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CLASSIFICATION OF 3-MANIFOLDS WITH CERTAIN SPINES(1)

BY

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ABSTRACT. Given the group presentation $\varphi = \langle a, b \mid a^m b^n, a^p b^q \rangle$ with $m, n, p, q \neq 0$, we construct the corresponding 2-complex K_φ. We prove the following theorems.

THEOREM 7. *K_φ is a spine of a closed orientable 3-manifold if and only if*
(i) $|m| = |p| = 1$ *or* $|n| = |q| = 1$, *or*
(ii) $(m, p) = (n, q) = 1$.
Further, if (ii) *holds but* (i) *does not, then the manifold is unique.*

THEOREM 10. *If M is a closed orientable 3-manifold having K_φ as a spine and $\lambda = |mq - np|$ then M is a lens space $L_{\lambda,k}$ where $(\lambda, k) = 1$ except when $\lambda = 0$ in which case $M = S^2 \times S^1$.*

It is well known that every connected compact orientable 3-manifold with or without boundary has a spine that is a 2-complex with a single vertex. Such 2-complexes correspond very naturally to group presentations, the 1-cells corresponding to generators and the 2-cells corresponding to relators. In the case of a closed orientable 3-manifold, there are equally many 1-cells and 2-cells in the spine, i.e., equally many generators and relators in the corresponding presentation. Given a group presentation one is motivated to ask the following questions: (1) Is the corresponding 2-complex a spine of a compact orientable 3-manifold? (2) If there are equally many generators and relators, is the 2-complex a spine of a closed orientable 3-manifold? (3) Can we say what manifold(s) have the 2-complex as a spine?

L. Neuwirth [3] has given an algorithm for deciding question (2). This algorithm can be modified in a rather trivial way to decide question (1). However, as Neuwirth himself points out [4, p. 182], the algorithm is limited in a practical sense since a computer program which follows it can handle presentations of length at most "about 30."

We investigate group presentations of the form $\varphi = \langle a, b \mid a^m b^n, a^p b^q \rangle$ with

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(1) The results of this paper are contained in the author's doctoral dissertation at Colorado State University. The author wishes to thank his advisor, R. P. Osborne, for his helpful guidance.

$m, n, p, q \neq 0$, and for these presentations we are able to answer completely all three questions above.

The answers to questions (1) and (2) are given in Theorems 6 and 7 and are based on results used in establishing Neuwirth's algorithm. Zieschang [7] has obtained similar results from the somewhat different point of view of simple closed curves and homeomorphisms on handlebodies. The answer to question (3) is given in Theorems 10 and 11 and is based on a technique we develop for changing the spine of a manifold. As an immediate corollary to Theorem 10 it follows that if φ presents the trivial group, then the manifold must be S^3, a result that was observed by Bing [1] and Zieschang [7].

The techniques used herein are far more useful than our results indicate. In terms of the Poincaré Conjecture no new direct information is obtained. However, in a series of papers under joint authorship with R. P. Osborne, techniques based on Theorems 5 and 9 and some results of Zieschang [8] will be developed. In the first of the series an efficient algorithm will be given to decide question (1) in most cases for 2-generator presentations. In future papers a new concept, called *Railroad Systems*, will be developed to aid in the study of all compact orientable 3-manifolds. It will also be shown that no 2-generator presentation whose relators contain 11 or fewer syllables can correspond to a spine of a counterexample to the Poincaré Conjecture.

1. **Preliminaries.** The results of this paper involving compact 3-manifolds, we work in the PL category. Due to E. Moise [2], this is no loss of generality.

Unless otherwise stated, we use the following notation:

M will denote a *compact connected orientable 3-dimensional manifold*; henceforth just *3-manifold* or *manifold*. We assume that M has a nonempty boundary unless it is specified that M is *closed*.

φ will denote a group presentation whose relators are not empty. In the case φ has m generators and n relators we write $\varphi = \langle a_1, \cdots, a_m | R_1, \cdots, R_n \rangle$. If φ has two generators, we will denote them by a and b.

We will use K to denote a *2-dimensional cell-complex* or *2-complex* having a single vertex Q. The notation K_φ will refer to the 2-complex constructed from the group presentation φ in the standard way, i.e., a single vertex, a 1-cell for each generator, and a 2-cell for each relator.

Given spaces $X \subset Y$ we use $\mathrm{cl}(X)$, $\mathrm{int}(X)$, $\mathrm{bd}(X)$, and $N(X)$ to denote respectively the *closure, interior, boundary,* and *regular neighborhood* of X in Y.

The n-sphere is denoted by S^n.

DEFINITION 1. Let P be a graph (i.e., a 1-dimensional cell-complex) with vertices in pairs denoted by $a^+, a^-, b^+, b^-, \cdots$, read "$a$-head," "$a$-tail," etc.

For each pair of vertices x^+ and x^-, suppose that there is a 1-1 correspondence B between the sets $E_{x^+} = \mathrm{bd}(N(x^+))$ and $E_{x^-} = \mathrm{bd}(N(x^-))$. Label the respective points of E_{x^+} and E_{x^-} by x_i^+ and x_i^-, $i = 1, \cdots, \mu_x$, so that $B(x_i^+) = x_i^- = B^{-1}(x_i^+)$. The graph P together with the 1-1 correspondence B will be called a *Presentation-graph*, or *P-graph*. We will denote the P-graph by just P, taking B as understood.

