.9) \quad \frac{(\lambda_{n_{s+1}} - \lambda_{n_s}) \log^{[q-1]} \lambda_{n_s}}{\log |a_{n_s}/a_{n_{s+1}}|} > (\nu - \epsilon) - \frac{C(\nu - \epsilon)}{\psi(n_s)}.$$

Taking limit inferior as $s \rightarrow \infty$ in (3.9), we get

$$(3.10) \quad \beta(\{n_s\}) \geq \nu.$$

Hence, from (3.6) and (3.10) we get equation (3.2) for $f(z)$. This completes the proof of Theorem 1.

The following corollary follows easily from the above theorem.

COROLLARY. *If $\{\lambda_{n_s}\}$ denotes the sequence of principal indices of $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ and $\rho(n_s)$ denotes the jump points of $N(r)$, then*

$$\nu = \liminf_{s \rightarrow \infty} \frac{\log^{[q-1]} \lambda_{n_s}}{\log \rho(n_s)}.$$

For $q = 2$, this corollary is due to Gray and Shah [3].

Sato's result (1.6), which gives S_q -order in terms of the coefficients of the entire series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ may be written as

$$\lambda = \limsup_{n \rightarrow \infty} \frac{n \log^{[q-1]} n}{\log |a_n|^{-1}} = \max_{\{n_m\}} \alpha^*(\{n_m\})$$

where

$$\alpha^*(\{n_m\}) = \limsup_{m \rightarrow \infty} \frac{n_m \log^{[q-1]} n_m}{\log |a_{n_m}|^{-1}}$$

and $\{n_m\}$ is a subsequence of $\{n\}$ and $\{a_{n_m}\}$ is the corresponding sequence of $\{a_n\}$ from $f(z)$. Thus, our result (3.1) for the lower S_q -order is an analogue of (1.6) for the case of S_q -order. Now, we investigate whether a result analogous to (3.2) holds for the case of S_q -order also, i.e., a relation of the type

$$\lambda = \max_{\{n_m\}} \beta^*(\{n_m\})$$

where

$$\beta^*(\{n_m\}) = \limsup_{m \rightarrow \infty} \frac{(\lambda_{n_{m+1}} - \lambda_{n_m}) \log^{[q-1]} \lambda_{n_{m+1}}}{\log |a_{n_m}/a_{n_{m+1}}|}$$

holds in the case of S_q -order. Very simple examples show that this need not hold in general. However in the next theorem, we obtain a result in this direction which holds under some restrictions and at the same time generalizes the results of Shah [10] and Bajpai [1] for $q = 2$ and for $q = 3, 4, 5, \dots$, respectively. These results are valid only when $\lambda_n = n$.

THEOREM 2. Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be an entire function of S_q -order λ , such that $\{\psi(n)\}$ forms a nondecreasing function of n for $n \geq n_0$, then

$$(3.11) \quad \lambda = \limsup_{n \rightarrow \infty} \frac{(\lambda_{n+1} - \lambda_n) \log^{[q-1]} \lambda_{n+1}}{\log |a_n/a_{n+1}|}.$$

PROOF. Let

$$\beta = \limsup_{n \rightarrow \infty} \frac{(\lambda_{n+1} - \lambda_n) \log^{[q-1]} \lambda_{n+1}}{\log |a_n/a_{n+1}|}.$$

As usual, first let $\beta > \epsilon > 0$, then for all $n \geq n_0(\epsilon)$, we have

$$(3.12) \quad (\beta + \epsilon) \log |a_n/a_{n+1}| > (\lambda_{n+1} - \lambda_n) \log^{[q-1]} \lambda_{n+1}.$$

Summing all the inequalities obtained from (3.12) after writing $n = n_0, n_0 + 1, \dots, m$, we get

$$(3.13) \quad (\beta + \epsilon) \log |a_{n_0}/a_{m+1}| > \sum_{n=n_0}^m (\lambda_{n+1} - \lambda_n) \log^{[q-1]} \lambda_{n+1} \\ \sim \lambda_{m+1} \log^{[q-1]} \lambda_{m+1}.$$

From (3.13) we easily get

$$(3.14) \quad \beta \geq \lambda.$$

Conversely, when $\{\psi(n)\}$ forms a nondecreasing function of n , then by writing

$$\log |a_{n_0}/a_{n+1}| = (\lambda_{n_0+1} - \lambda_{n_0})\psi(n_0) + \dots + (\lambda_{n+1} - \lambda_n)\psi(n)$$

we have

$$\log \left| \frac{a_{n_0}}{a_{n+1}} \right| \leq (\lambda_{n+1} - \lambda_{n_0})\psi(n) = \left(\frac{\lambda_{n+1} - \lambda_{n_0}}{\lambda_{n+1} - \lambda_n} \right) \log \left| \frac{a_n}{a_{n+1}} \right|.$$

Hence, we get

$$(3.15) \quad \beta = \limsup_{n \rightarrow \infty} \frac{(\lambda_{n+1} - \lambda_n) \log^{[q-1]} \lambda_{n+1}}{\log |a_n/a_{n+1}|} \leq \lambda.$$

By combining (3.14) with (3.15) we get (3.11). This completes the proof of the

theorem. The case $\beta = 0$ is obvious and $\beta = \infty$ can be treated similar to $\beta > 0$.

