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THE GEOMETRIC DIMENSION OF SOME VECTOR BUNDLES OVER PROJECTIVE SPACES(1)

BY

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ABSTRACT. We prove that in many cases the geometric dimension of the p-fold Whitney sum pH_k of the Hopf bundle H_k over quaternionic projective space QP^k is the smallest n such that for all $i \leqslant k$ the reduction of the ith symplectic Pontryagin class of pH_k to coefficients $\pi_{4i-1}((RP^\infty/RP^{n-1}) \wedge bo)$ is zero, where bo is the spectrum for connective KO-theory localized at 2. We immediately obtain new immersions of real projective space RP^{4k+3} in Euclidean space if the number of 1's in the binary expansion of k is between 5 and 8.

1. **Introduction.** In this paper we determine the geometric dimension (gd) of the stable class of many of the multiples pH_k of the Hopf bundle H_k over quaternionic projective space QP^k. This enables us immediately to obtain some new immersions of real projective spaces RP^{4k+3} in Euclidean space.

Let $\nu(2^a(2b + 1)) = a$, $\nu(p, i) = \nu\binom{p}{i}$, and let $\alpha(n)$ equal the number of 1's in the binary expansion of n. Let $P_n^m = RP^m/RP^{n-1}$ and $P_n = P_n^\infty$. bo denotes the spectrum for real connective K-theory localized at 2 [3]. $\pi_{4i-1}(P_n \wedge bo)$ is a finite cyclic group ([3] or [7]).

THEOREM 1.1. (a) *If* $\mathrm{gd}(pH_k) \leqslant n$, *then* $\nu(p, i) \geqslant \nu(\pi_{4i-1}(P_n \wedge bo))$ *for all* $i \leqslant k$.

(b) *The converse of* (a) *is true if* $n \geqslant 2k$ *and* $\pi^s_{4i-1}(P_n) \longrightarrow \pi_{4i-1}(P_n \wedge bo)$ *is injective for all* $i \leqslant k$.

REMARK 1.2. $\mathrm{gd}(pH_k) \geqslant 2k$, because if the stable class of pH_k contains a $2k$-plane bundle, it has nonzero Euler class, since $e^2 = p_k \neq 0$.

By using the computations of [7] or [3] and [11], the observation that $\binom{p}{i}$ is the coefficient of the symplectic Pontryagin class $e_i(pH_k)$, and an argument which is presented in §4 to eliminate one extraneous obstruction, Theorem 1.1 may be restated:

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THEOREM 1.3. (a) *If* $\mathrm{gd}(pH_k) \leqslant 4m + \epsilon$ $(\epsilon = 1, 2, 3)$, *then for all* $m + j \leqslant k$, $e_{m+j}(pH_k)$ *is divisible by* 2^{2j} *if j even*, $2^{2j+2-\epsilon}$ *if j odd.*

(b) *The converse is true if* $4m + \epsilon \geqslant 2k$ *and*

$$m \geqslant k - \begin{cases} 3 & \text{if } \epsilon = 1, \text{ and } \nu(p, m) \geqslant 3 \text{ if } m \text{ even,} \\ 4 & \text{if } \epsilon = 2, 3, \text{ and } \nu(p, m+1) \geqslant 3 \text{ if } m \text{ odd.} \end{cases}$$

This enables us to state the precise geometric dimension of many pH_k, for example $\mathrm{gd}(16H_4) = \mathrm{gd}(16H_5) = 14$ and $\mathrm{gd}(18H_{10}) = \mathrm{gd}(18H_{11}) = 39$. Part (a) is slightly stronger than the main theorem of [14]. We conjecture that part (b) is true without the condition $m \geqslant k - \{^3_4$.

We obtain new immersions of RP^{4k+3} by combining Theorem 1.3(b) with the map $RP^{4k+3} \longrightarrow QP^k$.

THEOREM 1.4. *If* $K \equiv 3$ (4), *then* $RP^K \subseteq R^{2K-D}$, *where D is given by the following table*:

$\alpha(K)$	6	7	8	9	10
D	10	12	13	14	17

These results for $\alpha(K) \geqslant 7$ are slightly stronger than those of Milgram [12]. The first new immersion result that we obtain is $RP^{191} \subseteq R^{370}$. These results are not quite as good as the conjectured best possible immersion dimension:

CONJECTURE 1.5. [7]. If $K \equiv 7$ (8) the smallest Euclidean space in which RP^K can be immersed as dimension

$$2K - 2\alpha(K) + \begin{cases} 0 & \text{if } \alpha(K) \equiv 0 \ (4), \\ 1 & \text{if } \alpha(K) \equiv 1, 2 \ (4), \\ -1 & \text{if } \alpha(K) \equiv 3 \ (4). \end{cases}$$

We feel that the conjectured immersions for $\alpha(K) \leqslant 10$ will follow quite readily from Theorem 1.3(b) together with computations of the indeterminacies in the modified Postnikov tower [6]. As these computations are extremely detailed, they will be deferred until a later paper. The negative part of this conjecture has been announced and retracted [4], [7]. It is hoped that the methods of this paper together with those of the last sections of [5] will produce a proof of the negative part of this conjecture.