Note that if K_φ is the 2-complex (with vertex Q) corresponding to a given group presentation φ, then $P_\varphi = \mathrm{bd}(N(Q))$ together with the permutation B as defined in [3] is a P-graph which we call the *P-graph corresponding to* φ. One can easily reconstruct K_φ, and hence φ, if P_φ is given; in fact if P is any P-graph one can construct φ so that $P = P_\varphi$.

The above definition merely formalizes a concept developed by L. Neuwirth. See [3] for details.

Let P be a P-graph and let $E = \bigcup\{E_x = \mathrm{bd}(N(x))\colon x \text{ a vertex of } P\}$. We define the permutation A on E so that for each $x \in E$, x and $A(x)$ are the two points of E lying on the same edge (or 1-cell) of P. Note that the P-graph is determined by the sets E_x and the permutations A and B defined on $E = \bigcup\{E_x\}$.

If P is embedded in S^2, then walking clockwise around each vertex of P induces a permutation C on E whose orbits are exactly the sets E_x.

DEFINITION 2. Given a P-graph P, a permutation C on E whose orbits are the sets E_x is called a *cyclic ordering of the edges around each vertex of* P, or a *cyclic ordering of* P. If P is embedded in S^2 and C is the cyclic ordering obtained as above (i.e., by walking clockwise around each vertex), then we will say that C *is consistent with an embedding of* P *in* S^2. If C satisfies the further condition that $BC = C^{-1}B$, then we will say that the embedding is *faithful*, that P is *faithfully embedded in* S^2 *with ordering* C, or that P *is faithfully embedded in* S^2. Given a group presentation φ, if there exists a faithful embedding of P_φ in S^2, we will say that φ *fits*.

Note that if P is a connected P-graph, then a faithful embedding of P is uniquely determined (up to a homeomorphism of S^2) by the cyclic ordering C.

We will use the following results which are derived in [3].

THEOREM 1. *Let* φ *be a group presentation. Then* K_φ *is a spine of a manifold* (*3-dimensional, connected, compact, orientable, with boundary*) *if and only if* φ *fits. The manifold is uniquely determined by the faithful embedding of* P_φ *in* S^2.

THEOREM 2. *Given a connected P-graph* P *a cyclic ordering* C *of* P *is* *consistent with an embedding (not necessarily faithful) of* P *in* S^2 *if and only if* $|C| - |A| + |AC| = 2$ *(the notation* $|\sigma|$ *denotes the number of orbits of the permutation* σ*).*

THEOREM 3. *Given a group presentation* φ *with equally many generators and relators and with* P_φ *connected, a necessary and sufficient condition that* K_φ *be a spine of a closed manifold is that* φ *fit and the permutation group generated by* AC *and* BC *be transitive.*

2. **Faithful embeddings of *P*-graphs in** S^2. We will use the following lemma which is an easy corollary of Theorem 2.

LEMMA. *Let* P *be a* *P-graph with one pair of vertices and no loops (a loop is an edge of a graph with both endpoints at the same vertex) and let* C *be a cyclic ordering of* P*. Then* C *is consistent with an embedding of* P *in* S^2 *if and only if* $AC = C^{-1}A$.

THEOREM 4. *Let* $\varphi = \langle a | a^m \rangle$ *with* $m > 0$. *Construct* P_φ *and assume the points of* E *are labeled so that* $A(a_i^+) = a_{i+1}^-$, $i = 1, \cdots, m$ *(subscripts taken modulo* m*). Then*

(i) *a cyclic ordering* C *of* P_φ *is consistent with a faithful embedding of* P_φ *in* S^2 *if and only if there exists an integer* k *with* $1 \leqslant k \leqslant m$ *and* $(k, m) = 1$ *such that* $C(a_i^+) = a_{i+k}^+$ *and* $C(a_i^-) = a_{i-k}^-$, $i = 1, \cdots, m$ *(subscripts modulo* m*), and*

(ii) *if* P_φ *is faithfully embedded in* S^2 *with ordering* C *(as defined in* (i)), *then the unique closed manifold determined (by Theorems* 1 *and* 3) *is the lens space* $L_{m,k}$.

PROOF. (i) If C is defined in terms of k as stated, then it is easy to verify that C has orbits E_{a^+} and E_{a^-} (so that C is a cyclic ordering), that $AC = C^{-1}A$ (so that P_φ is embeddable in S^2), and that $BC = C^{-1}B$ (so that the embedding is faithful). Conversely, if P_φ is faithfully embedded in S^2 with ordering C, we define k by $C(a_1^+) = a_{1+k}^+$. Then $C(a_2^+) = CBA(a_1^+) = BC^{-1}A(a_1^+) = BAC(a_1^+) = a_{2+k}^+$. We repeat this argument inductively to show $C(a_i^+) = a_{i+k}^+$. Since E_{a^+} is an orbit of C, it follows that $(m, k) = 1$. That C is defined as stated on E_{a^-} follows from $BC = C^{-1}B$.

(ii) See [6, p. 210] and [5] for discussions of lens spaces. The result is self-evident.