As a corollary to Theorem 2, we get the following result of Gray and Shah [3].

COROLLARY. If $\{\lambda_{n_s}\}$ denotes the sequence of principal indices of $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ and $\{\rho(n_s)\}$ denotes the jump points of $N(r)$, then

$$\lambda = \limsup_{s \rightarrow \infty} \frac{\log^{[q-1]} \lambda_{n_s+1}}{\log \rho(n_s)}.$$

THEOREM 3. Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be an entire function of S_q -order λ , then

$$(3.16) \quad \nu \leq \lambda \liminf_{n \rightarrow \infty} \frac{\log^{[q-1]} \lambda_n}{\log^{[q-1]} \lambda_{n+1}}.$$

Further, if $f(z)$ is of S_q -irregular growth and if $\nu < \mu < \lambda$ then $f(z)$ is of the form $g_\mu(z) + h_\mu(z)$ where $g_\mu(z)$ is of S_q -order $\leq \mu$ and $h_\mu(z) = \sum_{p=0}^{\infty} a_{m_p} z^{\lambda_{m_p}}$ satisfies the inequality

$$(3.17) \quad \nu \geq \mu \liminf_{p \rightarrow \infty} \frac{\log^{[q-1]} \lambda_{m_p}}{\log^{[q-1]} \lambda_{m_p+1}}.$$

For $q = 2$, this result is due to Whittaker [14].

PROOF. Let r_t be the value at which $N(r)$ jumps from a value $\leq \lambda_{n_t}$ to a value $\geq \lambda_{n_t+1}$. Then

$$\begin{aligned} \nu &\leq \liminf_{t \rightarrow \infty} \frac{\log^{[q-1]} N(r_t - 0)}{\log r_t} \\ &\leq \lambda \liminf_{t \rightarrow \infty} \frac{\log^{[q-1]} N(r_t - 0)}{\log^{[q-1]} N(r_t + 0)} \leq \lambda \liminf_{n \rightarrow \infty} \frac{\log^{[q-1]} \lambda_n}{\log^{[q-1]} \lambda_{n+1}}. \end{aligned}$$

This establishes (3.16). For establishing (3.17), let us define $g_\mu(z) = \sum' a_n z^{\lambda_n}$, where $\sum'$ denotes the summation over n , for which

$$(3.18) \quad (\log^{[q-2]} \lambda_n)^{-\lambda_n/\mu} \geq |a_n|.$$

Then $g_\mu(z)$ is of S_q -order $\leq \mu$. Further, define

$$h_\mu(z) = f(z) - g_\mu(z) = \sum_{p=0}^{\infty} a_{m_p} z^{\lambda_{m_p}}$$

where a_{m_p} satisfies the inequality complementary to (3.18). Hence

$$\begin{aligned} \frac{\log^{[q]} M(r)}{\log r} &\geq \frac{\log^{[q-1]} [\log |a_{m_p}| + \lambda_{m_p} \log r]}{\log r} \\ &\geq \frac{\log^{[q-1]} [-(\lambda_{m_p}/\mu) \log^{[q-1]} \lambda_{m_p} + \lambda_{m_p} \log r]}{\log r}. \end{aligned}$$

Write $r_p = (e \log^{[q-2]} \lambda_{m_p})^{1/\mu}$, then for $r_p \leq r \leq r_{p+1}$, we have

$$(3.19) \quad \frac{\log^{[q]} M(r)}{\log r} \geq \frac{\log^{[q-1]} \lambda_{m_p}/\mu}{\log r} \geq \mu \frac{\log^{[q-1]} \lambda_{m_p}}{\log^{[q-1]} \lambda_{m_{p+1}}} (1 + o(1)).$$

Taking limit inferior in (3.19), we ultimately have (3.17). This completes the proof of Theorem 3.

4. In this section we shall obtain the coefficient equivalents of (1.6) which in particular will include the results of Basinger [2]⁽²⁾ and generalize a result of Shah [11]. We note that our characterization in terms of the coefficients of (1.6) is not complete but is valid for a wider class of entire functions even when $q = 2$. We shall further obtain some relations involving type, lower type and coefficients of the power series which will generalize the results of Juneja [5] even for $q = 2$. Thus, we prove the following:

THEOREM 4. Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be an entire function of S_q -order λ , then

$$(4.1) \quad t \geq \max_{\{n_m\}} \liminf_{m \rightarrow \infty} |a_{n_m}|^{\lambda/\lambda_{n_m}} \log^{[q-2]} \frac{\lambda_{n_{m-1}}}{e\lambda}.$$

