

STIEFEL-WHITNEY HOMOLOGY CLASSES AND BORDISM

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ABSTRACT. We develop the theory of mod 2 Stiefel-Whitney homology classes for Euler polyhedra. We then describe a simple method of obtaining p.l. bordism theories. Finally, we define the ungraded bordism theory of Euler spaces and show that it is isomorphic to ordinary total homology.

In this paper we develop the theory of Stiefel-Whitney homology classes for a class of polyhedra more general than manifolds, christened Euler spaces by Halperin and Toledo [H-T].

Sullivan and I worked out the details of the first section after he returned from the Georgia conference where Cheeger presented a proof [Ch] of Whitney's unpublished result that the Stiefel-Whitney cohomology classes are dual to the skeleta of the derived subdivision of a triangulation of the manifold. Sullivan has since applied some of this work to analytic spaces [S]. Halperin and Toledo have generalized this work in several directions. They have shown that these classes are integral in certain cases and have extended the duality theorem to the non-compact case, too. Their argument, in contrast to that of Cheeger, is purely combinatorial. For simplicity, we will deal for the most part with only the mod 2 compact case.

The second part of the paper describes a simple method of obtaining p.l. bordism theories and computes some examples.

In the third section we define the ungraded bordism theory of Euler spaces and show that it is naturally isomorphic to ordinary total homology. From this it follows, for example, that a map from an Euler space is Euler space cobordant to a map from a disjoint union of manifolds.

Stiefel-Whitney homology classes. Recall that if X is a locally compact polyhedron, we can define the link of x in X , $Lk(x; X)$ up to p.l. equivalence as the $Lk(x; K)$ where K is any triangulation of X having x as a vertex. Recall also that if K is any triangulation of X with $x \in \overset{\circ}{A}$ and $A \in K$ with $\dim A = k$ then

$$Lk(x; X) \cong \partial A * Lk(A; K) \cong \Sigma^k Lk(A; K)$$

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where \cong denotes p.l. equivalence, $*$ denotes joining and Σ^k denotes k -fold suspension.

DEFINITION 1. A locally finite simplicial pair (K, K_0) is called an Euler pair if the following equivalent conditions hold:

(a) For $x \in |K - K_0|$, $\chi(\text{Lk}(x; |K|)) \equiv 0$. For $x \in |K_0|$, $\chi(\text{Lk}(x; |K_0|)) \equiv 0$ and $\chi(\text{Lk}(x; |K|)) \equiv 1$.

(b) For $A \in K - K_0$, $\chi(\text{Lk}(A; K)) \equiv 0$. For $A \in K_0$, $\chi(\text{Lk}(A; K_0)) \equiv 0$ and $\chi(\text{Lk}(A; K)) \equiv 1$.

Here χ is the Euler characteristic and \equiv is congruence mod 2.

A locally compact polyhedral pair is called an Euler pair if some triangulation and hence any triangulation of the pair is Euler. (Note that (a) is p.l. invariant.)

A locally finite simplicial complex K (resp. a locally compact polyhedron X) is called an Euler complex (resp. an Euler space) if the pair (K, \emptyset) (resp. the pair (X, \emptyset)) is Euler.

REMARKS. 1. If X is a compact polyhedron then $\chi(\Sigma X) \equiv \chi(X)$. This implies (a) \Leftrightarrow (b).

2. If K is a finite simplicial complex $\chi(K) \equiv \#K$ (=the number of simplices of K).

3. If (X, X_0) is Euler then $X - X_0$ and X_0 are Euler. The converse holds if $X_0 \subset X$ is locally collared. In fact, in this case Remark 1 implies that $X - X_0$ Euler is sufficient.

4. If X is a polyhedron, then define

$$X_0 = \{x \in X: \chi(\text{Lk}(x; X)) \equiv 1\}.$$

X_0 is a union of open simplices of any triangulation of X and, in fact, is a union of components of open intrinsic skeleta $I^i(X) - I^{i-1}(X)$ (cf. [A₁; pp. 421, 422]). Clearly, there exists a subpolyhedron Y of X such that (X, Y) is Euler iff X_0 is closed in X , and hence a subpolyhedron iff (X, X_0) is Euler.

All homology is singular homology with Z_2 coefficients.

PROPOSITION 1. (a) If K is a finite Euler complex and ηK is a derived subdivision of K , then $C_i^\eta(K)$ = sum of the i dimensional simplices of ηK is a cycle.

(b) If (K, K_0) is a finite Euler pair, then $C_i^\eta(K)$ is a relative cycle with boundary $C_{i-1}^\eta(K_0)$.

PROOF. If $\partial_0 \dots \partial_{i-1} = A$ is an $i-1$ simplex of ηK , where $\sigma_0 < \dots < \sigma_{i-1}$ in K , then the coefficient of A in $\partial C_i^\eta \equiv \#$ of i -simplices having A as a face, i.e. we must count how many σ 's can be inserted into the sequence for A to

obtain an i -simplex. Clearly, this number is equal to

$$\# \partial \sigma_0 + \sum_{j=1}^{i-1} \# \text{Lk}(\sigma_{j-1}, \partial \sigma_j) + \# \text{Lk}(\sigma_{i-1}, K).$$

Since the Euler number of a sphere is 0 mod 2, this coefficient is $\equiv \chi(\text{Lk}(\sigma_{i-1}K))$.

So if (K, K_0) is Euler this coefficient is 0 if $\sigma_{i-1} \in K - K_0$, i.e. if $A \in \eta K - \eta K_0$ and is 1 if $\sigma_{i-1} \in K_0$, i.e. if $A \in \eta K_0$.

REMARK. If ηK and $\bar{\eta} K$ are two different derived subdivisions of K obtained by starring the simplices of K at different internal points, then $\partial \rightarrow \bar{\partial}$, the obvious simplicial isomorphism carries $C_i^\eta(K, K_0)$ to $C_i^{\bar{\eta}}(K, K_0)$. The map is isotopic to the identity so the chains represent the same homology class. So we will drop the superscript η .

DEFINITION 2. If (K, K_0) is a finite Euler pair, then $w_i(K, K_0)$ is the homology class of $C_i(K, K_0)$.

PROPOSITION 2. The w_i 's are p.l. invariants, i.e. if (K, K_0) and (L, L_0) are two triangulations of the polyhedral Euler pair (X, X_0) then $w_i(K, K_0) = w_i(L, L_0)$ in $H_i(X, X_0)$ and so we can write $w_i(X, X_0)$ for these classes.

PROOF. If we subdivide K by a stellar subdivision at $\tau \in K$, we obtain K^* which is the same complex as K on $|K - \text{St}^0(\tau; K)|$. Choose derived ηK and ηK^* which agree there. We note that as singular chains the difference $C_i^\eta(K) - C_i^\eta(K^*)$ is carried by $|\text{St}(\tau, K)|$.

If $\tau \in K - K_0$ then $C_{i-1}^\eta(K_0) = C_{i-1}^\eta(K_0^*)$ and so $C_i^\eta(K) - C_i^\eta(K^*)$ is a cycle on K carried by the contractible space $|\text{St}(\tau; K)|$.

If $\tau \in K_0$ then $C_i^\eta(K) - C_i^\eta(K^*)$ is a relative cycle carried by the contractible pair $(\text{St}(\tau; K), \text{St}(\tau; K_0))$.

Now if K and L have a common subdivision it is not known whether they have a common stellar subdivision, but it is known e.g. [G, p. 41, Theorem II-17] that one can get from one to the other by a finite sequence of stellar subdivisions or inverse stellar subdivisions i.e. there exists $K = K_0, K_1, \dots, K_n = L$ such that for $i = 0, \dots, n-1$ either K_i is a stellar subdivision of K_{i+1} or K_{i+1} is a stellar subdivision of K_i . Note that any triangulation of X is a triangulation of the pair (X, X_0) because $\chi(\text{Lk}(x; X)) \equiv 0$ for $x \notin X_0$ and $\equiv 1$ for $x \in X_0$.

REMARK. If $\partial: H_i(X, X_0) \rightarrow H_{i-1}(X_0)$, Propositions 1 and 2 imply that $\partial w_i(X, X_0) = w_{i-1}(X_0)$.

PROPOSITION 3 (EXCISION). Let X_0, X_1, X_2, X_3 be closed subpolyhedra of a compact polyhedron X such that $X = X_1 \cup X_2$, $X_0 \subset X_3 \cup X_2$ and (X, X_3)

and (X_1, X_0) are Euler pairs; then in the diagram:

$$H_i(X_1, X_0) \xrightarrow{j_1^*} H_i(X, X_3 \cup X_2) \xleftarrow{j^*} H_i(X, X_3),$$

$$j_1 \cdot (w_i(X_1, X_0)) = j \cdot (w_i(X, X_3)).$$

PROOF. Triangulate to obtain K a complex on X and K_i with $|K_i| = X_i$, $i = 0, 1, 2, 3$. $C_i^\eta(K) - C_i^\eta(K_1)$ is carried by X_2 , i.e. if $\partial_0 \dots \partial_i \in \eta K - \eta K_1$ then $\sigma_i \in K_2$ and $\partial_0 \dots \partial_i$ is in X_2 .

