TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 205, 1975

LOCATION OF THE ZEROS OF A POLYNOMIAL RELATIVE TO CERTAIN DISKS

BY

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ABSTRACT. The zeros of the complex polynomial $P(z) = z^n + \Sigma\alpha_i z^{n-i}$ are studied under the assumption that some $|\alpha_k|$ is large in comparison with the other $|\alpha_i|$. It is shown under certain conditions that $P(z)$ has $n-k$ zeros in $|z| \leq m_-$ and k zeros in $|z| \geq m_+$, where $m_- < m_+ \leq |\alpha_k|^{1/k}$; and under suitably strengthened conditions, one of the k zeros of larger modulus is shown to lie in each of the k disks $|z - (-\alpha_k)^{1/k}| \leq R$, where $m_- + R < |\alpha_k|^{1/k}$.

1. **Notation and basic estimate.** If the complex polynomial

$$P(z) = z^n + \alpha_1 z^{n-1} + \cdots + \alpha_{n-1} z + \alpha_n$$

has a dominant coefficient, in the sense that some $|\alpha_k|$ is large in comparison with the other $|\alpha_i|$, then $P(z)$ has $n-k$ zeros near 0 and one zero near each of the k values of $(-\alpha_k)^{1/k}$. We shall establish some conditions under which precise estimates can be given. The first type (Estimate A in §2 below), which goes back in principle to a theorem of Pellet [3, p. 393], [1, p. 10], asserts the existence of a zero-free annulus $m_- < |z| < m_+$. The second type (Estimate B in §3) further asserts, under stronger conditions, the existence of k disks $|z - (-\alpha_k)^{1/k}| \leq R$, each one of which isolates a single zero of $P(z)$.

These results rest on the following simple observation. Let k denote an integer in the range $1 \leq k \leq n$, chosen now and fixed in the sequel. With $P(z)$ as above, suppose that $P(z) = 0$ and $z \neq 0$. Transposing all terms other than $z^n + \alpha_k z^{n-k}$, dividing by z^{n-k}, and applying the triangle inequality, we obtain

$$|z^k + \alpha_k| \leq \sum_{i \neq k} |\alpha_i|\, |z|^{k-i}.$$

For convenience in estimating the right side, we define

$$a = |\alpha_k|, \quad b = |\alpha_1| + \cdots + |\alpha_{k-1}|, \quad c = |\alpha_{k+1}| + \cdots + |\alpha_n|,$$

$$g(r) = \begin{cases} br + cr^{k-n} & (0 < r \leq 1), \\ br^{k-1} + cr^{-1} & (r \geq 1), \end{cases}$$

Received by the editors June 21, 1972.

AMS (MOS) subject classifications (1970). Primary 30A08, 26A78.

Key words and phrases. Dominant coefficient, zero-free annulus, disk which isolates a single zero.

where it is understood that $b = 0$ in case $k = 1$, and $c = 0$ in case $k = n$. Then as an immediate consequence of the above inequality, we obtain our basic estimate:

LEMMA 1. *If* $P(z) = 0$ *and* $|z| > 0$, *then*

$$|z^k + \alpha_k| \leqslant g(|z|). \tag{1}$$

In what follows, the quantities $P(z)$, k, a, b, c, and $g(r)$ will continue to have the meanings given above.

2. **Annuli which contain no zero.** The conclusions established in this section all have the same form, which we state in advance for ease of reference.

ESTIMATE A. $P(z)$ has $n - k$ zeros in the disk $|z| \leqslant m_-$ and k zeros in the region $|z| \geqslant m_+$, where $m_- < m_+ \leqslant a^{1/k}$.

Here of course zeros are being counted with their multiplicities. Theorems asserting the validity of Estimate A are deduced below from the following consequence of Lemma 1.

LEMMA 2. *Estimate* A *holds if* $r = m_-$ *and* $r = m_+$ *are two solutions of a relation of the form*

$$a = h(r), \quad \text{where } h(r) \geqslant r^k + g(r). \tag{2}$$

PROOF. If $P(z) = 0$ and $r = |z| > 0$, then by applying (1) to the inequality $|\alpha_k| \leqslant |-z^k| + |z^k + \alpha_k|$ we see that $a \leqslant r^k + g(r)$. By the convexity of $r^k + g(r)$, the last inequality fails to hold for r in any interval of the form $m_- < r < m_+$ with $m_\pm$ solutions of relation (2). Thus if such $m_\pm$ exist, the zeros of $P(z)$ are confined to the union of the two separate domains $|z| \leqslant m_-$ and $|z| \geqslant m_+$, which we may label I and E respectively.