PROOF. Let $\{n_m\}$ be a strictly increasing sequence of positive integers. Set

$$B \equiv B(\{n_m\}) = \liminf_{m \rightarrow \infty} |a_{n_m}|^{\lambda/\lambda_{n_m}} \log^{[q-2]} \frac{\lambda_{n_{m-1}}}{e\lambda}.$$

If $B = 0$, then $t \geq B$ is obvious. Let $B > \epsilon > 0$. Then for all sufficiently large m , we have,

$$(4.2) \quad [(B - \epsilon) \{\log^{[q-2]} \lambda_{n_{m-1}}/e\lambda\}^{-1}]^{(\lambda_{n_m}/\lambda)} < |a_{n_m}|.$$

Now Cauchy's inequality and (4.2) give

$$(4.3) \quad r^{\lambda_{n_m}} [(B - \epsilon) \{\log^{[q-2]} \lambda_{n_{m-1}}/e\lambda\}^{-1}]^{(\lambda/\lambda_{n_m})^{-1}} \leq M(r)$$

(2) This result was also known to all of us even for $q = 2$.

for all r and all sufficiently large m . Let us suppose that

$$\log P = \lambda_{n_m} \log r + \frac{\lambda_{n_m}}{\lambda} \log(B - \epsilon) - \frac{\lambda_{n_m}}{\lambda} \log^{[q-1]} \frac{\lambda_{n_m-1}}{e\lambda}.$$

Then, if

$$\log D + \log^{[q-1]} \lambda_{n_m-1}/e\lambda \leq \lambda \log r < \log D + \log^{[q-1]} \lambda_{n_m}/e\lambda$$

where $D = e/B$ if $q = 2$ and $D = 1/B$ if $q = 3, 4, 5, \dots$, we get

$$(4.4) \quad \log P \geq (\lambda_{n_m}/\lambda) \log [D(B - \epsilon)] > e \log [D(B - \epsilon)] \exp^{[q-2]} r^\lambda/D.$$

Hence, from (4.3) and (4.4) we obtain

$$(4.5) \quad \frac{\log^{[q-1]} M(r)}{r^\lambda} > \begin{cases} B \log(e(B - \epsilon)/B) & \text{if } q = 2, \\ B & \text{if } q = 3, 4, 5, \dots, \end{cases}$$

for all $r \geq r_0(\epsilon)$. Hence

$$\liminf_{r \rightarrow \infty} \frac{\log^{[q-1]} M(r)}{r^\lambda} = t \geq B \quad \text{for } q = 2, 3, 4, \dots.$$

Since $t \geq B(\{n_m\})$ for every sequence $\{n_m\}$, it follows that $t \geq \max_{\{n_m\}} B(\{n_m\})$. Hence the theorem.

As a corollary to Theorem 4, we get the following result which in particular for $q = 2$ is due to Basinger [2].

COROLLARY. Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be an entire function of S_q -order λ , such that

$$\liminf_{n \rightarrow \infty} \frac{\log^{[q-2]} \lambda_{n-1}}{\log^{[q-2]} \lambda_n} = L \quad (> 0),$$

then

$$t \geq L \liminf_{n \rightarrow \infty} |a_n|^{\lambda/\lambda_n} \log^{[q-2]} \lambda_n/e\lambda.$$

THEOREM 5. Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be an entire function of S_q -order λ , such that $\{\psi(n)\}$ forms a nondecreasing function of n for all $n \geq n_0$, then

$$(4.6) \quad t \leq \liminf_{n \rightarrow \infty} |a_n|^{\lambda/\lambda_n} \log^{[q-2]} \lambda_n/e\lambda.$$

For $q = 2$, this result is due to Basinger [2].

PROOF. First, let $0 < t < \infty$. Then, from Lemma 3, for all $r \geq r_0(\epsilon)$ and $t > \epsilon > 0$, we have

$$(4.7) \quad \log m(r) > \exp^{[q-2]} \{(t - \epsilon)r^\lambda\}.$$

If $a_{n_1} z^{\lambda_{n_1}}$ and $a_{n_2} z^{\lambda_{n_2}}$ denote two consecutive maximum terms of $f(z)$, then as $\{\psi(n)\}$ forms a nondecreasing function of n , we get, for $n_1 \leq n \leq n_2 - 1$, that

$$(4.8) \quad \psi(n_2 - 1) = \psi(n_2 - 2) = \cdots = \psi(n) = \cdots = \psi(n_1)$$

and

$$(4.9) \quad |a_n| r^{\lambda_n} = |a_{n_2}| r^{\lambda_{n_2}} \quad \text{for } r = e^{\psi(n)}.$$

Thus, from (4.7)–(4.9) we get

$$(4.10) \quad \lambda_n \psi(n) + \log |a_n| = \lambda_{n_2} \psi(n_2) + \log |a_{n_2}| > \exp^{[q-2]} \{(t - \epsilon) e^{\lambda \psi(n)}\}.$$

If $B^* = |a_n|^{\lambda/\lambda_n} \log^{[q-2]} \lambda_n / e\lambda$ then from (4.10) we get

$$(4.11) \quad B^* > e^{-\lambda \psi(n)} \exp \left[\frac{\lambda}{\lambda_n} \exp^{[q-2]} \{(t - \epsilon) e^{\lambda \psi(n)}\} \right] \log^{[q-2]} \frac{\lambda_n}{e\lambda}.$$