Letting $X_3 = \emptyset$ we obtain:

COROLLARY. Let X_0, X_1, X_2 be closed subpolyhedra of a compact Euler space X such that $X = X_1 \cup X_2$, $X_0 \subset X_2$ and (X_1, X_0) is an Euler pair, then in the diagram

$$H_i(X_1, X_0) \xrightarrow{j_1^*} H_i(X, X_2) \xleftarrow{j^*} H_i(X),$$

$$j_1 \cdot w_i(X_1, X_0) = j \cdot w_i(X).$$

PROPOSITION 4 (MAYER-VIETORIS). Let X_0, X_1, X_2, X_3 be closed subpolyhedra of a compact polyhedron X such that $X_0 = X_1 \cap X_2$, $X_1 \cup X_2 = X$ and $(X_1, X_0 \cup (X_3 \cap X_1))$, $(X_2, X_0 \cup (X_3 \cap X_2))$ and $(X_0, X_0 \cap X_3)$ are Euler pairs.

Then $(X_3 \cap X_2, X_3 \cap X_0)$, $(X_3 \cap X_1, X_3 \cap X_0)$ and (X, X_3) are Euler pairs, and the Mayer-Vietoris boundary $\Delta: H_i(X, X_3) \rightarrow H_{i-1}(X_0, X_0 \cap X_3)$ takes $w_i(X, X_3)$ to $w_{i-1}(X_0, X_0 \cap X_3)$.

$$\begin{array}{ccc} H_i(X, X_3) \rightarrow H_i(X, X_2 \cup X_3) & \xleftarrow{\cong} & H_i(X_1, X_0 \cup (X_1 \cap X_3)) \\ \Delta \downarrow & & \partial \downarrow \\ H_{i-1}(X_0, X_3 \cap X_0) \xrightarrow{\cong} H_{i-1}(X_0 \cup (X_1 \cap X_3), X_1 \cap X_3) & \xleftarrow{\quad} & H_{i-1}(X_0 \cup (X_1 \cap X_3)) \end{array}$$

PROOF. To see that $(X_3 \cap X_i, X_3 \cap X_0)$, $i = 1, 2$, and (X, X_3) are Euler pairs we look first at points $x \in X_0 \cap X_3$. $\text{Lk}(x; (X_3 \cap X_i) \cup X_0) = \text{Lk}(x; X_3 \cap X_i) \cup \text{Lk}(x; X_0)$ with intersection $\text{Lk}(x; X_0 \cap X_3)$. So $\chi(\text{Lk}(x; X_3 \cap X_i)) \equiv 1$. $\text{Lk}(x; X) = \text{Lk}(x; X_1) \cup \text{Lk}(x; X_2)$ with intersection $\text{Lk}(x; X_0)$ so $\chi(\text{Lk}(x; X)) \equiv 1$. For $x \in X_0 - X_3$, $\text{Lk}(x; X) = \text{Lk}(x; X_1) \cup \text{Lk}(x; X_2)$ with intersection $\text{Lk}(x; X_0)$ implies that $\chi(\text{Lk}(x; X)) \equiv 0$.

That $\Delta w_i(X, X_3) = w_{i-1}(X_0, X_3 \cap X_0)$ follows from Proposition 3, the remark after Proposition 2, and the Corollary to Proposition 3, via the commutative diagram above.

Letting $X_3 = \emptyset$ we obtain:

COROLLARY 1. *Let X_0, X_1, X_2 be closed subpolyhedra of a compact polyhedron X such that $X_0 = X_1 \cap X_2$, $X = X_1 \cup X_2$ and (X_i, X_0) ($i = 1, 2$) are Euler pairs.*

Then X is an Euler space and $\Delta w_i(X) = w_{i-1}(X_0)$ where Δ is the Mayer-Vietoris boundary:

$$H_i(X) \rightarrow H_i(X, X_2) \xleftarrow{\cong} H_i(X_1, X_0) \xrightarrow{\partial} H_{i-1}(X_0).$$

COROLLARY 2. *Let (X, X_0) be a compact Euler pair and $(I, \dot{I}) = ([0, 1], \{0, 1\})$, then $(X \times I, X \times I \cup X_0 \times I)$ is an Euler pair and*

$$\bar{\Delta} w_i(X \times I, X \times \dot{I} \cup X_0 \times I) = w_{i-1}(X, X_0)$$

where $\bar{\Delta}$ is given by:

$$\begin{array}{ccc} H_i(X \times I, X_0 \times I \cup X \times \dot{I}) & \xrightarrow{\partial} & H_{i-1}(X_0 \times I \cup X \times \dot{I}) \\ \bar{\Delta} \downarrow & & \downarrow \\ H_{i-1}(X, X_0) & \xrightarrow{\cong} & H_{i-1}(X_0 \times I \cup X \times \dot{I}, X_0 \times I \cup X \times 1) \end{array}$$

PROOF. Let $\bar{X}_1 = X \times [0, \frac{1}{2}]$, $\bar{X}_2 = X \times [\frac{1}{2}, 1]$, $\bar{X}_3 = X_0 \times I \cup X \times \dot{I}$, $\bar{X}_0 = X \times \frac{1}{2}$ and $\bar{X} = X \times I$. Using the addition formula for Euler characteristics again we see that Proposition 4 applies to this situation. It remains to identify Δ with $\bar{\Delta}$, which is easy using a map $r: I \rightarrow I$ with $r[0, \frac{1}{2}] = 0$ and $r(1) = 1$.

REMARK. Since $(X \times 0, X \times 0) \subset (\bar{X}_1, \bar{X}_3 \cap \bar{X}_1)$ and $(X \times 1, X \times 1) \subset (\bar{X}_2, \bar{X}_3 \cap \bar{X}_2)$ are homotopy equivalences, exactness of the Mayer-Vietoris sequence implies that $\bar{\Delta}$ is an isomorphism.

COROLLARY 3. *Let X be a compact Euler space with $\chi(X) \equiv 0$; then ΣX is an Euler space and (cX, X) is an Euler pair. The suspension isomorphism, $\sigma: H_i(X) \rightarrow H_{i-1}(\Sigma X)$ takes $w_i(X)$ to $w_{i+1}(\Sigma X)$, where σ is the inverse to the Mayer-Vietoris boundary for $(\Sigma X; c_1 X, c_2 X)$ (recall that the suspension is the union of two cones on X intersecting in their common base).*

Now we turn to the relation between these homology classes and mappings. Another way of stating Proposition 2 is that if $f: (X, X_0) \rightarrow (Y, Y_0)$ is a p.l. homeomorphism with (X, X_0) (and hence (Y, Y_0) also) an Euler pair, then $f_*(w_i(X, X_0)) = w_i(Y, Y_0)$, where f_* is the induced map on Z_2 homology.

This is pretty weak invariance. It is reasonable to suppose that the w_i 's are homotopy invariants. This is true, after all, of the Stiefel-Whitney classes on manifolds.

However, it is not true for Euler spaces. In fact let $X = S^2 \vee S^2$, i.e. the

union of two spheres along a point and let $f: X \rightarrow X$ by the union of $s: S^2 \rightarrow S^2 \vee S^2$ and $i_2: S^2 \rightarrow S^2 \vee S^2$ on the two pieces of the domain, s is the coproduct obtained by pinching the equator to a point and i_2 is the inclusion on the second term. $H_2(X) = Z_2 \oplus Z_2$ generated by z_1 and z_2 , the fundamental classes on the two spheres. $w_2(X) = z_1 + z_2$ and so $f_*w_2(X) = s_*(z_1) + i_{2*}(z_2) = z_1 + z_2 + z_2 = z_1$. Thus $f_*w_2(X) \neq w_2(X)$. Note that X is an Euler space as the link of the bad point is the disjoint union of two circles.

In the positive direction we have invariance under the appropriate notion of cobordism of maps.

DEFINITION 3. Let $(X, X_0), (Y, Y_0)$ be compact Euler pairs and (T, T_0) any topological pair, and let $f: (X, X_0) \rightarrow (T, T_0)$ and $g: (Y, Y_0) \rightarrow (T, T_0)$ be continuous.

A cobordism between f and g is a quintuple (W, W_0, W_1, i_X, i_Y) with $W \supset W_0 \supset W_1$ compact polyhedra, $i_X: X \rightarrow W_0$, $i_Y: Y \rightarrow W_0$ p.l. embeddings and a map $F: (W, W_1) \rightarrow (T, T_0)$ such that

(1) $i_X(X) \cap W_1 = i_X(X_0)$, $i_Y(Y) \cap W_1 = i_Y(Y_0)$, $i_X(X) \cap i_Y(Y) = \emptyset$ and $W_0 = i_X(X) \cup W_1 \cup i_Y(Y)$.

(2) (W, W_0) and $(W_1, i_X(X_0) \cup i_Y(Y_0))$ are Euler pairs.

(3) $F \cdot i_X = f$ and $F \cdot i_Y = g$.