Let $\varphi = \langle a | a^{m_1}, a^{m_2} \rangle$ with $m_1, m_2 > 0$. Then in the P-graph P_φ some of the points of E_{a^+} (of E_{a^-}) belong to edges corresponding to a^{m_1} and others to a^{m_2}. We denote these points by $a_{j,i}^+$ (by $a_{j,i}^-$), $i = 1, \cdots, m_j$,

$j = 1, 2$. Assume that $B(a_{j,i}^{+}) = a_{j,i}^{-}$ and that $A(a_{j,i}^{+}) = a_{j,i+1}^{-}$.

THEOREM 5. *Suppose* $\varphi = \langle a \mid a^{m_1}, a^{m_2} \rangle$ *and construct* P_φ *as above. Then the cyclic ordering* C *is consistent with a faithful embedding of* P_φ *in* S^2 *if and only if* $m_1 = m_2$ *and for some integers* k *and* r *we have*

$$C(a_{1,i}^{+}) = a_{2,i+r}^{+}, \qquad C(a_{2,i}^{+}) = a_{1,i-r+k}^{+},$$
$$C(a_{1,i}^{-}) = a_{2,i+r-k}^{-}, \qquad C(a_{2,i}^{-}) = a_{1,1-r}^{-},$$

where $(k, m_1) = 1$ *(all second subscripts modulo* m_1*).*

PROOF. If C is defined in terms of k and r as stated, then one can show easily that C is consistent with a faithful embedding of P_φ in S^2. Conversely, assume that C is consistent with a faithful embedding. Using $AC = C^{-1}A$ and $BC = C^{-1}B$, we argue that if, for some i and j, $C(a_{1,i}^{+}) = a_{1,j}^{+}$, then the orbit of C containing $a_{1,i}^{+}$ cannot contain any point of the form $a_{2,s}^{+}$. Since E_{a^+} is an orbit of C, this is a contradiction implying in particular that $C(a_{1,1}^{+}) = a_{2,1+r}^{+}$ for some r. Write $C^s(a_{1,1}^{+}) = a_{t,u}^{+}$ to define t and u as functions of $s = 1, 2, \cdots, m_1 + m_2$. An argument similar to the above shows that $C(a_{2,i}^{+}) = a_{2,j}^{+}$ is also impossible for any i and j. Thus $t = 1$ or 2 if s is even or odd respectively, and $m_1 = m_2$. If $\varphi_j = \langle a \mid a^{m_j} \rangle, j = 1, 2$, then P_{φ_j} inherits a faithful embedding in S^2 from the given one of P_φ. The corresponding cyclic ordering C_j of P_{φ_j} must satisfy $C_j = C^2$. Let $k = u(2) - 1$. Then by Theorem 4(i), it follows that $u(s) = 1 + ks/2 \pmod{m_1}$ for s even. We argue inductively that $C(a_{1,i+1}^{+}) = CBA(a_{1,i}^{+}) = BAC(a_{1,i}^{+}) = a_{2,i+1+r}^{+}$, so that $u(s) = u(s-1) + r \pmod{m_1}$ for s odd. The theorem follows.

DEFINITION 3. The integer k of Theorems 4 and 5 is called the *gap.*

THEOREM 6. *Let* $\varphi = \langle a, b \mid a^m b^n, a^p b^q \rangle$ *with* $m, n, p, q \neq 0$. *Then* φ *fits if and only if one of the following holds*:

(i) $|m| = |p| = 1$ *or* $|n| = |q| = 1$,

(ii) $(m, p) = (n, q) = 1$, *or*

(iii) $m = \epsilon p$ *and* $n = \epsilon q$ *where* $\epsilon = \pm 1$.

Further, if (ii) *holds but* (i) *does not, then the faithful embedding of* P_φ *in* S^2 *is unique (up to a homeomorphism of* S^2*).*

PROOF. We may assume without loss of generality that $m, n, p > 0$. (We can arrange this by interchanging the generators and/or replacing one or both generators by their respective inverses.) We consider the cases $q > 0$ and $q < 0$ separately.

Case ($q > 0$). We may assume that

$$\begin{aligned} A(a_i^+) &= b_1^-, & i &= m,\\ &= b_{n+1}^-, & i &= m+p,\\ &= a_{i+1}^-, & &\text{otherwise for } 1 \leqslant i \leqslant m+p. \end{aligned}$$

$$\begin{aligned} A(a_i^-) &= b_n^+, & i &= 1,\\ &= b_{n+q}^+, & i &= m+1,\\ &= a_{i-1}^+, & &\text{otherwise for } 1 \leqslant i \leqslant m+p. \end{aligned}$$

$$\begin{aligned} A(b_i^+) &= a_1^-, & i &= n,\\ &= a_{m+1}^-, & i &= n+q,\\ &= b_{i+1}^-, & &\text{otherwise for } 1 \leqslant i \leqslant n+q. \end{aligned}$$

$$\begin{aligned} A(b_i^-) &= a_m^+, & i &= 1,\\ &= a_{m+p}^+, & i &= n+1,\\ &= b_{i-1}^+, & &\text{otherwise for } 1 \leqslant i \leqslant n+q. \end{aligned}$$

Assume that φ fits and that m and p are not both 1 and that n and q are not both 1. Then P_φ is embedded in S^2 with some ordering C and either $C(a_m^+) = a_{m+p}^+$ or $C(a_{m+p}^+) = a_m^+$. Reverse the orientation of S^2 if necessary so that $C(a_m^+) = a_{m+p}^+$. Then $C(b_{n+1}^-) = b_1^-$. We also have one of the following (which we will consider as separate subcases):