If $P_t(z)$ is the polynomial obtained by multiplying each coefficient α_i of $P(z)$ other than α_k by a parameter t, $0 \leqslant t \leqslant 1$, then the corresponding $g_t(r)$ is dominated by $g(r)$ for every $r > 0$. Hence the zeros $z_i(t)$ of $P_t(z)$ also belong to $I \cup E$. But the $z_i(t)$ are continuous in t, and since precisely $n - k$ of them are in the connected component I when $t = 0$, the same must be true when $t = 1$. Q.E.D.

Lemma 2 is essentially the above-mentioned result of Pellet, adapted to the present context. Our first application of it gives conditions for the existence of $m_\pm$ to the right of $r = 1$.

THEOREM 2.1. *Assume* $1 < a \leqslant 1 + b + c$, *and* $D > 0$, *where*

$$D = \frac{1}{4}(a^{1/k} + b)^2 - \frac{a^{1/k} - 1}{a - 1}(ab + c).$$

Then Estimate A *holds with*

$$m_{\pm} = \tfrac{1}{2}(a^{1/k} - b) \pm D^{1/2}. \tag{3}$$

PROOF. Multiplying the equation $a = r^k + g(r)$ for $r \geqslant 1$ by r, we obtain an equation equivalent to the pair

$$(s - a)(r + b) = -(ab + c), \tag{4}$$

$$s = r^k. \tag{5}$$

Over the interval $1 \leqslant r \leqslant a^{1/k}$, the convex curve (5) lies on or below the chord C joining the points (1, 1) and $(a^{1/k}, a)$. Substitution of the equation of C into (4) leads to a quadratic in r, whose solutions are the $m_{\pm}$ of (3).

It is clear from (3) that $m_- < m_+ < a^{1/k}$, and one finds that $m_- \geqslant 1$ is equivalent to

$$\frac{a^{1/k} - 1}{a - 1}(1 + b + c - a) \geqslant 0,$$

which holds under the present assumptions. Thus $m_{\pm}$ belong to the interval $1 \leqslant r \leqslant a^{1/k}$ where the chord C lies on or above (5), so that the quadratic which they satisfy is an equation of the type (2), and Lemma 2 applies. Q.E.D.

Next we deal with a case in which m_- and m_+ lie on opposite sides of $r = 1$.

THEOREM 2.2. *Assume* $1 + b + c < a$. *Then Estimate* A *holds with*

$$m_- = (c/(a - b - 1))^{1/(n-k)}, \tag{6}$$

$$m_+ = ((a - c)/(b + 1))^{1/k}, \tag{7}$$

and also with m_- *given by* (6) *and* m_+ *by* (3).

PROOF. The function $r^k + g(r)$ is everywhere dominated by

$$h(r) = \begin{cases} 1 + b + cr^{k-n} & (r \leqslant 1), \\ (1 + b)r^k + c & (r \geqslant 1). \end{cases}$$

Thus the equation $a = h(r)$, which has solutions $m_{\pm}$ as in (6) and (7) under the present assumptions, is of the form (2), and Lemma 2 applies.

Over the interval $1 \leqslant r \leqslant a^{1/k}$, we can also dominate $s = r^k$ by the chord used in the previous proof, and so the present theorem remains true with m_+ given by (3) instead of (7). Q.E.D.

The case $b = 0$ of Theorem 2.2 strengthens a result of Parodi [2, pp. 139–140].

It remains to consider the possibility that $m_\pm$ are both to the left of $r = 1$:

THEOREM 2.3. *Assume* $c < a \leqslant 1 + b + c$, *and* $b + 2d^{1/2} < a$, *where*

$$d \geqslant \begin{cases} a(c/a)^{k/(n-k)} & \text{if } k < \tfrac{1}{2}n, \\ \min\{1, a^{2-n/k}\}c & \text{if } k \geqslant \tfrac{1}{2}n. \end{cases}$$

Then Estimate A *holds with*

$$m_\pm = \left\{\frac{1}{2}(a-b) \pm \left[\frac{1}{4}(a-b)^2 - d\right]^{1/2}\right\}^{1/k}. \tag{8}$$

PROOF. Let r_* be the solution of $cr_*^{k-n} = a$, or $r_* = 0$ in case $c = 0$, and consider the interval

$$r_* \leqslant r \leqslant \min\{1, a^{1/k}\}. \tag{9}$$

In this interval we have $br \leqslant b$, and $cr^{k-n} \leqslant dr^{-k}$ by the definition of d in the statement of the theorem. Hence $b + dr^{-k} \geqslant g(r)$ over the interval (9), and the equation

$$a = r^k + b + dr^{-k} \tag{10}$$

is of the type (2) there.