REMARK. Clearly, if f and g are homotopic then they are cobordant, see Corollary 2 of Proposition 4.

PROPOSITION 5. If $f: (X, X_0) \rightarrow (T, T_0)$ and $g: (Y, Y_0) \rightarrow (T, T_0)$ be cobordant. Then

$$f_*w_i(X, X_0) = g_*w_i(Y, Y_0) \quad \text{for all } i.$$

PROOF. Using p.l. invariance of the w_i 's we shall regard i_X and i_Y as inclusions. We consider the map of the triples induced by $F: (W, W_0, W_1) \rightarrow (T, T, T_0)$.

$$\begin{array}{ccccccc}
 H_{i+1}(W, W_0) & \xrightarrow{\partial} & H_i(W_0) & \rightarrow & H_i(W_0, W_1) & \xrightarrow{\cong} & \begin{array}{c} H_i(W_0, W_1 \cup Y) \\ \oplus \\ H_i(W_0, W_1 \cup X) \end{array} & \xleftarrow{\cong} & \begin{array}{c} H_i(X, X_0) \\ \oplus \\ H_i(Y, Y_0) \end{array} \\
 F_* \downarrow & & F_* \downarrow & & F_* \downarrow & & & \nwarrow f_* + g_* & \\
 H_{i+1}(T, T) & \xrightarrow{\partial} & H_i(T) & \rightarrow & H_i(T, T_0) & & & &
 \end{array}$$

Note that $(X \cup Y, X_0 \cup Y_0) \subset (W_0, W_1)$, $(X, X_0) \subset (W_0, W_1 \cup Y)$ and $(Y, Y_0) \subset (W_0, W_1 \cup X)$ are excisions. Now by the Corollary to Proposition 3 applied twice, $w_i(W_0)$ on the one hand and $w_i(X, X_0) + w_i(Y, Y_0)$ on the other,

map to the same thing in $H_i(W_0, W_1 \cup Y) \oplus H_i(W_0, W_1 \cup X)$. Since $\partial w_{i+1}(W, W_0) = w_i(W_0)$ and since $H_*(T, T) = 0$, it follows that $f_*w_i(X, X_0) + g_*w_i(Y, Y_0) = 0$. So $f_*w_i(X, X_0) = g_*w_i(Y, Y_0)$.

As an illustration we will look at an example.

DEFINITION 4. Let $f: X_1 \rightarrow X_2$ be a proper p.l. map, of Euler spaces, with $f(X_1) = X_2$.

(a) Call f even if, for all $x \in X_2$, $\chi(f^{-1}(x)) \equiv 0$.

(b) Call f odd if, for all $x \in X_2$, $\chi(f^{-1}(x)) \equiv 1$.

We will show that (a) if $f: X_1 \rightarrow X_2$ is odd then f is cobordant to 1_{X_2} , and (b) if f is even then f is cobordant to $\emptyset: \emptyset \rightarrow X_2$, the empty map. It follows that odd maps preserve the Stiefel-Whitney homology classes and even maps kill them.

We will use material on general complexes and mapping cylinders as discussed in [A₂].

LEMMA. Let \mathfrak{D}_i be finite general complexes on compact polyhedra X_i , $i = 1, 2$, and $\gamma: \mathfrak{D}_1 \rightarrow \mathfrak{D}_2$ an isomorphism of general complexes. Assume that K is a constant such that $\chi(\gamma(\sigma)) = K\chi(\sigma)$ for all $\sigma \in \mathfrak{D}_1$. Then $\chi(X_2) = K\chi(X_1)$.

PROOF. By induction on the number of elements in the complexes. Let σ be a free element of \mathfrak{D}_1 , i.e. $\sigma' \in \mathfrak{D}_1$ implies $\sigma' \cap \sigma \subset \partial\sigma$ or $\sigma' = \sigma$, then $\gamma\sigma$ is free in \mathfrak{D}_2 . Let $\mathfrak{D}'_1 = \mathfrak{D}_1 - \sigma$, $\mathfrak{D}'_2 = \mathfrak{D}_2 - \gamma\sigma$, $\mathfrak{B}_1 = \{\text{all faces of } \sigma\} \cup \{\sigma\}$, $\mathfrak{B}_2 = \{\text{all faces of } \gamma\sigma\} \cup \{\gamma\sigma\}$.

If $\mathfrak{B}_i = \mathfrak{D}_i$ then $\chi(X_2) = \chi(\gamma\sigma) = K\chi(\sigma) = K\chi(X_1)$.

If \mathfrak{B}_i and \mathfrak{D}'_i are proper subsets of \mathfrak{D}_i , $i = 1, 2$, then by induction hypothesis $\chi(\mathfrak{B}_2) = K\chi(\mathfrak{B}_1)$, $\chi(\mathfrak{D}'_2) = K\chi(\mathfrak{D}'_1)$ and $\chi(\mathfrak{B}_2 \cap \mathfrak{D}'_2) = \chi(\mathfrak{B}_2 \cap \mathfrak{D}'_2) = K\chi(\mathfrak{B}_1 \cap \mathfrak{D}'_1) = K\chi(\mathfrak{B}_1 \cap \mathfrak{D}'_1)$; and so the result follows from the addition formula for Euler characteristic.

REMARK. In the above proof and hence in the above theorem “=” can be replaced by “ $\equiv \text{mod } n$ ” for any n .

This lemma is motivated by and can be used to prove the well-known fact that if $\pi: E \rightarrow B$ is a p.l. fibration with E compact and fibre F , then $\chi(E) = \chi(F)\chi(B)$. In particular if E admits a Z_2 symmetry, then $\chi(E) \equiv 0$. Thus, for a polyhedron to be Euler it is sufficient that the links of points admit p.l. Z_2 symmetries.

Given $f: K_1 \rightarrow K_2$ simplicial map either even or odd, of locally finite Euler complexes, we examine the simplicial mapping cylinder C_f . We utilize the general complexes on K_1 , K_2 and C_f given by the map developed by Cohen and discussed in [A₂, pp. 404, 407–408, 410]. For $\sigma \in K_2$, $\text{Lk}(\sigma; C_f) \cong \dot{Q}(\sigma; f) \cup D(\sigma; f)$ (intersection $\dot{D}(\sigma, f)$). The map $Q(\sigma'; f) \rightarrow D(\sigma'; K_2)$ is an isomorphism of

general complexes on $\dot{Q}(\sigma; f)$ and $\dot{D}(\sigma; K_2)$ as σ' varies with $\sigma < \sigma'$. Thus, by the above lemma $\chi(\dot{Q}(\sigma; f)) = \chi(\dot{D}(\sigma; K_2)) \equiv 0$.

Now $D(\sigma; f)$ is a regular neighborhood of $f^{-1}(\hat{\sigma})$ and so $\chi(D(\sigma; f)) \equiv \epsilon$ where $\epsilon = 0$ or 1 according as f is even or odd. The map $D(\sigma'; f) \rightarrow D(\sigma'; K_2)$ with $\sigma' > \sigma$ is an isomorphism of general complexes on $\dot{D}(\sigma; f)$ and $\dot{D}(\sigma; K_2)$ and hence by the lemma $\chi(\dot{D}(\sigma; f)) \equiv \epsilon \chi(\dot{D}(\sigma; K_2)) \equiv 0$.

Thus by the addition formula for Euler characteristics we have $\chi(\text{Lk}(\sigma; C_f)) \equiv \chi(D(\sigma; f)) \equiv \epsilon$.

Thus, for points $x \in |K_2|$,

$$\chi(\text{Lk}(x; C_f)) \equiv \epsilon \quad \text{and} \quad \chi(\text{Lk}(x; K_2)) \equiv 0.$$

For $x \in |K_1|$,

$$\chi(\text{Lk}(x; C_f)) \equiv 1 \quad \text{and} \quad \chi(\text{Lk}(x; K_1)) \equiv 0$$

as K_1 is collared in C_f . For $x \in C_f - |K_1| \cup |K_2|$, $\chi(\text{Lk}(x; C_f)) \equiv 0$ because the collar can be extended to include x so that such a point, looking locally, thinks it is inside $K_1 \times I - K_1 \times \dot{I}$ which is Euler.

In short, if f is even $(C_f, |K_1|)$ is an Euler pair and if f is odd $(C_f, |K_1| \cup |K_2|)$ is an Euler pair.

We can also show the converse, more precisely we show that if $\chi(\text{Lk}(\sigma, C_f))$ is the same (mod 2) for all $\sigma \in K_2$ then $\chi(f^{-1}(\eta\sigma))$ is the same for all $\sigma \in K_2$.

Note that the above proof shows that if $\chi(f^{-1}(\eta\sigma'))$ is constant mod 2 for all $\sigma' > \sigma$, then $\chi(\text{Lk}(\sigma; C_f)) \equiv \chi(f^{-1}(\eta\sigma))$.