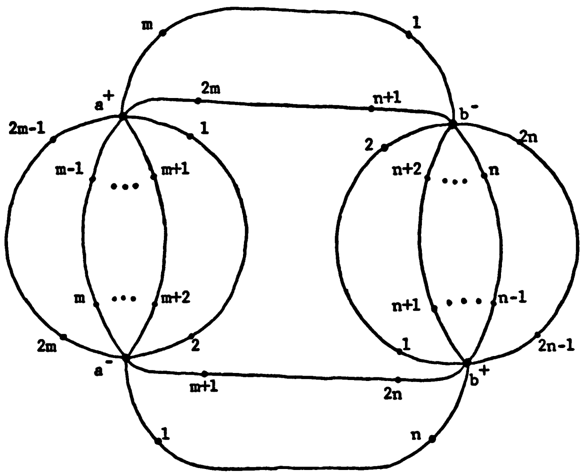
(α) $\quad C(a_{m+1}^-) = a_1^-$ and $C(b_n^+) = b_{n+q}^+$, or

(β) $\quad C(a_1) = a_{m+1}^-$ and $C(b_{n+q}^+) = b_n^+$.

In either subcase we construct a new P-graph P_a whose vertices are a^+ and a^- and whose edges are those of P_φ which connect a^+ and a^- along with two additional edges (connecting a^+ with a^-), each passing through two of the four points a_1^-, a_{m+1}^-, a_m^+, and a_{m+p}^+. If the cyclic ordering for P_a agrees with that for P on $E_{a^+} \cup E_{a^-}$ and is consistent with an embedding of P_a in S^2, then the manner in which this is done is unique and yields a faithful embedding. In particular, we must have $A(a_1^-) = a_m^+$ and $A(a_{m+1}^-) = a_{m+p}^+$ in subcase (α), while subcase (β) implies $A(a_1^-) = a_{m+p}^+$ and $A(a_{m+1}^+) = a_m^-$. In a similar manner we construct the P-graph P_b with vertices b^+ and b^-.

In subcase (α) we observe that P_a is the P-graph for $\langle a \,|\, a^m, a^p \rangle$ and conclude from Theorem 5 that $m = p$. Moreover, P_b is the P-graph for $\langle b | b^n, b^q \rangle$

so that $n = q$. Before proceeding with subcase (β) we observe (Figure 1) that condition (iii) is sufficient for P_φ to be faithfully embeddably in S^2.



Faithful embedding of P-graph for $< a,b \mid a^m b^n, a^m b^n >$.

FIGURE 1

To check that a drawing such as this exhibits a faithful embedding one must verify three things:

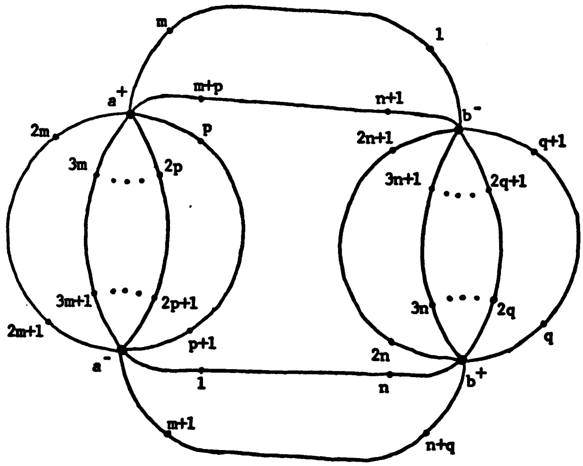
(1) That the two labeled points of E on each edge are consistent with the original assumptions about the definition of A.

(2) That C is consistent with an embedding in S^2. (This is the reason for the drawing. Without it we would have the tedious task of verifying that Theorem 2 is satisfied.)

(3) That the embedding is faithful, i.e., that $BC = C^{-1}B$. Since we defined B as we did (so that $B(a_i^-) = a_i^+$, etc.), this can be done rather easily by walking clockwise around each "head" vertex and comparing the sequence of subscripts obtained with that obtained when walking counterclockwise around the corresponding "tail" vertex.

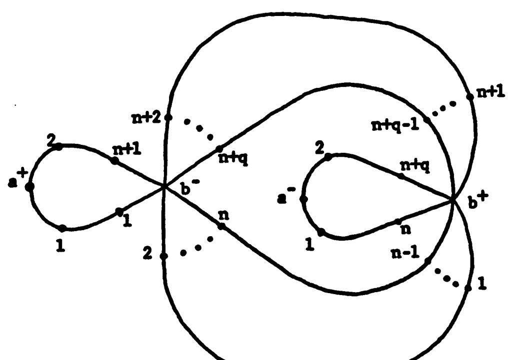
Henceforth we will use such drawings for this purpose, indicating only the subscripts for the points of E.

In subcase (β) we observe that P_a and P_b are the P-graphs for $\langle a \mid a^{m+p} \rangle$ and $\langle b \mid b^{n+q} \rangle$ in which C must be defined with gaps p and n respectively. We infer from Theorem 4 that $(p, m + p) = (n, n + q) = 1$, i.e., that $(m, p) = (n, q) = 1$. That (ii) is sufficient for faithful embeddability is verified in Figure 2. Note also that this embedding is unique.



Faithful embedding for $< a,b \mid a^m b^n, a^p b^q >$ with $m,n,p,q > 0$ and $(m,p) = (n,q) = 1$.