The solutions of (10) are just the $m_\pm$ of (8), and it is clear from this formula that $m_-^k < m_+^k \leqslant a$. The condition $m_+^k \leqslant 1$ is readily seen to be equivalent to $a \leqslant 1 + b + d$, which obviously holds if $a \leqslant 1$ and follows from $a \leqslant 1 + b + c$, $c \leqslant d$, in case $a > 1$. On the other hand, $m_\pm$ as solutions of (10) must exceed the solution r of $a = dr^{-k}$, which in turn is at least r_* by the definition of d. Thus $m_\pm$ belong to the interval (9), where their equation (10) is of the form (2), and so Lemma 2 applies. Q.E.D.

3. **Disks which contain a single zero.** The conclusions established in this section are all of the following form:

ESTIMATE B. $P(z)$ has $n - k$ zeros in the disk $|z| \leqslant m_-$ and one zero in each of the k disjoint disks $|z - (-\alpha_k)^{1/k}| \leqslant R$, where $m_- + R < a^{1/k}$ $(a = |\alpha_k|)$.

Again we begin by establishing a preliminary consequence of Lemma 1.

LEMMA 3. *Suppose that Estimate* A *holds with* $m_\pm$ *given as in Lemma* 2. *Suppose that, for some upper bound* M *on the moduli of the zeros of* $P(z)$,

$$g(M) \leqslant a - m_+^k. \tag{11}$$

Then Estimate B *holds with the given* m_- *and with*

(12) $$R = a^{1/k} - m_+,$$

provided also that, in case $k \geqslant 3$,

(13) $$R < a^{1/k}\sin(\pi/k).$$

PROOF. Under the present conditions, the k zeros of $P(z)$ of largest modulus satisfy $m_+ \leqslant |z| \leqslant M$, and so by inequality (1) of Lemma 1 they must lie in the region

$$|z^k + \alpha_k| \leqslant \max\{g(r): m_+ \leqslant r \leqslant M\}.$$

Since g is a convex function, this maximum is in fact $\max\{g(m_+), g(M)\}$. But the definition of m_+ by means of inequality (2) implies that $g(m_+) \leqslant a - m_+^k$, and $g(M)$ admits the same bound by assumption (11). Hence the k zeros of $P(z)$ of largest modulus lie in $|z^k + \alpha_k| \leqslant S = a - m_+^k$.

We claim that this region is covered by the union of k closed disks centered at the k values of $(-\alpha_k)^{1/k}$, with common radius $R = a^{1/k} - (a - S)^{1/k} = a^{1/k} - m_+$, i.e., the R of (12). To see this, it is enough to consider the case $\alpha_k = -1$, for the general case follows from it by a dilation and a rotation. Any z satisfying $|z^k - 1| \leqslant S$ can be written

$$z = w(1 + te^{iu})^{1/k} \qquad (0 \leqslant t \leqslant S,\ 0 \leqslant u \leqslant 2\pi),$$

where $w^k = 1$ and the kth root is taken with its principal value. Following a suggestion by D. Boyd, we expand with the binomial series, estimate by the triangle inequality, and observe that the binomial coefficients alternate in sign, to obtain

$$|z - w| \leqslant \sum_{m \geqslant 1} (-1)^{m+1} \binom{1/k}{m} t^m$$

$$= 1 - (1 - t)^{1/k} \leqslant 1 - (1 - S)^{1/k},$$

and the claim is verified.

The covering disks obviously do not meet if $k = 2$, while in case $k \geqslant 3$ their pairwise disjointness is ensured by assumption (13). Since $m_- + R < m_+ + R = a^{1/k}$, each of these disks is disjoint from $|z| \leqslant m_-$; and a continuity argument similar to the one in the proof of Lemma 2 shows that each one of them contains precisely one zero of $P(z)$. Q.E.D.