Now, assume K_2 is finite dimensional and let n be the maximum codimension such that $\chi(f^{-1}(\eta\sigma))$ is constant mod 2 for σ of codimension $< n$. So if $\chi(f^{-1}(\eta\sigma))$ is not constant for all σ , then there exist σ_1 and σ_2 of codimension $\leq n$ such that $\chi(f^{-1}(\eta\sigma_1)) \not\equiv \chi(f^{-1}(\eta\sigma_2))$. Since $\sigma' > \sigma_i$ for $i = 1$ or 2 implies codimension $\sigma' < n$ it follows that

$$\chi(\text{Lk}(\sigma_1; C_f)) \neq \chi(\text{Lk}(\sigma_2; C_f)).$$

If K_2 is not finite dimensional then choose σ_1 and σ_2 with $\chi(f^{-1}(\eta\sigma_1)) \not\equiv \chi(f^{-1}(\eta\sigma_2))$ and apply the above inductive argument on codimension to the simplices interior to $\text{St}(\sigma_1; K_2) \cup \text{St}(\sigma_2; K_2)$, noting that, since K_2 is locally finite, this is a finite complex.

In sum we have proved:

PROPOSITION 6. *Let $f: X_1 \rightarrow X_2$ be an onto, proper p.l. map of Euler spaces and let C_f be a mapping cylinder obtained by triangulating f .*

(a) *(C_f, X_1) is an Euler pair iff f is an even map.*

(b) *$(C_f, X_1 \cup X_2)$ is an Euler pair iff f is an odd map.*

PROOF. If either (C_f, X_1) or $(C_f, X_1 \cup X_2)$ is an Euler pair then $\chi(\text{Lk}(x; C_f))$ is constant for x in X_2 and so $\chi(f^{-1}(x))$ is constant for $x \in X_2$ so f is either even or odd. Since $\chi(\text{Lk}(\sigma; C_f)) \equiv \chi(f^{-1}(\eta\sigma))$ in these cases, the possibilities pair up properly.

In private correspondence, McCrory has shown me a proof of this result which requires only that the mapping cylinder of f be triangulable with point-inverses as subcomplexes, instead of requiring that f itself be a p.l. map.

Let $p_f: C_f \rightarrow X_2$ be any retraction extending f on X_1 .

COROLLARY 1. (a) If $f: X_1 \rightarrow X_2$ is an even map then $(C_f, X_1, \emptyset; \text{inclusion}_{X_1}, \emptyset)$ and the map p_f is a cobordism of f and the empty map.

(b) If $f: X_1 \rightarrow X_2$ is an odd map then $(C_f, X_1 \cup X_2, \emptyset; \text{inclusion}_{X_1}, \text{inclusion}_{X_2})$ and the map p_f is a cobordism of f and 1_{X_2} .

COROLLARY 2. (a) If $f: X_1 \rightarrow X_2$ is an even map then $f_* w_i(X_1) = 0$ for all i .

(b) If $f: X_1 \rightarrow X_2$ is an odd map then $f_* w_i(X_1) = w_i(X_2)$ for all i .

Z_2 -bordism theories. Let \mathfrak{F}^n , $n \in \mathbb{Z}$, be a sequence of classes of compact p.l. pairs satisfying the following four axioms (write $X \in \mathfrak{F}^n$ for $(X, \emptyset) \in \mathfrak{F}^n$):

AXIOM 1. If $(X, X_0) \cong (Y, Y_0)$ and $(X, X_0) \in \mathfrak{F}^n$ then $(Y, Y_0) \in \mathfrak{F}^n$.

AXIOM 2. $(X, X_0) \in \mathfrak{F}^n$ implies $X_0 \in \mathfrak{F}^{n-1}$ and $(X \times I, X \times \dot{I} \cup X_0 \times I) \in \mathfrak{F}^{n+1}$.

AXIOM 3 (CUTTING). Let $(X, X_0) \in \mathfrak{F}^n$, A a closed subpolyhedron of X , (V, V_0) a regular neighborhood pair for A in (X, X_0) . If $Y = \overline{X - V}$ and $Y_0 = \overline{X_0 - V_0}$, then $(Y, \dot{V} \cup Y_0) \in \mathfrak{F}^n$, $(V, \dot{V} \cup V_0) \in \mathfrak{F}^n$ and $(\dot{V}, \dot{V} \cap Y_0) \in \mathfrak{F}^{n-1}$.

AXIOM 4 (PASTING). Let $X \supset X_0 \supset X_1$, $Y \supset Y_0 \supset Y_1$ and $Z \supset Z_0 \supset \emptyset$ be p.l. triples with X and Y disjoint and $i_X: Z \rightarrow X_0$, $i_Y: Z \rightarrow Y_0$ p.l. embeddings such that

(a) $X_0 = i_X(Z) \cup X_1$, $Y_0 = i_Y(Z) \cup X_1$ and $i_X^{-1}(X_1) = i_Y^{-1}(Y_1) = Z_0$.

(b) If $(X, X_0), (Y, Y_0) \in \mathfrak{F}^n$ and $(X_1, i_X(Z_0)), (Y_1, i_Y(Z_0)), (Z, Z_0) \in \mathfrak{F}^{n-1}$, then $(X \cup_Z Y, X_1 \cup_{Z_0} Y_1) \in \mathfrak{F}^n$.

REMARKS. 1. Letting $A = X$ in 3 implies $\emptyset \in \mathfrak{F}^n$ provided \mathfrak{F}^n is not empty. For convenience we will assume in addition $\emptyset \in \mathfrak{F}^n$ for all n .

2. For the definition and theory of regular neighborhood pairs we refer the reader to Cohen's paper [C]. In particular, if (V, V_0) is a regular neighborhood pair of A in (X, X_0) , then V_0 is a regular neighborhood of $A \cap X_0$ in X_0 . Hence, in Axiom 3, it follows that $(Y_0, Y_0 \cap \dot{V}), (V_0, V_0 \cap \dot{V}) \in \mathfrak{F}^{n-1}$ by Axiom 3 applied to $A \cap X_0$ in (X_0, \emptyset) . We recall that $\dot{V}_0 = \dot{V} \cap X_0 = V_0 \cap Y_0 = V_0 \cap \dot{V} = \dot{V} \cap Y_0$ from the simplicial definitions [C, pp. 192–194]. Also, (\dot{V}, \dot{V}_0) is

collared in (V, V_0) and (Y, Y_0) , and bicollared in (X, X_0) . This relative version of [C, p. 203, Theorem 5.3] does not seem to appear anywhere. The proof is the same as Cohen's absolute theorem, except that the equation on p. 203 of [C] where a common factor S is pulled off of various links becomes a pair (S, S_0) pulled from link pairs. Then one applies the relative theorem [A₁, p. 455, Theorem 2], in place of Cohen's theorem [C, p. 199, Proposition 4.2].

DEFINITION 5. (a) Let (X, X_0) and $(Y, Y_0) \in \mathfrak{F}^n$; then a cobordism \mathfrak{A} between (X, X_0) and (Y, Y_0) is a quintuple $(W, W_0, W_1; i_X, i_Y)$ where $W \supset W_0 \supset W_1$ and $i_X: X \rightarrow W_0$ and $i_Y: Y \rightarrow W_0$ are p.l. embeddings, satisfying:

(1) $i_X(X) \cap W_1 = i_X(X_0)$, $i_Y(Y) \cap W_1 = i_Y(Y_0)$, $i_X(X) \cap i_Y(Y) = \emptyset$ and $W_0 = i_X(X) \cup W_1 \cup i_Y(Y)$.

(2) $(W, W_0) \in \mathfrak{F}^{n+1}$ and $(W_1, i_X(X_0) \cup i_Y(Y_0)) \in \mathfrak{F}^n$.

(b) Let (X, X_0) and $(Y, Y_0) \in \mathfrak{F}^n$. Let (X, X_0) and $(Y, Y_0) \in \mathfrak{F}^n$. Let $f: (X, X_0) \rightarrow (T, T_0)$ and $g: (Y, Y_0) \rightarrow (T, T_0)$ be continuous maps to a topological pair (T, T_0) (T_0 is closed in T), then a cobordism V between f and g is a pair (\mathfrak{A}, F) where $\mathfrak{A} = (W, W_0, W_1; i_X, i_Y)$ is a cobordism between (X, X_0) and (Y, Y_0) and $F: (W, W_1) \rightarrow (T, T_0)$ is a continuous map such that $F \cdot i_X = f$ and $F \cdot i_Y = g$.

With slight abuse of notation we shall usually regard i_X and i_Y as inclusions and write $\mathfrak{A} = (W, W_0, W_1)$ when i_X and i_Y are clear.

By Axiom 2, the cobordism relations are reflexive and by the pasting axiom they are transitive. They are clearly symmetric and hence are equivalence relations. We denote by $\Omega_n(T, T_0)$ the set of equivalence classes of maps $f: (X, X_0) \rightarrow (T, T_0)$ $(X, X_0) \in \mathfrak{F}^n$ (equivalence is cobordism of maps). Under disjoint union (possible by pasting with $Z = \emptyset$) this is a group with zero element $\emptyset: (\emptyset, \emptyset) \rightarrow (T, T_0)$ and satisfying $f + f = 0$ by the cobordism $((X \times I, X \times \dot{I} \cup X_0 \times I, X_0 \times I), f \cdot \pi_X)$.