FIGURE 2



Faithful embedding for $< a,b \mid ab^n, ab^q >$ with $n,q > 0$.

FIGURE 3

To finish case $(q > 0)$ see Figure 3 for verification that condition (i) is sufficient for faithful embeddability. (Assume without loss of generality that $m = p = 1$ and $n, q > 0$.)

Case $(q < 0)$. First replace q by $-q$ so that we have $\varphi = \langle a, b \mid a^m b^n, a^p b^{-q} \rangle$ with $m, n, p, q > 0$. We may assume that

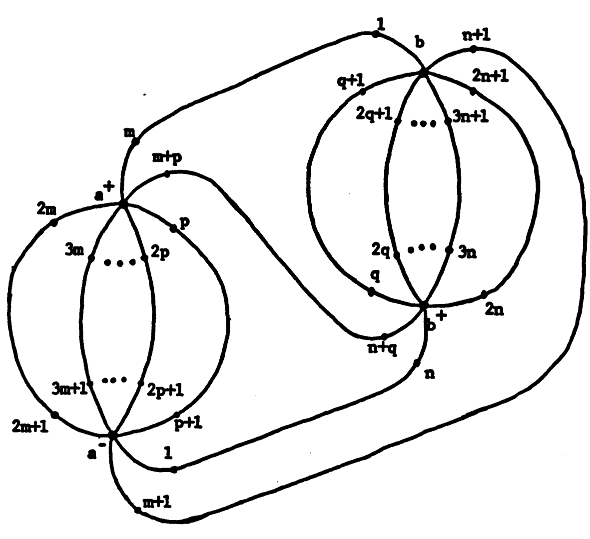
$$
\begin{aligned}
A(a_i^+) &= b_1^-, && i = m,\\
&= b_{n+q}^+, && i = m+p,\\
&= a_{i+1}^-, && \text{otherwise for } 1 \leqslant i \leqslant m+p.
\end{aligned}
$$

$$
\begin{aligned}
A(a_i^-) &= b_n^+, && i = 1,\\
&= b_{n+1}^-, && i = m+1,\\
&= a_{i-1}^-, && \text{otherwise for } 1 \leqslant i \leqslant m+p.
\end{aligned}
$$

$$
\begin{aligned}
A(b_i^+) &= a_1^-, && i = n,\\
&= a_{m+p}^+, && i = n+q,\\
&= b_{i+1}^-, && \text{otherwise for } 1 \leqslant i \leqslant n+q.
\end{aligned}
$$

$$
\begin{aligned}
A(b_i^-) &= a_m^+, && i = 1,\\
&= a_{m+1}^-, && i = n+1,\\
&= b_{i-1}^+, && \text{otherwise for } 1 \leqslant i \leqslant n+q.
\end{aligned}
$$

Assume that φ fits and that m and p are not both 1 and that n and q are not both 1. Then P_φ is faithfully embedded in S^2 with some ordering C. If we construct P_a and P_b as before, we obtain the respective faithfully em-



Faithful embedding for $< a,b \mid a^m b^n, a^p b^{-q} >$ with $m,n,p,q > 0$ and $(m,p) = (n,q) = 1$.

FIGURE 4

bedded P-graphs of $\langle a \mid a^{m+p}\rangle$ and $\langle b \mid b^{n+q}\rangle$ with gaps p and q respectively. Hence $(m, p) = (n, q) = 1$. This defines C uniquely, and we observe (Figure 4) that it is consistent with a faithful embedding of P_φ in S^2.

Finally, we assume that condition (i) holds with $m, p = 1$ and $n > 0$, $q < 0$. Then K_φ is a subdivision of $K_{\varphi'}$, where $\varphi' = \langle b \mid b^{n+q}\rangle$, so that φ fits, by Theorem 4. In fact, K_φ is a spine of a lens space.

THEOREM 7. *If* $\varphi = \langle a, b \mid a^m b^n, a^p b^q\rangle$ *with* $m, n, p, q \neq 0$, *then* K_φ *is a spine of a closed manifold if and only if*

(i) $|m| = |p| = 1$ *or* $|n| = |q| = 1$ *or*

(ii) $(m, p) = (n, q) = 1$.

Further, if (ii) *holds but* (i) *does not, then the manifold is unique.*

PROOF. In view of Theorems 3 and 6, it suffices to consider the following cases:

(a) $m = n = p = q = 1$,

(b) $m = p > 1$ and $n = q > 1$,

(c) $m = p = 1, n > 0, q > 0$, and n and q not both 1,

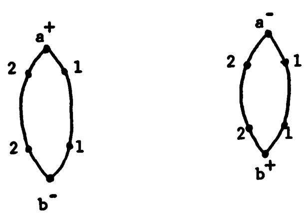
(d) $m = p = 1, n > 0, q < 0$, and n and $-q$ not both 1,

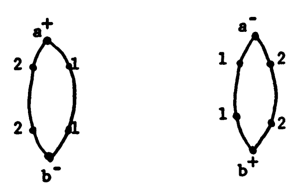
(e) $(m, p) = (n, q) = 1, m, n, p, q > 0, m$ and p not both 1, and n and q not both 1, and

(f) $(m, p) = (n, q) = 1, m, n, p > 0,\ q < 0,\ m$ and p not both 1, and n and $-q$ not both 1.