We note that the minimum value of the function

$$P(r) \equiv \log^{[q-2]} \frac{\lambda_n}{e\lambda} r^{-\lambda} \exp \left[\frac{\lambda}{\lambda_n} \exp^{[q-2]} \{(t - \epsilon) r^\lambda\} \right]$$

is obtained by solving the following equation

$$(4.12) \quad \lambda_n = \lambda r^\lambda (t - \epsilon) \prod_{m=1}^{q-2} \exp^{[m]} \{(t - \epsilon) r^\lambda\}$$

where the product Π is understood to have a value one when $q = 2$. Let r_n be the value of r , for which this minimum is attained. It is easily seen that

$$(4.13) \quad P(r_n) \geq \begin{cases} (t - \epsilon)(1 - o(1)) \exp(o(1)) & \text{if } q = 3, 4, 5, \dots, \\ (t - \epsilon) & \text{if } q = 2. \end{cases}$$

From (4.11)–(4.13) for $0 < t < \infty$, we get $\liminf_{n \rightarrow \infty} B^* \geq t$. In the case when $t = 0$, (4.6) is obvious, while if $t = \infty$, then (4.11)–(4.13) are obtained by replacing t by an arbitrary large number t^* , which ultimately by making $t^* \rightarrow \infty$, leads to $\liminf_{n \rightarrow \infty} B^* = \infty$. This completes the proof of the theorem.

By combining the corollary to Theorem 4 with Theorem 5, we get the following theorem, which in particular generalizes a result of Shah [11].

THEOREM 6. Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be an entire function of S_q -order λ , such that

(i) $\{\psi(n)\}$ forms a nondecreasing function of n , and

(ii) $\log^{[q-2]} \lambda_n \sim \log^{[q-2]} \lambda_{n+1}$ as $n \rightarrow \infty$;

then $t = \liminf_{n \rightarrow \infty} |a_n|^{\lambda/\lambda_n} \log^{[q-2]} \lambda_n / e\lambda$.

Now we shall prove our main theorem of this section which is a coefficient equivalent of (1.6) for all those entire functions for which the principal indices

$\{\lambda_{n_s}\}$ satisfy an asymptotic relation $\log^{[q-2]}\lambda_{n_s} \sim \log^{[q-2]}\lambda_{n_{s+1}}$. It will be of interest to see if this result holds without this restriction.

THEOREM 7. Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be an entire function of S_q -order λ and lower S_q -type t , for which $\{\lambda_{n_k}\}$ denotes the sequence of principal indices. If $\log^{[q-2]}\lambda_{n_k} \sim \log^{[q-2]}\lambda_{n_{k+1}}$, as $k \rightarrow \infty$, then

$$(4.13') \quad t = \max_{\{n_m\}} \left\{ \liminf_{m \rightarrow \infty} |a_{n_m}|^{\lambda/\lambda_{n_m}} \log^{[q-2]} \frac{\lambda_{n_m-1}}{e\lambda} \right\}.$$

PROOF. Consider the function $g(z) = \sum_{k=0}^{\infty} a_{n_k} z^{\lambda_{n_k}}$ where $\{\lambda_{n_k}\}$ denotes the sequence of principal indices of $f(z)$. It is easily seen that $g(z)$ is an entire function and that $f(z)$ and $g(z)$ have the same maximum term and rank for every value of r . So, by Lemmas 1 to 3, it follows that $f(z)$ and $g(z)$ have the same S_q -order λ and lower S_q -type t . Since $\{\lambda_{n_k}\}$ denotes the sequence of principal indices it also follows that $\{\psi(n_k)\}$ is a nondecreasing function of k . Thus, from Theorem 6, we get

$$(4.14) \quad t = \liminf_{k \rightarrow \infty} |a_{n_k}|^{\lambda/\lambda_{n_k}} \log^{[q-2]} \frac{\lambda_{n_k-1}}{e\lambda}.$$

But from Theorem 4, we have

$$(4.15) \quad t \geq \max_{\{n_m\}} \left\{ \liminf_{m \rightarrow \infty} |a_{n_m}|^{\lambda/\lambda_{n_m}} \log^{[q-2]} \frac{\lambda_{n_m-1}}{e\lambda} \right\}.$$

Hence, from (4.14) and (4.15), we get (4.13') for $f(z)$. This completes the proof of the theorem.