The boundary $\partial: \Omega_n(T, T_0) \rightarrow \Omega_{n-1}(T_0)$ is defined by $\partial[f] = [f|X_0]$ where $f: (X, X_0) \rightarrow (T, T_0)$.

If $F: (T, T_0) \rightarrow (R, R_0)$ is continuous then $F_*[f] = [F \cdot f]$ defines $F_*: \Omega_n(T, T_0) \rightarrow \Omega_n(R, R_0)$.

PROPOSITION 7. Ω_* is a homology theory.

PROOF. The homotopy axiom is clear using the usual product cobordism.

EXCISION. Let U be an open set of T with $\bar{U} \subset \text{interior } T_0$. Then we show that $\Omega_n(T - U, T_0 - U) \xrightarrow{\cong} \Omega_n(T, T_0)$ is an isomorphism.

Let $f: (X, X_0) \rightarrow (T, T_0)$ and let $T_1 = \bar{U}$, $U_0 = \text{interior } T_0$; then $f^{-1}T_1 \subset f^{-1}(U_0)$ and we can find A a closed subpolyhedron of X with $f^{-1}T_1 \subset A \subset f^{-1}(U_0)$ and hence a regular neighborhood pair (V, V_0) of A with $V \subset f^{-1}(U_0)$.

If we let $Y = \overline{X - V}$ and $Y_0 = \overline{X_0 - V_0}$, then the restriction of f, f_1 maps $(Y, Y_0 \cup \dot{V})$ into $(T - T_1, T_0 - T_1)$, and we claim that, regarded as a map into $(T, T_0), f_1$ is cobordant to f . Let F be the restriction of $f \cdot \pi_X$ and use the cobordism

$$(X \times [0, \frac{1}{2}] \cup Y \times [\frac{1}{2}, 1],$$

$$X \times 0 \cup X_0 \times [0, \frac{1}{2}] \cup V \times \frac{1}{2} \cup (Y_0 \cup \dot{V}) \times [\frac{1}{2}, 1] \cup Y \times 1,$$

$$X_0 \times [0, \frac{1}{2}] \cup V \times \frac{1}{2} \cup (Y_0 \cup \dot{V}) \times [\frac{1}{2}, 1]).$$

Note that $f_1(Y) \subset T - T_1$ and if $f(X) \subset T - U$ then this is a cobordism in $(T - U, T_0 - U)$.

This shows i_* is onto.

Now let $f: (X, X_0) \rightarrow (T - U, T_0 - U)$ such that $i_*[f] = 0$. By the above note we can assume that $f(X) \subset T - T_1$. Let $((W, W_0, W_1), F)$ be a cobordism of f with 0 in (T, T_0) , i.e. $W_0 = X \cup W_1$, $X \cap W_1 = X_0$, and $F: (W, W_1) \rightarrow (T, T_0)$ with $F|X = f$. Since $f(X) \subset T - T_1$, $F^{-1}(T_1) \cap X = \emptyset$; so we can first choose A and then a regular neighborhood pair (V, V_0) of A so that

$$F^{-1}(T_1) \subset V \subset F^{-1}(U_0) \cap (W - X).$$

Then the restriction of F to $(\overline{W - V}, \overline{W_0 - V_0} \cup \dot{V}, \overline{W_1 - V_0} \cup \dot{V})$ gives a cobordism of f to 0 in $(T - U, T_0 - U)$.

EXACTNESS OF THE DIAGRAM $\dots \Omega_n(T_0) \xrightarrow{i_*} \Omega_n(T) \xrightarrow{j_*} \Omega_n(T, T_0) \xrightarrow{\partial} \Omega_{n-1}(T_0) \xrightarrow{i_*} \Omega_{n-1}(T).$

Exactness at $\Omega_{n-1}(T_0)$: If $f: (X, X_0) \rightarrow (T, T_0)$, then $((X, X_0, \emptyset), f)$ is a cobordism of $f: X_0 \rightarrow T$ and 0. Hence $i_*\partial[f] = 0$.

If $i_*[f] = 0$ with $f: X_0 \rightarrow T_0$, then $((W, W_0, W_1), F)$ maps W_1 to \emptyset and so $W_1 = \emptyset$, i.e. $X_0 = W_0$ and $F: (W, X_0) \rightarrow (T, T_0)$ or $\partial[F] = [f]$.

Exactness at $\Omega_n(T, T_0)$: If $f: X \rightarrow T$, i.e. $f: (X, \emptyset) \rightarrow (T, \emptyset)$ then $\partial j_*[f] = 0$ by definition. If $f: (X, X_0) \rightarrow (T, T_0)$ with $\partial[f] = 0$ then there exists as in the previous argument, a pair $(Y, X_0) \in \mathfrak{F}^n$ with $F: Y \rightarrow T_0$ extending f . By pasting $X' = X \cup_{X_0} Y \in \mathfrak{F}^n$ and $f \cup F: X' \rightarrow T$ is cobordant to f in (T, T_0) . The cobordism is

$$(X \times [0, \frac{1}{2}] \cup X' \times [\frac{1}{2}, 1],$$

$$X \times 0 \cup X_0 \times [0, \frac{1}{2}] \cup Y \times \frac{1}{2} \cup X' \times 1, X_0 \times [0, \frac{1}{2}] \cup Y \times \frac{1}{2})$$

together with the map $f \cup F \cdot \pi_{X'}$.

Exactness at $\Omega_n(T)$: If $f: X \rightarrow T_0$ then $((X \times I, X \times \dot{I}, X \times 1), f \cdot \pi_X)$ is a cobordism of f to 0 in $\Omega_n(T, T_0)$, i.e. $j_* \cdot i_* = 0$. If $j_*[f] = 0$ where $f: X \rightarrow T$ then there exists $((W, W_0, W_1), F)$ with $X \subset W_0$ and $W_1 \cap X = \emptyset$, and F extends f mapping $(W, W_1) \rightarrow (T, T_0)$. Hence it can also be regarded as a cobord-

ism of $[F|W_1] + f$ with 0 in $\Omega_n(T)$. So $[f] = i_*[F|W_1]$.

REMARKS. 1. Compactness was assumed to prevent Ω_* from always being 0; it is not used in the proof, but for most examples if $f: (X, X_0) \rightarrow (T, T_0)$ then $((X \times [0, \infty), X \times 0 \cup X_0 \times [0, \infty), X_0 \times [0, \infty)), f \cdot \pi)$ would be a cobordism to 0 if noncompact pairs were allowed. Proper maps only extend the result.

2. If $\{\mathfrak{F}^n\}$ satisfies Axioms 1–4, then we can define $\mathfrak{F}_c^n = \{(X, X_0) \in \mathfrak{F}^n: X_0 \subset X \text{ is collared}\}$.

One can show that $\{\mathfrak{F}_c^n\}$ still satisfies Axioms 1–4.

This is clear for Axioms 1 and 2. For Axiom 3, we first show that for $x \in \dot{V}_0$, $\text{Lk}(x; (\dot{V}, \dot{V}_0)) \cong (c\text{Lk}(x; \dot{V}_0), \text{Lk}(x; \dot{V}_0))$, where (cX, X) for X a compact polyhedron, denotes the cone pair with base X and vertex c . By [A₁, p. 424, Corollary 10] it suffices to show that the suspension of the two sides are isomorphic. By Remark 2 after Axioms 1–4, (\dot{V}, \dot{V}_0) is bicollared in (X, X_0) . So, $\Sigma \text{Lk}(x; (\dot{V}, \dot{V}_0)) \cong \text{Lk}(x; (X, X_0))$. On the other hand, $\Sigma(c\text{Lk}(x; \dot{V}_0), \text{Lk}(x; \dot{V}_0)) \cong (c\Sigma \text{Lk}(x; \dot{V}_0), \Sigma \text{Lk}(x; \dot{V}_0)) \cong (c\text{Lk}(x; X_0), \text{Lk}(x; X_0))$. Now these pairs are isomorphic because $X_0 \subset X$ is collared. Thus, $\dot{V}_0 \subset \dot{V}$ is collared (cf. [C, p. 205, Theorem 5.3]). Now because $(\dot{V}, \dot{V}_0) \subset (V, V_0)$ is collared we have, for $x \in V_0$:

$$\text{Lk}(x; (V, V_0, \dot{V}, \dot{V}_0)) \cong (c_1 c \text{Lk}(x; \dot{V}_0), c_1 \text{Lk}(x; \dot{V}_0), c \text{Lk}(x; \dot{V}_0), \text{Lk}(x; \dot{V}_0)).$$

So $\text{Lk}(x; (V, \dot{V} \cup V_0)) \cong (c' \Sigma \text{Lk}(x; V_0), \Sigma \text{Lk}(x; V_0))$ because $[c_1, c] = c' * \{c_1, c\}$ for c' in the interior of $[c_1, c]$. Thus, $\dot{V} \cup V_0 \subset V$ is collared. Similarly, because $(\dot{V}, \dot{V}_0) \subset (Y, Y_0)$ is collared, $\text{Lk}(x; (Y, Y_0 \cup \dot{V})) \cong (c \Sigma \text{Lk}(x; \dot{V}_0), \Sigma \text{Lk}(x; \dot{V}_0))$ and so $Y_0 \cup \dot{V} \subset Y$ is collared.