Our first result derived from this lemma applies, in case $k \geqslant 2$, only to lacunary polynomials $P(z)$:

THEOREM 3.1. *Assume* $a > 1$, $b = 0$, *and* $D > 0$, *where*

$$D = \frac{1}{4} a^{2/k} - \frac{a^{1/k} - 1}{a - 1} c.$$

Then Estimate B *holds, with*

$$m_- = \begin{cases} \tfrac{1}{2} a^{1/k} - D^{1/2} & \text{if } a \leqslant 1 + c, \\ (c/(a-1))^{1/(n-k)} & \text{if } a > 1 + c, \end{cases}$$

$$R = \tfrac{1}{2} a^{1/k} - D^{1/2},$$

provided also that, in case $k \geqslant 3$,

$$R < a^{1/k} \sin(\pi/k).$$

PROOF. With the present assumptions we can apply Theorem 2.1 if $a \leqslant 1 + c$ and Theorem 2.2 if $a > 1 + c$, to establish Estimate A with m_- given by (3) or (6) and m_+ given by (3). The theorem then follows from Lemma 3 as soon as assumption (11) is established. Let M be any upper bound on the moduli of the zeros of $P(z)$. Then $M \geqslant m_+$, and since the assumption that $b = 0$ makes g a decreasing function, we have $g(M) \leqslant g(m_+)$, from which (11) follows as before. Q.E.D.

The case $k = 1$ of Theorem 3.1, in which the assumption $b = 0$ is redundant, refines an estimate due to Parodi [2, pp. 76, 77].

With only a slight loss in precision for small values of k, the next result avoids the restriction to lacunary polynomials.

THEOREM 3.2. *Assume* $1 + 2b < \min\{a^{1/k}, a + b - c\}$. *Then Estimate* B *holds, with*

$$m_- = \left(\frac{c}{a - b - 1}\right)^{1/(n-k)}, \quad R = a^{1/k} - \left(a - \frac{ab + c}{1 + b}\right)^{1/k},$$

provided also that, in case $k \geqslant 3$, $R < a^{1/k} \sin(\pi/k)$.

PROOF. Since the assumption implies $1 + b + c < a$, Theorem 2.2 establishes Estimate A with m_- given by (6) and m_+ by (7), i.e. by

$$m_+^k = (a - c)/(1 + b) = a - (ab + c)/(1 + b). \tag{14}$$

Thus the theorem will follow from Lemma 3 as soon as assumption (11) is verified.

To that end, we observe that if $P(z) = 0$ and $r = |z| > 0$, then (1) and the general inequality $|z^k| \leq |-\alpha_k| + |z^k + \alpha_k|$ together imply that $r^k \leq a + g(r)$. But the ratio of $a + g(r)$ to r^k is a combination of negative powers of r, and hence decreases steadily from $+\infty$ to 0 as r increases from 0. Thus there is precisely one value $r = r_0 > 0$ for which this ratio is unity, i.e. for which

$$r^k = a + g(r), \tag{15}$$

and according to the observation above, $|z| \leq r_0$ if $P(z) = 0$. To summarize, we may say that *M is an upper bound on the moduli of the zeros of $P(z)$ if M exceeds the solution r_0 of* (15), *and that this condition $M \geq r_0$ is equivalent to*

$$g(M) \leq M^k - a. \tag{16}$$

In the present situation, with $a + b + c > 1$, the solution r_0 of (15) exceeds 1, and in this region (15) is equivalent to the pair of equations

$$(s - a)(r - b) = ab + c, \tag{17}$$

$$s = r^k. \tag{18}$$

Since $b < a^{1/k}$ by assumption, the point

$$(r, s) = (a^{1/k}, a + (ab + c)/(a^{1/k} - b)) \tag{19}$$

on the curve (17) lies above the curve (18). But (17) decreases and (18) increases with increasing r, and so the point (19) lies above the unique point (r_0, r_0^k) with $r_0 > 1$ where (17) and (18) intersect. Thus $s > r_0^k$ and we may take $M = s^{1/k}$ with the s of (19). By (16), and in view of the assumption that $1 + b < a^{1/k} - b$, this M satisfies

$$g(M) \leq \frac{ab + c}{a^{1/k} - b} \leq \frac{ab + c}{1 + b}, \tag{20}$$

which by (14) is the desired inequality (11). Q.E.D.