THEOREM 8. Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be an entire function of S_q -order λ , ($0 < \lambda < \infty$), S_q -type T and lower S_q -type t ; then

$$(4.16) \quad \eta R \leq t \leq T \leq Q$$

where

$$Q = \limsup_{n \rightarrow \infty} \left| \frac{a_n}{a_{n-1}} \right|^{\lambda/(\lambda_n - \lambda_{n-1})} \log^{[q-2]} \frac{\lambda_n}{\lambda},$$

$$R = \liminf_{n \rightarrow \infty} \left| \frac{a_n}{a_{n-1}} \right|^{\lambda/(\lambda_n - \lambda_{n-1})} \log^{[q-2]} \frac{\lambda_{n-1}}{\lambda}$$

and

$$\eta = \begin{cases} \exp [\liminf_{n \rightarrow \infty} (\lambda_{n-1}/\lambda_n) - 1] & \text{if } q = 2, \\ 1 & \text{if } q = 3, 4, 5, \dots \end{cases}$$

For $q = 2$ and $\lambda_n = n$ this result is due to Juneja [5].

PROOF. First, let $0 < R \leq Q < \infty$, then for every $\epsilon > 0$ and for sufficiently large $n \geq n_0(\epsilon)$ such that $R > \epsilon > 0$, we have

$$(4.17) \quad \begin{aligned} R - \epsilon &< |a_n/a_{n-1}|^{\lambda/(\lambda_n - \lambda_{n-1})} \log^{[q-2]} \lambda_{n-1}/\lambda, \\ Q + \epsilon &> |a_n/a_{n-1}|^{\lambda/(\lambda_n - \lambda_{n-1})} \log^{[q-2]} \lambda_n/\lambda. \end{aligned}$$

Writing (4.17) for $m = n_0, n_0 + 1, \dots, n$ and multiplying together, we get

$$(4.18) \quad \begin{aligned} (\lambda_n - \lambda_{n_0-1}) \log(R - \epsilon) - \sum_{m=n_0}^n (\lambda_m - \lambda_{m-1}) \log^{[q-1]} \frac{\lambda_{m-1}}{\lambda} \\ < \lambda \log \left| \frac{a_n}{a_{n_0-1}} \right| \\ < \log(Q + \epsilon) - \sum_{m=n_0}^n (\lambda_m - \lambda_{m-1}) \log^{[q-1]} \frac{\lambda_m}{\lambda}. \end{aligned}$$

Now, if $q = 3, 4, 5, \dots$, and $n(t) = \lambda_m$, if $\lambda_{m-1} < t \leq \lambda_m$, then

$$(4.19) \quad \begin{aligned} & - \sum_{m=n_0}^n (\lambda_m - \lambda_{m-1}) \log^{[q-1]} \frac{\lambda_{m-1}}{\lambda} \\ &= \lambda_{n_0-1} \log^{[q-1]} \frac{\lambda_{n_0-1}}{\lambda} - \lambda_n \log^{[q-1]} \frac{\lambda_{n-1}}{\lambda} \\ & \quad + \sum_{m=n_0+1}^n \left(\log^{[q-1]} \frac{\lambda_{m-1}}{\lambda} - \log^{[q-1]} \frac{\lambda_{m-2}}{\lambda} \right) \lambda_{m-1} \\ &= -\lambda_n \log^{[q-1]} \frac{\lambda_{n-1}}{\lambda} + \int_{\lambda_{n_0+1}}^{\lambda_{n-1}} n(t) d \log^{[q-1]} \frac{t}{\lambda} + \lambda_{n_0-1} \log^{[q-1]} \frac{\lambda_{n_0-1}}{\lambda} \\ &> -\lambda_n \log^{[q-1]} \frac{\lambda_{n-1}}{\lambda} + \frac{\lambda_{n-1} - \lambda_{n_0+1}}{\prod_{m=1}^{q-2} \log^{[m]} \lambda_{n-1}/\lambda} + \lambda_{n_0-1} \log^{[q-1]} \frac{\lambda_{n_0-1}}{\lambda} \\ &= -\lambda_n \log^{[q-1]} \frac{\lambda_{n-1}}{\lambda} + o(\lambda_n). \end{aligned}$$

And, if $q = 2$, then for all $n \geq n_0(\epsilon)$, we have

$$(4.20) \quad \begin{aligned} & - \sum_{m=n_0}^n (\lambda_m - \lambda_{m-1}) \log \frac{\lambda_{m-1}}{\lambda} \\ &> -\lambda_n \log \frac{\lambda_{n-1}}{\lambda} + (\lambda_{n-1} - \lambda_{n_0+1}) + o(\lambda_n) \\ &> -\lambda_n \log \frac{\lambda_{n-1}}{\lambda} + \lambda_n(\log \eta + 1 - \epsilon) + o(\lambda_n). \end{aligned}$$