The rationale behind this conjecture is the belief that only *bo*-primary homotopy of V_n should obstruct multiples of the line bundle over projective spaces and that these obstructions should be given in terms of the symplectic Pontryagin classes. Some attempts have been made to show the latter directly using K-theoretic e_i-classes. In this paper we have shown that as far as divisibility by 2 is concerned, the e_i could be the obstructions.

The argument can be sketched as follows.

THEOREM 1.6. *There are bundles*

$$V_n \wedge bo \longrightarrow E_n^o \longrightarrow BSp, \quad V_n \wedge bu \longrightarrow E_n^u \longrightarrow BSp$$

such that

(i) *there are pairings of bundles*

$$E_n^o \times E_m^o \longrightarrow E_{n+m}^o, \qquad E_n^u \times E_m^u \longrightarrow E_{n+m}^u;$$

(ii) *if $QP^k \xrightarrow{f} BSp$ classifies a stable real vector bundle θ, then* $\mathrm{gd}(\theta) \leqslant n$ *implies f lifts to E_n^o, and if f lifts to E_n^o and $n \geqslant 2k$ and $\pi_{4i-1}(P_n) \longrightarrow \pi_{4i-1}(P_n \wedge bo)$ is injective for all $i \leqslant k$, then* $\mathrm{gd}(\theta) \leqslant n$.

DEFINITION 1.7. $N(p, k)$ is the smallest N such that

$$\nu(p, i) \geqslant \nu(\pi_{4i-1}(P_N \wedge bo)) \text{ for all } i \leqslant k.$$

$M(p, k)$ is the smallest M such that $\nu(p, i) \geqslant \nu(\pi_{4i-1}(P_M \wedge bu))$ for all $i \leqslant k$.

Theorem 1.1 then follows from Theorem 1.6 and

THEOREM 1.8. *pH_k lifts to $E^o_{\max(2k,N(p,k))}$ but not to $E^o_{\max(2k,N(p,k))-1}$.*

This is proved inductively by noting that if $p = \Sigma_{i=1}^r 2^{e_i}$, with all e_i distinct if p is not a power of 2, and $e_1 = e_2 = i$ if $p = 2^{i+1}$, then pH_k is classified by the composite

$$\begin{aligned} QP^k &\xrightarrow{\Delta} (QP^k \times \cdots \times QP^k)^{(4k)} \\ &= \bigcup QP^{k_1} \times \cdots \times QP^{k_r} \xrightarrow{\times 2^{e_i}H} BSp \times \cdots \times BSp \longrightarrow BSp. \end{aligned}$$

where Δ is a skeletal map homotopic to the iterated diagonal, and the union is taken over all ordered r-tuples $\langle k_1, \ldots, k_r\rangle$ such that $\Sigma k_i = k$. Liftings of $\times 2^{e_i}H_{k_i}$ to $E^u_{M(p,k)}$ are found by using the liftings of the factors and the pairing of Theorem 1.6(i). These fit together to give a lifting of pH_k to $E^u_{M(p,k)}$. The results for lifting to E_N^o are obtained by using the maps $E_N^o \longrightarrow E_N^u$.

2. Construction of some spaces and maps. Let $\widetilde{BSp}_n$ be the fibered product defined by the diagram

$$\begin{array}{ccc} \widetilde{BSp}_n & \longrightarrow & BO_n \\ \downarrow & & \downarrow \\ BSp & \longrightarrow & BO. \end{array}$$

We wish to form the fiberwise smash product [10] of $\widetilde{BSp}_n$ with the bo-spectrum. In order to form the fiberwise smash, the original bundle must have a section, which we obtain by applying fiberwise unreduced suspension, but then we must be careful in order to have a fiberwise map from the original bundle into the new one.

Let b_l indicate the lth space in a connective ringed Ω-spectrum b [17]. Then there is a map $S^l \longrightarrow b_l$, the unit of the ring, which commutes with $\epsilon: b_l \longrightarrow \Omega b_{l+1}$ and with the pairing $\mu: b_{l_1} \wedge b_{l_2} \longrightarrow b_{l_1+l_2}$. Let $x_l \in \Omega^l b_l$ correspond to this map. Let $F \longrightarrow E \xrightarrow{p} B$ be a fiber bundle. Form

$$\Omega^l_B((S^n_B E) \wedge{}_B b_l) \xrightarrow{\overline{p}_n} B$$

as in [10]. Let $P^n_B\Omega^l_B((S^n_B E) \wedge{}_B b_l)$ be the space of maps

$$\sigma: I^n \longrightarrow \Omega^l_B((S^n_B E) \wedge{}_B b_l)$$

such that $\overline{p}_n\sigma$ is constant and for $\overline{s} \in \dot{I}^n$, $\overline{t} \in I^l$, $\sigma(\overline{s})(\overline{t}) = [\overline{s}, p^{-1}\overline{p}_n(\sigma(I^n)), x_l(\overline{t})]$. Then there is a fiberwise map $i_E: E \longrightarrow P^n_B\Omega^l_B((S^n_B E) \wedge{}_B b_l)$ given by $i_E(e)(\overline{s}, \overline{t}) = [\overline{s}, e, x_l(\overline{t})]$. Also, there are obvious maps