For Axiom 4, regarding the embeddings as inclusions, we have, for $x \in Z_0$, that $\text{Lk}(x; (X_1, Z_0))$, $\text{Lk}(x; (Y_1, Z_0))$ and $\text{Lk}(x; (Z, Z_0))$ are cone pairs as above and we denote the cone vertices by c_1, c_2 and c_3 respectively. Then:

$$\begin{aligned} \text{Lk}(x; (X, X_0)) &\cong (c_4 \text{Lk}(x; X_0), \text{Lk}(x; X_0)) \\ &\cong (c_4 * \{c_1, c_3\}, \{c_1, c_3\}) * \text{Lk}(x; Z_0), \end{aligned}$$

$$\begin{aligned} \text{Lk}(x; (Y, Y_0)) &\cong (c_5 \text{Lk}(x; Y_0), \text{Lk}(x; Y_0)) \\ &\cong (c_5 * \{c_2, c_3\}, \{c_2, c_3\}) * \text{Lk}(x; Z_0). \end{aligned}$$

When we paste, the links are pasted along $\text{Lk}(x; Z) \cong c_3 * \text{Lk}(x; Z_0)$:

$$\begin{aligned} \text{Lk}(x; (X \cup_Z Y, X_1 \cup_{Z_0} Y_1)) \\ \cong (c_4 * \{c_1, c_3\} \cup c_5 * \{c_2, c_3\}, \{c_1, c_2\}) * \text{Lk}(x; Z_0). \end{aligned}$$

Since $(c_4 * \{c_1, c_3\} \cup c_5 * \{c_2, c_3\}, \{c_1, c_2\}) \cong (c_3 * \{c_1, c_2\}, \{c_1, c_2\})$, the link pair is a cone pair and so the result of the pasting is collared.

Furthermore, the bordism groups remain the same. Thus, if $f: (X, X_0) \rightarrow$

(T, T_0) with $(X, X_0) \in \mathfrak{F}^n$ then by Axioms 1 and 4, $(\bar{X}, \bar{X}_0) \in \mathfrak{F}^n$ where $\bar{X} = X \cup X_0 \times [0, 1]$ with $X_0 \subset X$ identified with $X_0 \times 0$ and $\bar{X}_0 = X_0 \times 1$. $\bar{X}_0 \subset \bar{X}$ is collared and if $r: (\bar{X}, \bar{X}_0) \rightarrow (X, X_0)$ is the retraction obtained by flattening the collar, then $F = f \cdot r \cdot \pi_{\bar{X}}$ together with

$$(\bar{X} \times [0, \frac{1}{2}] \cup X \times [\frac{1}{2}, 1], \bar{X} \times 0 \cup \bar{X}_0 \times [0, \frac{1}{2}] \cup X_0 \times I \times \frac{1}{2} \\ \cup X_0 \times [\frac{1}{2}, 1] \cup X \times 1, \bar{X}_0 \times [0, \frac{1}{2}] \cup X_0 \times I \times \frac{1}{2} \cup X_0 \times [\frac{1}{2}, 1])$$

is a cobordism between the maps $f: (X, X_0) \rightarrow (T, T_0)$ and $f \cdot r: (\bar{X}, \bar{X}_0) \rightarrow (T, T_0)$. Similarly, by attaching a collar to the cobordism, we can show that f cobordant to 0 with respect of $\{\mathfrak{F}^n\}$ implies $f \cdot r$ cobordant to 0 with respect to $\{\mathfrak{F}_c^n\}$.

So we can regard the condition " $X_0 \subset X$ is collared" as a convenient normalization condition which does not affect the bordism groups.

We say that a bordism sequence $\{\mathfrak{F}^n\}$ has collared boundaries if $(X, X_0) \in \mathfrak{F}^n$ implies that $X_0 \subset X$ is collared. We now characterize bordism sequences with collared boundaries in two important special cases.

DEFINITION 6. A locally finite complex K is called totally n -dimensional if it satisfies the equivalent conditions:

(a) Every simplex of K is a (not necessarily proper) face of some n -simplex of K .

(b) $K = \text{Closure}(I^n(|K|) - I^{n-1}(|K|))$.

Since the latter condition is p.l. invariant we can call a locally compact polyhedron X totally n -dimensional if the above holds for some (and hence for any) triangulation of X .

PROOF OF EQUIVALENCE. For any locally finite complex K with n -skeleton K^n , we have:

$$K^n - K^{n-1} \subset I^n(K) - I^{n-1}(K) \subset \text{Closure}(K^n - K^{n-1}).$$

The left inequality follows from the definition [A₁, pp. 420–421] of intrinsic dimension. For the right, if σ is a simplex of $I^n(K) - I^{n-1}(K)$, then since the latter is an open n -manifold, σ is the face of some n -simplex. It follows that the closures of $K^n - K^{n-1}$ and $I^n(K) - I^{n-1}(K)$ are the same.

REMARK. Let X be totally n -dimensional. If $X_0 \subset X$ is collared, then X_0 is totally $(n-1)$ -dimensional. If V is a subpolyhedron of X and is the closure of its interior, then V is totally n -dimensional. Putting these two results together, if $x \in X$, then $\text{Lk}(x; X)$ is totally $(n-1)$ -dimensional.

DEFINITION 7. Let $\{\mathfrak{F}^n\}$ be a bordism sequence with collared boundaries. Call $\{\mathfrak{F}^n\}$ ungraded if $\mathfrak{F}^n = \mathfrak{F}^m$ for all m and n . Call $\{\mathfrak{F}^n\}$ dimension graded if

$(X, X_0) \in \mathfrak{F}^n$ implies $X = \emptyset$ or X is totally n -dimensional.

REMARK. In either case, if we let $\mathfrak{F} = \bigcup_n \mathfrak{F}^n$ then \mathfrak{F} characterizes $\{\mathfrak{F}^n\}$, that is:

Ungraded case. $(X, X_0) \in \mathfrak{F}^n$ iff $(X, X_0) \in \mathfrak{F}$.

Dimension graded case. $(X, X_0) \in \mathfrak{F}^n$ iff $(X, X_0) \in \mathfrak{F}$ and $X = \emptyset$ or X is totally n -dimensional.

DEFINITION 8. A class \mathfrak{D} of compact polyhedra is called a class of singularities if it satisfies:

- (a) $\emptyset \in \mathfrak{D}$.
- (b) $X \in \mathfrak{D}$ and $X \cong Y$ implies $Y \in \mathfrak{D}$.
- (c) $X \in \mathfrak{D}$ iff $\Sigma X \in \mathfrak{D}$.

Since the suspensions of \emptyset are spheres, \mathfrak{D} includes all spheres.

For \mathfrak{D} a class of singularities define:

$$\mathfrak{F}_{\mathfrak{D}} = \{(X, X_0): (X, X_0) \text{ is a compact pair with } X_0 \subset X \text{ collared and } x \in X - X_0 \text{ implies } \text{Lk}(x; X) \in \mathfrak{D}\}.$$

$$\mathfrak{F}_{\mathfrak{D}}^n = \{(X, X_0) \in \mathfrak{F}_{\mathfrak{D}} \text{ with } (X, X_0) \text{ totally } n\text{-dimensional or } X = \emptyset\}.$$

PROPOSITION 8. Let \mathfrak{D} be a class of singularities, then $\mathfrak{F}_{\mathfrak{D}}$ is an ungraded bordism sequence and $\{\mathfrak{F}_{\mathfrak{D}}^n\}$ is a dimension graded bordism sequence.

PROOF. The preservation of collaring by the operations of the axioms is proved as in Remark 2 following Proposition 7. In the dimension graded case, the preservation of total n -dimensionality is easy (cf. the Remark following Definition 6). The links remain in \mathfrak{D} because if $X_0 \subset X$ is bicollared and $x \in X_0$, then $\text{Lk}(x; X) \cong \Sigma \text{Lk}(x; X_0)$ and so $\text{Lk}(x; X) \in \mathfrak{D}$ iff $\text{Lk}(x; X_0) \in \mathfrak{D}$. The details are left to the reader.

DEFINITION 9. Let $\{\mathfrak{F}^n\}$ be a bordism sequence and let $\mathfrak{F} = \bigcup_n \mathfrak{F}^n$. We define the singularity class $\mathfrak{D}_{\mathfrak{F}}$ by the following equivalent equations:

$$\begin{aligned} \mathfrak{D}_{\mathfrak{F}} &= \{\text{compact polyhedra } X: \Sigma X \in \mathfrak{F}\} \cup \{\emptyset\} \\ &= \{X: \exists Y \in \mathfrak{F}, y \in Y \text{ such that } \text{Lk}(y; Y) \cong X\} \cup \{\emptyset\} \\ &= \{\text{compact polyhedra } X: (cX, X) \in \mathfrak{F}\} \cup \{\emptyset\}. \end{aligned}$$

PROOF OF EQUIVALENCE. Since $\text{Lk}(v; \Sigma X) \cong X$ for v a suspension vertex, the first set is contained in the second. On the other hand, if $Y \in \mathfrak{F}$, then the regular neighborhood and boundary of y in Y is isomorphic to $(c\text{Lk}(y; Y), \text{Lk}(y; Y))$ and so the second set is contained in the third by Axioms 1 and 3. Finally, $\Sigma X \cong cX \cup c'X$, union along X , and so the third set is contained in the first, by Axiom 4.