In all cases except (b) we show that K_φ is a spine of a closed manifold.

Case (a). Theorem 3 does not apply since P_φ is not connected. This being the only such case we consider it separately. Assume that $A(a_i^+) = b_i^-$ and $A(a_i^-) = b_i^+$, $i = 1, 2$. Then P_φ has two faithful embeddings in S^2, as shown in Figure 5. In both of these cases one could appeal to [3]. We note here only that Figure 5a corresponds to the closed manifold $S^2 \times S^1$ while Figure 5b corresponds to the solid torus with a "bubble."





Two faithful embeddings for < a,b|ab,ab >.

FIGURE 5a

FIGURE 5b

Case (b). We have two copies of the same relator, i.e.,

$$\varphi = \langle a, b \,|\, a^m b^n, a^m b^n \rangle \quad \text{with } m, n > 0.$$

Assume that

$$\begin{aligned} A(a^+_{j,i}) &= a^-_{j,i+1}, \quad & i &= 1, \cdots, m-1, \\ &= b^-_{j,1}, & i &= m, \\ A(b^+_{j,i}) &= b^-_{j,i+1}, & i &= 1, \cdots, n-1, \\ &= a^-_{j,1}, & i &= n, \end{aligned}$$

where $j = 1$ or 2 to indicate which relator is involved. This together with $A^2 = 1$ defines A on all of E. Assume that C is consistent with a faithful embedding and construct the faithfully embedded P-graphs P_a and P_b (as in the proof of Theorem 6). Then for some s, t, u, v we have

$$C(a^+_{1,i}) = a^+_{2,i+u}, \qquad C(a^+_{2,i}) = a^+_{1,i-u+s}$$

(second subscripts taken modulo m), and

$$C(b^+_{1,i}) = b^+_{2,i+v}, \qquad C(b^+_{2,i}) = b_{1,i-v+t}$$

(second subscripts modulo n). Since $BC = C^{-1}B$, this defines C on E. Define the sets

$$O_{j,a,\epsilon} = \{a^\epsilon_{j,i}: i = 1, \cdots, m\} \quad \text{and} \quad O_{j,b,\epsilon} = \{b^\epsilon_{j,i}: i = 1, \cdots, n\}$$

for each $j = 1$ or 2 and each $\epsilon = +$ or $-$. Let $U = O_{1,a,+} \cup O_{2,a,-} \cup O_{1,b,+} \cup O_{2,b,-}$ and $V = E \sim U$. Then each of the orbits of AC and of BC is contained in one of the two sets U or V. Hence AC and BC do not generate a transitive permutation group. By Theorem 3, P_φ is not a spine of a closed manifold.

In cases (c), (e), and (f), one can show that AC and BC generate a transitive group. We take C to be as defined in Theorem 6, referring to Figures 3, 2, and 4 respectively. Case (d) corresponds to a spine of a lens space, as we have seen.

3. **Classifying the closed manifolds.** Note that all the presentations considered in Theorem 7 which correspond to spines of closed manifolds present cyclic groups. One is thus led to conjecture that the manifolds having such spines must be lens spaces, as indeed we will show. First, we give two theorems which allow changes from one spine to another in a manifold.

Let φ be a group presentation such that K_φ is a spine of a manifold M. Suppose that $R = aa^{-1}R'$ is a relator of φ where R' is not empty and let φ' be the presentation obtained from φ by replacing R with R'. Note that

P_φ has a loop λ at a^+. Since K_φ is a spine of M, there is a corresponding faithful embedding of P_φ in S^2, and $\mathrm{cl}(\lambda)$ is a simple closed curve in S^2.

THEOREM 8. *Let φ be a presentation as described in the above paragraph. If one of the two complementary domains of λ in S^2 does not intersect P_φ, then $K_{\varphi'}$ is also a spine of M.*

Theorems like this are known that are in some ways more general. For example, see [7], [8]. We offer the following

PROOF. Let Q be the vertex of K_φ. Since $P_\varphi = K_\varphi \cap \mathrm{bd}(N(Q))$, we view $K_\varphi \cap N(Q)$ as a cone of P_φ over Q which is embedded in the cone $N(Q)$ of $S^2 = \mathrm{bd}(N(Q))$ over Q. For any set $X \subset S^2$ let $X \circ Q$ denote the cone of X over Q. See Figure 6 for the following construction.

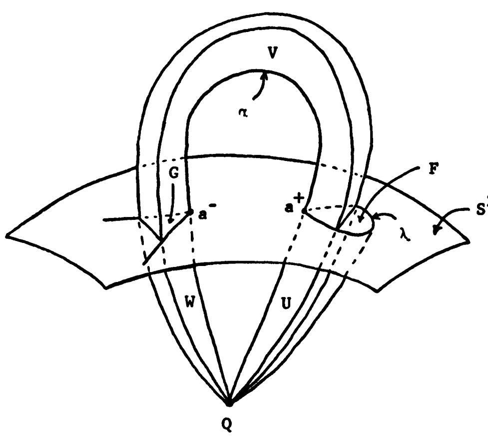


FIGURE 6

Let F be the component of $S^2 \sim \mathrm{cl}(\lambda)$ not intersecting P_φ, and let $U = \mathrm{int}(F \circ Q)$. Let α denote the 1-cell of K_φ containing the vertices a^+ and a^- of P_φ. Let V denote the (unique) component of $\mathrm{cl}(N(\alpha) \sim N(Q)) \sim K_\varphi$ that intersects F. Then $V \cap N(Q)$ has two components, one a subset of F, and the other we call G. Let $W = \mathrm{int}(G \circ Q)$ and note that $Z = \mathrm{int}(U \cup V \cup W)$ is a 3-cell. Considering $K_\varphi \cup \mathrm{cl}(Z)$ we collapse Z across $\mathrm{int}(\mathrm{bd}(Z) \cap D)$ where D denotes the 2-cell of K_φ corresponding to R.