Further,

$$\begin{aligned}
 & - \sum_{m=n_0}^n (\lambda_m - \lambda_{m-1}) \log^{[q-1]} \frac{\lambda_m}{\lambda} \\
 & = -\lambda_n \log^{[q-1]} \frac{\lambda_n}{\lambda} + \lambda_{n_0-1} \log^{[q-1]} \lambda_{n_0} \\
 (4.21) \quad & + \sum_{m=n_0+1}^n \left(\log^{[q-1]} \frac{\lambda_m}{\lambda} - \log^{[q-1]} \frac{\lambda_{m-1}}{\lambda} \right) \lambda_{m-1} \\
 & = -\lambda_n \log^{[q-1]} \frac{\lambda_n}{\lambda} + \lambda_{n_0-1} \log^{[q-1]} \lambda_{n_0} \\
 & + \int_{\lambda_{n_0+1}}^{\lambda_n} n(t) d \log^{[q-1]} \frac{t}{\lambda}
 \end{aligned}$$

where $n(t) = \lambda_{m-1}$ if $\lambda_{m-1} \leq t < \lambda_m$. Then, if $q = 3, 4, 5, \dots$

$$\begin{aligned}
 I & = \int_{\lambda_{n_0+1}}^{\lambda_n} n(t) d \log^{[q-1]} \frac{t}{\lambda} \\
 (4.22) \quad & = \left(\int_{\lambda_{n_0+1}}^{[\sqrt{\lambda_n}]} + \int_{[\sqrt{\lambda_n}]}^{\lambda_n} \right) n(t) d \log^{[q-1]} \frac{t}{\lambda} \\
 & < \frac{[\sqrt{\lambda_n}] - \lambda_{n_0+1}}{\prod_{m=1}^{q-2} \log^{[q-2]} (\lambda_{n_0+1}/\lambda)} + \frac{\lambda_n - [\sqrt{\lambda_n}]}{\prod_{m=1}^{q-2} \log^{[q-1]} ([\sqrt{\lambda_n}]/\lambda)} = o(\lambda_n)
 \end{aligned}$$

and for $q = 2$, we get

$$(4.23) \quad I = \int_{\lambda_{n_0+1}}^{\lambda_n} \frac{n(t)}{t} dt < (\lambda_n - \lambda_{n_0+1}).$$

From (4.18)–(4.23) we get (4.16) whenever $0 < R \leq Q < \infty$. In the case when $R = 0$, (4.16) is obvious. If $R = \infty$, then (4.18)–(4.23) are obtained exactly as above by replacing $R - \epsilon$ by any arbitrary large number R^* , which ultimately leads to $t = \infty$. In the case when $Q = \infty$, then (4.16) is obvious. This completes the proof of Theorem 8.

THEOREM 9. Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be an entire function of S_q -order λ and S_q -type T such that $\{\psi(n)\}$ forms a nondecreasing function of n ; then

$$(4.24) \quad \begin{cases} T \leq Q \leq eT, & \text{if } q = 2, \\ T = Q & \text{if } q = 3, 4, 5, \dots \end{cases}$$

In the case when $\lambda_n = n$ and $q = 2$, Theorem 8 is due to Juneja [5].

PROOF. If $T = \infty$, then (4.24) is obvious in view of Theorem 3. Let $0 \leq T < \infty$. Then, for every $\epsilon > 0$, and for sufficiently large $n \geq n_0(\epsilon)$, we have

$$(4.25) \quad \frac{\lambda}{\lambda_n} \log |a_n| < \log(T + \epsilon) - \log^{[q-1]} \frac{\lambda_n}{e\lambda}.$$

Writing

$$(4.26) \quad \begin{aligned} \log |a_n| &= \log |a_{n_0}| - \sum_{m=n_0}^{n-1} (\lambda_{m+1} - \lambda_m) \psi(m) \\ &> \log |a_{n_0}| - (\lambda_n - \lambda_{n_0}) \psi(n-1) \quad (\text{by hypothesis}) \\ &= \log |a_{n_0}| + \left(\frac{\lambda_n - \lambda_{n_0}}{\lambda_n - \lambda_{n-1}} \right) \log \left| \frac{a_n}{a_{n-1}} \right|. \end{aligned}$$

Hence, from (4.25) and (4.26), we get

$$(4.27) \quad \log(T + \epsilon) > \log^{[q-1]} \frac{\lambda_n}{e\lambda} + \frac{\lambda}{\lambda_n} \left[\log |a_{n_0}| + \left(\frac{\lambda_n - \lambda_{n_0}}{\lambda_n - \lambda_{n-1}} \right) \log \left| \frac{a_n}{a_{n-1}} \right| \right]$$

for all $n \geq n_0(\epsilon)$. By taking exponentiation of (4.27), we have

$$(T + \epsilon) > |a_{n_0}|^{\lambda/\lambda_n} \left| \frac{a_n}{a_{n-1}} \right|^{\lambda(\lambda_n - \lambda_{n_0})/\lambda_n(\lambda_n - \lambda_{n-1})} \log^{[q-2]} \frac{\lambda_n}{e\lambda}$$

for all $n \geq n_0(\epsilon)$. Now, by taking limit superior, we ultimately get

$$(4.28) \quad T \geq \limsup_{n \rightarrow \infty} \left| \frac{a_n}{a_{n-1}} \right|^{\lambda/(\lambda_n - \lambda_{n-1})} \log^{[q-2]} \frac{\lambda_n}{e\lambda}.$$

Thus, by combining (4.16) with (4.28), we get (4.24) whenever $0 \leq T < \infty$. This completes the proof of the theorem.