REMARK. If $\mathfrak{F} \neq \{\emptyset\}$ then $\mathfrak{D}_{\mathfrak{F}}$ is a singularity class. For if $X \in \mathfrak{D}_{\mathfrak{F}}$ and $X \neq \emptyset$ then $\text{Lk}((y, \frac{1}{2}), Y \times I) \cong \Sigma \text{Lk}(y, Y)$ implies that $\Sigma X \in \mathfrak{D}_{\mathfrak{F}}$ by Axiom 2. If $\Sigma X \in \mathfrak{D}_{\mathfrak{F}}$, then $(c\Sigma X, \Sigma X) \in \mathfrak{F}$ and so $\Sigma X \in \mathfrak{F}$ by Axiom 2 and so $X \in \mathfrak{D}_{\mathfrak{F}}$. Finally, \mathfrak{F} contains some nonempty polyhedron and so $\mathfrak{D}_{\mathfrak{F}}$ contains some sphere. By suspending and desuspending it contains all spheres. Thus, $\mathfrak{D}_{\mathfrak{F}}$ satisfies (a), (b) and (c) above.

PROPOSITION 9. (a) Let $\{\mathfrak{F}^n\}$ be an ungraded bordism sequence and let $\mathfrak{D} = \mathfrak{D}_{\mathfrak{F}}$. Then $\mathfrak{F}_{\mathfrak{D}} = \mathfrak{F}$.

(b) Let $\{\mathfrak{F}^n\}$ be a dimension graded bordism sequence and let $\mathfrak{D} = \mathfrak{D}_{\mathfrak{F}}$. Then $\mathfrak{F}_{\mathfrak{D}}^n = \mathfrak{F}^n$ for all n .

PROOF. It suffices to show that $X \in \mathfrak{F}_{\mathfrak{D}}$ implies $X \in \mathfrak{F}$ in (a), and $X \in \mathfrak{F}_{\mathfrak{D}}^n$ implies $X \in \mathfrak{F}^n$ in (b). The result for pairs then follows because if $(X, X_0) \in \mathfrak{F}_{\mathfrak{D}}^n$ then the double consisting of two copies of X pasted along X_0 is in $\mathfrak{F}_{\mathfrak{D}}^n$ and hence is in \mathfrak{F}^n . But since $X_0 \subset X$ is collared, the pair (X, X_0) is isomorphic to the regular neighborhood and boundary of X in the double. So $(X, X_0) \in \mathfrak{F}^n$ by Axioms 1 and 3. The result for pairs is obtained similarly in case (a).

We prove the result in the dimension graded case. The proof is easier in the ungraded case as we can just forget about the grading, but otherwise it is the same.

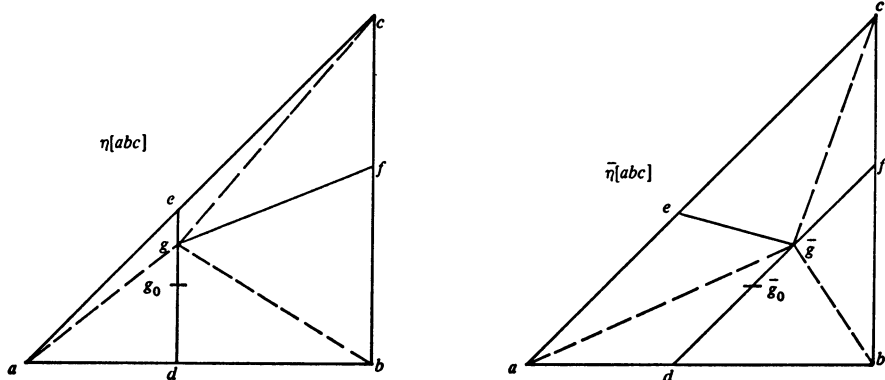
Let K be a finite complex with $|K| \in \mathfrak{F}_{\mathfrak{D}}^n$ and let $K_1 \subset K_2 \subset \dots \subset K_N = K$ be a sequence of subcomplexes so that K_1 is a vertex of K and $K_{i+1} = K_i \cup \sigma_{i+1}$. We show by induction that if V_i is a regular neighborhood of K_i in K , then $(V_i, \dot{V}_i) \in \mathfrak{F}^n$. For $i = N$ this says that $|K| \in \mathfrak{F}^n$. Note that by the remark after Definition 6, the V_i 's are totally n -dimensional. By uniqueness of regular neighborhoods and Axiom 1, either all of the regular neighborhoods of K_i lie in \mathfrak{F}^n or none of them do.

Initial step. For any point $x \in K$, if V is a regular neighborhood of x , then $(V, \dot{V}) \cong (c\text{Lk}(x; |K|), \text{Lk}(x; |K|))$. $\text{Lk}(x; |K|) \in \mathfrak{D}$ because $K \in \mathfrak{F}_{\mathfrak{D}}^n$ and so the pair is in \mathfrak{F}^n by Definition 9. This applies, in particular to the vertex in K_1 .

Inductive step. We will show that a regular neighborhood V_{i+1} of K_{i+1} can be obtained as a union of V_i and U_{i+1} , where V_i is a regular neighborhood of K_i , U_{i+1} is a regular neighborhood of the barycentre of σ_{i+1} , and $V_i \cap U_{i+1} = \dot{V}_i \cap \dot{U}_{i+1}$ is a regular neighborhood of something both in \dot{V}_i and \dot{U}_{i+1} . Then, V_i is in \mathfrak{F}^n by induction hypothesis, U_{i+1} is in \mathfrak{F}^n by the initial step and so $(V_{i+1}, \dot{V}_{i+1}) \in \mathfrak{F}^n$ by Axioms 3 and 4.

To prove the regular neighborhood result we have to do a little work with Cohen's parametrization lemma [C, p. 194, Lemma 2.14]. Let K' be the barycentric subdivision of K and let $p: K' \rightarrow [abc]$ be a simplicial map to a two simplex with p mapping K'_i to a , the barycentre of σ_{i+1} to b and the remaining

vertices of K' to c . We use two derived subdivisions of $[abc]$ so that $\pi: \eta[abc] \rightarrow \eta[ab]$ and $\bar{\pi}: \bar{\eta}[abc] \rightarrow \eta[ab]$ are simplicial, where π and $\bar{\pi}$ are the simplicial retractions of $[abc]$ onto $[ab]$ with $\pi(c) = b$ and $\bar{\pi}(c) = a$. We obtain corresponding derived subdivisions $\eta K'$ and $\bar{\eta} K'$ so that $p: \eta K' \rightarrow \eta[abc]$ and $p: \bar{\eta} K' \rightarrow \bar{\eta}[abc]$ are still simplicial. See the figure below:



Applying the parametrization lemma to the diagram for $\eta[abc]$, we get:

$$(p^{-1}(adge), p^{-1}(dge)) = (V_i, \dot{V}_i), \quad (p^{-1}(bdgf), p^{-1}(dgf)) = (U_{i+1}, \dot{U}_{i+1}),$$

$$p^{-1}(dg) = V_i \cap U_{i+1} = \dot{V}_i \cap \dot{U}_{i+1}.$$

Furthermore, $p^{-1}(dg_0)$ is a regular neighborhood of $p^{-1}(d)$ in V_i with boundary $p^{-1}(g_0)$. If we choose g_0 as the vertex for $[abc]$ instead of g and correspondingly change the choices of vertices for those simplices of K' which map onto $[abc]$, then we obtain different derived subdivisions $\eta_0 K$ and $\eta_0 [abc]$. Of course, $\eta_0 K'$ is simplicially isomorphic to $\eta K'$ and because e, g, g_0, d are colinear, this isomorphism takes $(p^{-1}(dg_0e), p^{-1}(dg_0), p^{-1}(g_0))$ onto $(p^{-1}(dge), p^{-1}(dg), p^{-1}(g))$ and is the identity on $p^{-1}(d)$. Hence, $p^{-1}(dg)$ is a regular neighborhood with boundary $p^{-1}(g)$ of $p^{-1}(d)$ in $p^{-1}(dge)$, i.e. $V_i \cap U_{i+1}$ is a regular neighborhood in \dot{V}_i . Next, use the simplicial isomorphism of $\eta K'$ with $\bar{\eta} K'$ and a similar sliding argument on the colinear points d, \bar{g}_0, \bar{g}, f to get that $p^{-1}(dg)$ is a regular neighborhood of $p^{-1}(d)$ in $p^{-1}(dgf)$, i.e. $V_i \cap U_{i+1}$ is a regular neighborhood of $p^{-1}(d)$ in \dot{U}_{i+1} .