What remains is a spine of M that is a 2-complex K with vertex Q and the same 1-cells as K_φ. The 2-cells of K are the same as those of K_φ except that D is replaced by $D' = (D \cup \mathrm{bd}(Z)) \sim \mathrm{int}(\mathrm{bd}(Z) \cap D) \sim Q$, which corresponds to R'. Hence $K = K_{\varphi'}$ establishing the theorem.

Now let φ be a group presentation corresponding to a spine of a manifold M.

Let R_1 and R_2 be two relators of φ. For each $i = 1, 2$, let e_i and $e_i' = A(e_i)$ be the two points of E on the edge of P_φ corresponding to the space between the last and first letters of the relator R_i. We label these points so that one starts at e_i and traces through P_φ corresponding to R_i before passing e_i' and returning to e_i.

THEOREM 9. *Let φ be a presentation as described in the above paragraph. Suppose that there are two "parallel" arcs γ and δ in S^2 (in which P_φ is faithfully embedded) whose interiors do not intersect P_φ, γ connecting e_1 with e_2' and δ connecting e_1' with e_2. Let φ' be the presentation obtained from φ by replacing R_1 with R_1R_2. Then $K_{\varphi'}$ is a spine of M.*

PROOF. Let D_1 and D_2 denote the 2-cells of K_φ corresponding to R_1 and R_2 respectively. We construct in $\operatorname{int}(M)$ a 2-cell D_2^* which is "parallel" to D_2 so that (1) $K_\varphi \cup D_2^*$ is a 2-complex K_{φ^*} where φ^* is the presentation φ with an extra relator R_2^* that is an exact copy of R_2, and (2) $\operatorname{cl}(D_2^* \cup D_2)$ bounds a 3-cell component U of $\operatorname{int}(M) \sim K_{\varphi^*}$. For convenience we arrange it so that γ and δ intersect U. Denote the points $\gamma \cap D_2^*$ and $\delta \cap D_2^*$ by $e_2'^*$ and e_2^* respectively and let $\gamma^* = \gamma \sim (U \cup D^2)$ and $\delta^* = \delta \sim (U \cup D^2)$.

The four points $e_1, e_1', e_2^*, e_2'^*$, together with the two arcs γ^* and δ^* and the "middle segments" of the two appropriate edges of P_{φ^*}, bound a disc F in $S^2 \sim P_{\varphi^*}$. Let $V = \operatorname{int}(F \circ Q)$ (see Figure 7). Considering $K_\varphi \cup \operatorname{cl}(U) \cup \operatorname{cl}(V)$, we collapse V across $\operatorname{int}(\operatorname{bd}(V) \cap D_1)$, and then collapse U across $\operatorname{int}(\operatorname{bd}(V) \cap D_2^*)$. The 2-complex K that remains has vertex Q and

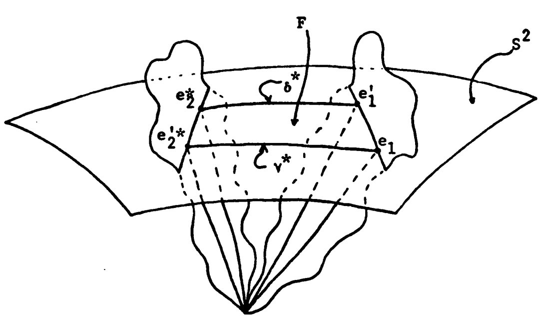


FIGURE 7

the same 1-cells as K_φ. Moreover, the 2-cells of K are those of K_φ except that D_1 is replaced by $D_1' = (D_1 \cup D_2^* \cup \operatorname{bd}(V)) \sim \operatorname{int}(\operatorname{cl}(V) \cap (D_1 \cup D_2^*)) \sim Q$, which corresponds to R_1R_2. Hence $K = K_{\varphi'}$, and the theorem is proved.

THEOREM 10. *Let* $\varphi = \langle a, b \mid a^m b^n, a^p b^q \rangle, m, n, p, q \neq 0$. *Let* $\lambda = |mq - np|$ *and suppose that* K_φ *is a spine of a closed orientable 3-manifold* M. *If* $\lambda = 0$, *then* $M = S^2 \times S^1$. *If* $\lambda > 0$, *then* M *is a lens space* $L_{\lambda, k}$ *where* $(\lambda, k) = 1$.

PROOF. Referring to Theorem 7 we see that K_φ is a spine of some M if and only if

(i) $|m| = |p| = 1$ or $|n| = |q| = 1$, or

(ii) $(m, p) = (n, q) = 1$.