The two preceding estimates apply when $a > 1$. Our final result applies to the complementary case.

THEOREM 3.3. *Assume $c < a^{n/k} \leq 1$ and $b + 2d^{1/2} < a$, and assume $b^2 \leq d$ if $a + b + c > 1$, where*

$$d = \begin{cases} a(c/a)^{k/(n-k)} & \text{if } k < \tfrac{1}{2}n, \\ c & \text{if } k \geq \tfrac{1}{2}n. \end{cases}$$

Then Estimate B *holds, with*

$$m_- = \{\tfrac{1}{2}(a-b) - [\tfrac{1}{4}(a-b)^2 - d]^{1/2}\}^{1/k},$$
$$R = a^{1/k} - (a - b - 2d/(a-b))^{1/k},$$

provided also that, in case $k \geq 3$, $R < a^{1/k}\sin(\pi/k)$.

PROOF. The assumptions imply those of Theorem 2.3, and so Estimate A holds with

$$m_\pm = \{\tfrac{1}{2}(a-b) \pm [\tfrac{1}{4}(a-b)^2 - d]^{1/2}\}^{1/k}.$$

Since the square root can be estimated as

$$[\tfrac{1}{4}(a-b)^2 - d]^{1/2} \geq \tfrac{1}{2}(a-b) - 2d/(a-b) > 0,$$

Estimate A holds a fortiori with the given m_- and with $m_+ > m_-$ given by

$$m_+^k = a - b - 2d/(a-b). \tag{21}$$

Thus the theorem will follow from Lemma 3 as soon as assumption (11) can be verified.

The present assumptions imply that $b < a^{1/k}$, so that in case $a + b + c > 1$ we can use the upper bound M of the previous proof. According to (20), this M satisfies $g(M) \leq (ab + c)/(a^{1/k} - b)$. But $a \leq 1$ makes $a \leq a^{1/k}$, $c < a$ makes $c \leq d$, and $b^2 \leq d$ holds in the present case by assumption. Hence

$$g(M) \leq \frac{ab + c}{a - b} = b + \frac{c + b^2}{a - b} \leq b + \frac{2d}{a - b},$$

and in view of (21), this establishes (11) in case $a + b + c > 1$.

If on the other hand $a + b + c \leq 1$, then the solution r_0 of equation (15) in the previous proof lies to the left of 1; and we claim in this case that an upper bound $M \geq r_0$ is defined by

$$M^k = a + b + d'/(a+b), \tag{22}$$

where

$$d' = \begin{cases} a^{2-n/k}c & \text{if } k < \tfrac{1}{2}n, \\ c & \text{if } k \geq \tfrac{1}{2}n. \end{cases} \tag{23}$$

Indeed, if $M^k \geq 1$ there is nothing to prove. Otherwise M lies in the interval $(a+b)^{1/k} \leq r \leq 1$, where $bM \leq b$ and $cM^{k-n} \leq d'M^{-k} \leq d'/(a+b)$. Then we obtain

$$g(M) = bM + cM^{k-n} \leqslant b + d'/(a+b) = M^k - a \tag{24}$$

by (22), and according to (16) and the remark above it, our claim is proved. Moreover, by (23), $d' = d$ if $k \geqslant \frac{1}{2}n$, while if $k < \frac{1}{2}n$ one checks that $d' < d$ is equivalent to the inequality $c \leqslant a^{n/k}$, which holds by assumption. Thus $d' \leqslant d$ for any choice of k, and comparing (24) with (21), we conclude that inequality (11) holds as well in case $a + b + c \leqslant 1$. Q.E.D.

If $k = n$ and $a + b > 1$, Theorem 3.3 makes no assertion, since here $d = c = 0$ and the extra hypothesis $b^2 \leqslant d$ cannot be met. However, if we assume $b < \frac{1}{2}a$, then Estimate A holds with

$$m_+ = a - ab/(a - b) > 0$$

instead of (21), and inequality (11) can be verified just as in the proof above. Thus in the case $k = n$, $a + b > 1$, we may replace the assumptions of Theorem 3.3 by $2b < a \leqslant 1$ and retain the conclusion with

$$R = a^{1/n} - (a - ab/(a - b))^{1/n}.$$

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