REMARKS. 1. This result is really a negative one. It shows that all of the examples are of a particular type. The culprit is the pasting axiom. To get more general bordism theories, one must introduce additional structure, e.g. orientation, on the pairs and only allowing pastings which preserve the additional structure, e.g. opposite orientations on Z . In these cases, the structures induced on $X \times 0$ and $X \times 1$ from $X \times I$ are often different so that one does not obtain a Z_2 theory any more.

2. If $X \in \mathfrak{D}$ implies X is totally n -dimensional for $n = \dim X$, then the

ungraded bordism group $\Omega(T, T_0)$, associated to the ungraded bordism family \mathfrak{F}_g is just the direct sum $\sum_n \Omega_n(T, T_0)$ where $\Omega_n(T, T_0)$ are the bordism group associated to the dimension graded family $\{\mathfrak{F}_g^n\}$. This is left to the reader with the hint that if X is a polyhedron such that the $\text{Lk}(x; X)$ is totally n_x -dimensional for each $x \in X$ (with n_x varying with x), then each component of X is totally n -dimensional (with n varying with the choice of component). Hence, any element of \mathfrak{F}_g is a disjoint union of elements of $\bigcup_n \mathfrak{F}_g^n$.

Note that if we begin with a dimension graded family $\{\mathfrak{F}^n\}$ then the corresponding family \mathfrak{D}_g satisfies the above condition.

In general, however, if \mathfrak{D} does not satisfy this condition then the map $\sum_n \Omega_n \rightarrow \Omega$ induced by the inclusion of $\bigcup_n \mathfrak{F}_g^n$ in \mathfrak{F}_g , need not be injective. For example, look at the Euler space bordism groups of the next section.

Euler space cobordism. We will now turn to an example of a Z_2 bordism theory which comes from Euler spaces.

Let $\mathfrak{F}^E = \{(X, X_0) \text{ is a compact Euler pair}\}$. If we let $\mathfrak{F}^n = \mathfrak{F}^E$ for all $n \in Z$ we obtain a sequence satisfying Axioms 1, 2, 3, 4 and hence an ungraded bordism theory Ω^E with exact triangle for topological pair (T, T_0) :

$$\begin{array}{ccc} \Omega^E(T_0) & \xrightarrow{i_*} & \Omega^E(T) \\ & \searrow \partial & \swarrow j_* \\ & \Omega^E(T, T_0) & \end{array}$$

PROPOSITION 10. *There is a natural homomorphism*

$$S: \Omega^E(T, T_0) \rightarrow H_*(T, T_0) \quad (\text{total } Z_2 \text{ homology})$$

defined by $S[f] = \sum_i f_ w_i(X, X_0)$ where $f: (X, X_0) \rightarrow (T, T_0)$, i.e. S maps $[f]$ to the image under f_* of the total Stiefel-Whitney homology class.*

PROOF. S is well defined by Proposition 5. It is clearly a natural homomorphism commuting with boundaries.

PROPOSITION 11. (a) X and $Y \in \mathfrak{F}$ are cobordant if and only if $\chi(X) \equiv \chi(Y)$.

(b) For compact p.l. pairs (T, T_0) , S is an isomorphism.

PROOF. (1) $S: \Omega^E(p) \rightarrow H_*(p) = Z_2$ (p a point) is onto: Let $f: X \rightarrow p$ be the identity map on p . $w_0(p) \neq 0$.

(2) If $\chi(X) \equiv \chi(Y)$ then the pair $(cX \cup_c cY, X \cup Y)$ is a cobordism between X and Y , since $\chi(X \cup Y) \equiv 0$.

(3) $S: \Omega^E(p) \rightarrow H_*(p) = Z_2$ is iso: By (1) it is onto and by (2) there is at most one nonzero class in $\Omega^E(p)$.

(4) If there existed X cobordant to Y with $\chi(X) \neq \chi(Y)$ then, by (2), $\Omega^E p = 0$ which is impossible by (3). This proves (a).

(5) $S: \Omega^E(E^n, S^{n-1}) \rightarrow H_*(E^n, S^{n-1})$ is iso by induction on n using naturality of S with respect to the Mayer-Vietoris boundary, cf. Proposition 4.

(6) $S: \Omega^E(K) \rightarrow H_*(K)$ is iso for any finite complex K by induction on the number of cells. $K = K_0 \cup \sigma$ with $\sigma \cap K_0 = \partial\sigma$. Then by inductive hypothesis S is iso on K_0 and by excision and (5) it is iso on (K, K_0) and so S is iso on K by the five lemma.

(7) $S: \Omega^E(K, K_0) \rightarrow H_*(K, K_0)$ is iso for any finite complex pair (K, K_0) by the five lemma and (6) for K and K_0 separately.

COROLLARY 1. *If (X, X_0) is a compact Euler pair, then $\chi(X_0) \equiv 0$.*

Another corollary is the converse of Proposition 5.

COROLLARY 2. *If $f: (X, X_0) \rightarrow (T, T_0)$ and $g: (Y, Y_0) \rightarrow (T, T_0)$ are maps of compact Euler pairs to compact p.l. pairs, then if $f_* w_i(X, X_0) = g_* w_i(Y, Y_0)$ for all i , then f and g are cobordant.*

George Cooke showed me how to contrast this with manifold cobordism. f and g are cobordant iff the generalized Stiefel-Whitney numbers are equal (see [C-F; p. 47, Theorem 17.2]), i.e. iff

$$\langle f_* z \cup w^{i_1} \cup \dots \cup w^{i_k}, \mu_X \rangle = \langle g_* z \cup w^{i_1} \cup \dots \cup w^{i_k}, \mu_Y \rangle$$

where μ_X and μ_Y are the fundamental homology classes, \cup is cup product and z is an arbitrary cohomology class of (T, T_0) of codimension $i_1 + \dots + i_k$. The w^i 's are the usual Stiefel-Whitney cohomology classes, which by Cheeger, Halperin, Toledo etc. are dual to the w_i 's. Since \cap = intersection is dual to cup product, we have

$$\begin{aligned} \langle f_* z \cup w^{i_1} \cup \dots \cup w^{i_k}, \mu_X \rangle &= \langle f_* z, w_{i_1} \cap \dots \cap w_{i_k} \rangle \\ &= \langle z, f_*(w_{i_1} \cap \dots \cap w_{i_k}) \rangle. \end{aligned}$$

Thus, f and g are Euler space cobordant iff $f_* w_i = g_* w_i$ for all i . They are manifold cobordant iff $f_*(w_{i_1} \cap \dots \cap w_{i_k}) = g_*(w_{i_1} \cap \dots \cap w_{i_k})$. In the first case the Stiefel-Whitney homology classes are related and in the second the generated intersection algebra classes are all related.

Finally, using the fact that ordinary cobordism maps onto homology by evaluation on the fundamental class, we can find a cobordism between any Euler space and a disjoint union of manifolds.

PROPOSITION 12. (a) *If $z \in H_n(T)$ then there exists X a disjoint union of compact manifolds of dimension $\leq n$ and a map $f: X \rightarrow T$ such that $z = S[f]$.*

(b) If X is a compact Euler space then there exists Y a disjoint union of manifolds of dimension $\leq \dim X$ and a map $r: Y \rightarrow X$ such that r is cobordant to 1_X .

PROOF. (a) By induction on n . For $n = 0$ it is obvious. There exists a manifold X_0 of dimension n and a map $f_0: X_0 \rightarrow T$ such that $f_0 \cdot \mu_{X_0} = f_0 \cdot w_n(X_0) = z$. So $z + S[f_0] = \sum_{i < n} f_0 \cdot w_i(X_0)$. By induction, there exists X_1 a disjoint union of manifolds and $f_1: X_1 \rightarrow T$ such that $S[f_1] = z + S[f_0]$. Let $X = X_0 \cup X_1$ and $f = f_0 \cup f_1$.

(b) Apply (a) to the classes of $S[1_X]$, for the existence of $r: Y \rightarrow X$. The cobordism follows from Corollary 2 of Proposition 9. Note that $\dim Y \leq \max\{i: w_i(X) \neq 0\}$.

Finally, we contrast the ungraded theory we have been looking at with the dimension graded bordism theory obtained from the class \mathfrak{D} of Euler spaces with even Euler characteristic via Proposition 8. Note that if X_1 and X_2 are totally n -dimensional Euler spaces with the same Euler characteristic mod 2, then the cobordism $cX_1 \cup cX_2$ is totally $(n+1)$ -dimensional. Hence, by Corollary 1, we can compute the coefficient groups: $\Omega_n(\text{point}) = \mathbb{Z}_2$ for all $n \geq 0$. This is suggestive of K -theory. At any rate notice that $\Omega \neq \sum_n \Omega_n$.

NOTES ADDED IN PROOF. 1. 12(b) extends to pairs (X, X_0) by removing a neighborhood of $r^{-1}(c)$ from $r: Y \rightarrow X \cup cX_0$.

2. Sullivan and the referee showed me that 12(b) and the product formula for S-W classes on manifolds yield the corresponding result for Euler spaces.

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