REMARK. It is evident from (4.16) that if $\log^{[q-2]} \lambda_{n-1} \sim \log^{[q-2]} \lambda_n$ and $0 < R = Q < \infty$, then $f(z)$ is of perfectly S_q -regular growth, of S_q -order λ and S_q -type Q . However the converse of this need not be true in general as can be seen from the following example. If $f(z) = e^{z^2} + e^z$ then $f(z)$ is of S_q -order, $\lambda = \nu = 2$, $T = t = 1$ and $R = 0$ while $Q = \infty$.

5. In this section we shall prove some theorems involving the coefficients of the entire series and growth numbers. It will be seen that these theorems generalize the results of Juneja [4] even for $q = 2$. Further, we shall also prove a decomposition theorem for entire functions which are of S_q -regular growth but

not of perfectly S_q -regular growth. Such a theorem is not known even for $q = 2$.

THEOREM 10. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be an entire function of S_q -order λ , S_q -growth number μ and lower S_q -growth number δ such that $\psi(n)$ forms a strictly increasing function of n , then*

$$(5.1) \quad \mu = BQ \text{ and } \delta = BR$$

where $B = \lambda$ if $q = 2$, $B = 1$ if $q = 3, 4, 5, \dots$, and Q and R are defined as in Theorem 8.

This theorem generalizes an earlier result of Juneja [4] which is valid only when $\lambda_n = n$ and $q = 2$.

PROOF. Since $\{\psi(n)\}$ forms a strictly increasing sequence of n , the λ_n th term will be the maximum term for $|z| = r$, if and only if,

$$(5.2) \quad N(r) = \lambda_n \text{ and } m(r) = |a_n| r^{\lambda_n}$$

for $\psi(n-1) \leq \log r < \psi(n)$. Then, for given $\epsilon > 0$, and for sufficiently large $n \geq n_0(\epsilon)$, we have

$$(5.3) \quad (\mu - \epsilon) < r^{-\lambda} \log^{[q-2]} \lambda_n$$

and

$$(5.4) \quad (\delta + \epsilon) > r^{-\lambda} \log^{[q-2]} \lambda_n$$

for all r satisfying $\psi(n-1) \leq \log r < \psi(n)$. Then from (5.3) and (5.4)

$$(5.5) \quad (\mu - \epsilon) < |a_n/a_{n-1}|^{\lambda/(\lambda_n - \lambda_{n-1})} \log^{[q-2]} \lambda_n,$$

$$(\delta + \epsilon) \geq |a_{n+1}/a_n|^{\lambda/(\lambda_{n+1} - \lambda_n)} \log^{[q-2]} \lambda_n.$$

Hence on proceeding to limits, we get

$$(5.6) \quad \mu \leq BR \text{ and } \delta \geq BQ.$$

Further, from (1.7) we have $\log^{[q-2]} N(r) < (\mu + \epsilon)r^\lambda$ for a sequence of values $r = r_1, r_2, \dots, r_n \rightarrow \infty$. Hence, (5.2), for n 's corresponding to these values of r_n 's, yields

$$\log^{[q-2]} \lambda_n < (\mu + \epsilon)r_n^\lambda < (\mu + \epsilon)|a_n/a_{n+1}|^{\lambda/(\lambda_{n+1} - \lambda_n)}.$$

This gives

$$(5.7) \quad BR \leq \mu.$$

Similarly, it can be shown that

$$(5.8) \quad BQ \geq \delta.$$

From (5.6)–(5.8) we get (5.1).

THEOREM 11. Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be an entire function of S_q -irregular growth, then $t = \mu = R = 0$.

For $q = 2$, this result is due to Srivastava and Singh [12]. Proof of this theorem is easy so we omit it.

THEOREM 12. Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be an entire function of S_q -order λ , S_q -growth number δ ($< \infty$), lower S_q -growth number μ and S_q -type T , and lower S_q -type t ; then

$$(5.9) \quad \mu \leq \delta \liminf_{n \rightarrow \infty} \frac{\log^{[q-2]} \lambda_n}{\log^{[q-2]} \lambda_{n+1}}.$$

Further, if $0 < t < \eta < T$ ($< \infty$), then $f(z)$ can be written in the form $f(z) = g_{\eta}(z) + h_{\eta}(z)$ where $g_{\eta}(z)$ is of S_q -order $\leq \lambda$ and of S_q -type $\leq \eta$ if of S_q -order λ , and $h_{\eta}(z) = \sum_{p=0}^{\infty} a_{n_p} z^{\lambda_{n_p}}$ satisfies the inequality

$$(5.10) \quad t \geq \eta \liminf_{p \rightarrow \infty} \frac{\log^{[q-2]} \lambda_{n_p}}{\log^{[q-2]} \lambda_{n_p+1}}.$$