In case (i) we assume without loss of generality that $m = p = 1$, and observe that $\lambda = |n - q|$ and that K_φ is homeomorphic to $K_{\varphi'}$ where $\varphi' = \langle b \mid b^n b^{-q} \rangle$ (K_φ being a subdivision of $K_{\varphi'}$). Now we apply Theorem 8 as necessary, i.e., if n and q have the same sign, to obtain a spine corresponding to $\langle b \mid bb^{-1} \rangle$ if $\lambda = 0$ (compare case (a) in the proof of Theorem 7) or $\langle b \mid b^\lambda \rangle$ if $\lambda > 0$ (see Theorem 4).

In case (ii) we assume that (i) does not hold and proceed by induction using Theorems 9 and 8 to obtain a spine of M which satisfies case (i). Assuming that $m, n > 0$, our induction step will be that if $\varphi' = \langle a, b \mid a^{m-p} b^{n-q}, a^p b^q \rangle$, then $K_{\varphi'}$ is also a spine of M. Then, since $(m, p) = 1$, we may repeat the induction step (interchanging the relators as necessary at each step) until a presentation satisfying (i) is obtained.

To verify the induction step we observe that φ may be rewritten as $\varphi = \langle a, b \mid a^m b^n, b^{-q} a^{-p} \rangle$. Referring to Figure 2 or 4 as appropriate we see that Theorem 9 applies. Thus we obtain a spine corresponding to $\langle a, b \mid a^m b^n b^{-q} a^{-p}, b^{-q} a^{-p} \rangle$, which can be rewritten as $\langle a, b \mid a^{-p} a^m b^n b^{-q}, a^p b^q \rangle$. Theorem 8 now applies since $|n|$ and $|q|$ are not both 1, and so we perform the indicated cancellations to obtain the spine corresponding to $\varphi' = \langle a, b \mid a^{m-p} b^{n-q}, a^p b^q \rangle$. This completes the proof.

We now have the following immediate result which has been observed by Bing [1] and Zieschang [7].

COROLLARY. *If* M *is a closed orientable 3-manifold having a spine corresponding to* $\langle a, b \mid a^m b^n, a^p b^q \rangle$ *where* $|mq - np| = 1$, *then* $M = S^3$.

Recall from Theorem 7 that the manifold M is unique if (ii) holds but (i) does not. We now give an algorithm for determining the lens space exactly. We do this by following the proof of Theorem 10 and establishing the gap.

After the last application of the induction step, we have a spine of M corresponding to $\varphi = \langle a, b \mid a^m b^n, a^p b^q \rangle$ where either $|m| = |p| = 1$ or $|n| = |q| = 1$. Assume that $|m| = |p| = 1$, i.e., that M has a spine corresponding to $\langle a, b \mid ab^n, ab^q \rangle$. The corresponding faithful embedding of P_φ in S^2 is such that an arc γ exists in S^2 with endpoints a^+ and a^- whose interior does not intersect P_φ. To see this we consider the situation immediately preceding the last application of the induction step. The corresponding presentation must be of the form $\langle a, b \mid ab^{n^*}a^2b^{q^*}\rangle$. The corresponding P-graph has exactly one edge connecting a^+ with a^-. Following Theorems 9 and then 8, as the induction step calls for, one can show that a neighborhood of this edge contains such an arc γ.

Now if $\varphi' = \langle b \mid b^n b^{-q}\rangle$, then K_φ and $K_{\varphi'}$ are homeomorphic so that $K_{\varphi'}$ is also a spine of M. The corresponding faithful embedding of $P_{\varphi'}$ is obtained from that of P_φ by merely "erasing" the labels on the points of the form $a^\pm$ and $a_i^\pm$ where $i = 1, 2$. If n and q have opposite sign, then the gap is either n or q (modulo $|n-q|$), both choices giving the same lens space. If n and q have the same sign, then a tedious but straightforward argument following the cancellations will also give this result.

Now to the presentation $\langle a, b \mid a^m b^n, a^p b^q\rangle$ we correspond the matrix $\left[\begin{smallmatrix} m & n \\ p & q\end{smallmatrix}\right]$. (We assume $(m, p) = (n, q) = 1$.) Certain matrix operations will yield new matrices corresponding to spines of the same manifold.

Among them are (1) interchange of rows or columns, and (2) multiplication of a row or column by -1. These correspond respectively to (1) an interchange of relators or generators and (2) replacement of a relator or generator by its inverse. In addition, the induction step of Theorem 10 allows (3) subtraction of one row from the other under the condition that both entries in at least one column have the same sign.

In summary, we have proved

THEOREM 11. *Let* $\varphi = \langle a, b \mid a^m b^n, a^p b^q\rangle$ *where* $m, n, p, q \neq 0$, $(m, p) = (n, q) = 1$, $|m|$ *and* $|p|$ *are not both* 1 *and* $|n|$ *and* $|q|$ *are not both* 1. *Operate on the matrix*

$$\begin{bmatrix} m & n \\ p & q \end{bmatrix}$$

using the above operations to obtain a matrix of the form

$$\begin{bmatrix} 1 & n' \\ 1 & q' \end{bmatrix}.$$

Let $\lambda = |mq - np| = |n' - q'|$ *and choose* k *so that* $0 \leqslant k \leqslant \lambda$ *and* $k \equiv$

n' (mod λ)· *Then* K_φ *is a spine of a unique closed orientable 3-manifold which is the lens space* $L_{\lambda,k}$ *if* $\lambda \neq 0$ *or* $S^2 \times S^1$ *if* $\lambda = 0$.

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