PROOF. Let r_t be the value at which $N(r)$ jumps from a value $\leq \lambda_{n_t}$ to a value $\geq \lambda_{n_t+1}$. Then

$$\begin{aligned} \mu &\leq \liminf_{t \rightarrow \infty} \frac{\log^{[q-2]} N(r_t - 0)}{r_t^{\lambda}} \\ &\leq \limsup_{t \rightarrow \infty} \frac{\log^{[q-2]} N(r_t + 0)}{r_t^{\lambda}} \liminf_{t \rightarrow \infty} \frac{\log^{[q-2]} N(r_t - 0)}{\log^{[q-2]} N(r_t + 0)} \\ &\leq \delta \liminf_{n \rightarrow \infty} \frac{\log^{[q-2]} \lambda_n}{\log^{[q-2]} \lambda_{n+1}}. \end{aligned}$$

This proves the first part of the theorem. For proving the second part, let $g_{\eta}(z) = \sum' a_n z^{\lambda_n}$ where $\sum'$ denote the summation over those n 's for which

$$(5.11) \quad |a_n| \leq (\eta / \log^{[q-2]} \lambda_n / e^{\lambda})^{\lambda_n / \lambda}.$$

Then, $g_{\eta}(z)$ is of S_q -order $\leq \lambda$, and S_q -type $\leq \eta$ if $g_{\eta}(z)$ is of S_q -order λ . Define

$$h_\eta(z) = f(z) - g_\eta(z) = \sum_{p=0}^{\infty} a_{n_p} z^{\lambda_{n_p}}$$

where a_{n_p} 's satisfy the reverse of the inequality (5.11). If $M(r, h_\eta) = \max_{|z|=r} |h_\eta(z)|$, then

$$\begin{aligned} \log^{[q-1]} M(r, h_\eta) &> \log^{[q-1]} \{ |a_{n_p}| r^{\lambda_{n_p}} \} \\ &\geq \log^{[q-2]} \left[\frac{\lambda_{n_p}}{\lambda} \log \eta - \frac{\lambda_{n_p}}{\lambda} \log^{[q-1]} \frac{\lambda_{n_p}}{e\lambda} + \lambda_{n_p} \log r \right] \end{aligned}$$

for all r and p . Let $\log r_p = X + \lambda^{-1} \log^{[q-1]} \lambda_{n_p} / e\lambda$, then, if $r_p \leq r \leq r_{p+1}$, we get

$$\begin{aligned} \log^{[q-1]} M(r, h_\eta) &\geq \log^{[q-2]} [(X + \lambda^{-1} \log \eta) \lambda_{n_p}] \\ &= \frac{(e^{-X} r_{p+1})^\lambda \log^{[q-2]} [(X + \lambda^{-1} \log \eta) \lambda_{n_p}]}{\log^{[q-2]} (\lambda_{n_{p+1}} / e\lambda)}. \end{aligned}$$

From this we get

$$t = \liminf_{r \rightarrow \infty} \frac{\log^{[q-1]} M(r)}{r^\lambda} \geq \begin{cases} e^{-X\lambda} \liminf_{p \rightarrow \infty} \frac{\log^{[q-2]} \lambda_{n_p}}{\log^{[q-2]} \lambda_{n_{p+1}}} & \text{if } q = 3, 4, 5, \dots, \\ e(X\lambda + \log \eta) e^{-X\lambda} \liminf_{p \rightarrow \infty} \frac{\lambda_{n_p}}{\lambda_{n_{p+1}}} & \text{if } q = 2. \end{cases}$$

By choosing

$$X = \begin{cases} -\lambda^{-1} \log \eta & \text{if } q = 3, 4, 5, \dots, \\ (1 - \log \eta) / \lambda & \text{if } q = 2, \end{cases}$$

we get (5.10). This completes the proof of the theorem.

6. We have seen in Theorem 11 that if $f(z)$ is of S_q -irregular growth then lower S_q -type is zero. In such a case it is natural to seek a constant ρ ($0 < \nu \leq \rho < \lambda$) for which

$$(6.1) \quad \liminf_{r \rightarrow \infty} \frac{\log^{[q-1]} M(r)}{r^\rho} = A_\rho$$

and A_ρ is finite and nonzero. We shall call A_ρ to be S_q - ρ -type for the entire function $f(z)$ and whenever $\rho = \nu$ this constant will be called S_q - ν -type. For

$q = 2$, this concept is due to Srivastava and Singh [12]. Coefficient equivalent of (6.1) is as follows:

THEOREM 13. Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be an entire function of S_q - ρ -type A_ρ , then

$$A_\rho = \max_{\{n_m\}} \left\{ \liminf_{m \rightarrow \infty} |a_{n_m}|^{\rho/\lambda_{n_m}} \log^{[q-2]} \frac{\lambda_{n_m-1}}{e\rho} \right\}$$

provided the principal indices $\{\lambda_{n_s}\}$ satisfy the asymptotic relation $\log^{[q-2]} \lambda_{n_s} \sim \log^{[q-2]} \lambda_{n_{s+1}}$ as $s \rightarrow \infty$.

We omit the proof of this theorem as it follows on the same lines as the proof of Theorem 7.

Analogues of Theorems 4 to 6 can also be obtained in terms of A_ρ .

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