$$P^n_B\Omega^l_B((S^n_B E) \wedge{}_B b_l) \longrightarrow P^n_B\Omega^{l+1}_B((S^n_B E) \wedge{}_B b_{l+1})$$

so that we can form

$$(E \wedge{}_B b)_n = \lim_l P^n_B\Omega^l_B((S^n_B E) \wedge{}_B b_l).$$

Note that

$$\text{fiber}\,((E \wedge{}_B b)_n \longrightarrow B) = \lim P^n\Omega^l(S^n F \wedge b_l) \simeq \lim \Omega^{l+n}(\Sigma^n F \wedge b_l)$$
$$\simeq \lim_l \Omega^{l+n}(F \wedge b_{l+n}) = F \wedge b,$$

since $\Sigma^n b_l \longrightarrow b_{l+n}$ is a $(2l-n)$-equivalence. (Here $P^n\Omega^l(S^n F \wedge b_l)$ is the set of maps $\sigma: I^n \times I^l \longrightarrow S^n F \wedge b_l$ such that $\overline{s} \in \dot{I}^n$ implies $\sigma(\overline{s}, \overline{t}) = [\overline{s}, F, x_l(\overline{t})]$ and $\overline{t} \in \dot{I}^l$ implies $\sigma(\overline{s}, \overline{t}) = *$. The equivalence with $\Omega^{n+l}(\Sigma^n F \wedge b_l)$ is obtained by reducing the suspensions.)

Given a pairing of fiber bundles

$$\begin{array}{ccc} E_1 \times E_2 & \xrightarrow{m} & E_3 \\ \Big\downarrow{\scriptstyle p_1 \times p_2} & & \Big\downarrow{\scriptstyle p_3} \\ B_1 \times B_2 & \longrightarrow & B_3 \end{array}$$

we define maps

$$\overline{m}\colon ((S_{B_1}^{n_1}E_1) \wedge_{B_1} b_{l_1}) \times ((S_{B_2}^{n_2}E_2) \wedge_{B_2} b_{l_2}) \longrightarrow ((S_{B_3}^{n_1+n_2}E_3) \wedge_{B_3} b_{l_1+l_2}),$$

$$\begin{array}{c} P_{B_1}^{n_1}\Omega_{B_1}^{l_1}((S_{B_1}^{n_1}E_1) \wedge_{B_1} b_{l_1}) \times P_{B_2}^{n_2}\Omega_{B_2}^{l_2}((S_{B_2}^{n_2}E_2) \wedge_{B_2} b_{l_2}) \\ \Big\downarrow \overline{\overline{m}} \\ P_{B_3}^{n_1+n_2}\Omega_{B_3}^{l_1+l_2}((S_{B_3}^{n_1+n_2}E_3) \wedge_{B_3} b_{l_1+l_2}) \end{array}$$

by

$$\overline{m}([\overline{s}_1, e_1, y_1], [\overline{s}_2, e_2, y_2]) = [(\overline{s}_1, \overline{s}_2), m(e_1, e_2), \mu(y_1, y_2)],$$

$$\overline{\overline{m}}(\sigma_1, \sigma_2)(\overline{s}_1, \overline{s}_2, \overline{t}_1, \overline{t}_2) = \overline{m}(\sigma_1(\overline{s}_1, \overline{t}_1), \sigma_2(\overline{s}_2, \overline{t}_2)).$$

Since $P_B^n\Omega_B^l((S_B^nE) \wedge_B b_l) \longrightarrow (E \wedge_B b)_n$ is an $(l-n)$-equivalence, by choosing l_1 and l_2 large, $\overline{\overline{m}}$ may be regarded as defining a bundle pairing

$$(E_1 \wedge_{B_1} b)_{n_1} \times (E_2 \wedge_{B_2} b)_{n_2} \longrightarrow (E_3 \wedge_{B_3} b)_{n_1+n_2}$$

through any finite skeleta, which is compatible with m, since $i_{E_3}m = \overline{\overline{m}}(i_{E_1} \times i_{E_2})$.

Let

$$E_n^o = (\widetilde{BSp}_n \wedge_{BSp} bo)_n \quad \text{and} \quad E_n^u = (\widetilde{BSp}_n \wedge_{BSp} bu)_n.$$

Thus, there are commutative diagrams of fiber bundles:

$$\begin{array}{ccc} V_n & \longrightarrow & V_n \wedge bo \\ \downarrow & & \downarrow \\ \widetilde{BSp}_n & \longrightarrow & E_n^o \\ & \searrow \quad \swarrow & \\ & BSp & \end{array} \qquad \begin{array}{ccc} V_n & \longrightarrow & V_n \wedge bu \\ \downarrow & & \downarrow \\ \widetilde{BSp}_n & \longrightarrow & E_n^u \\ & \searrow \quad \swarrow & \\ & BSp & \end{array}$$

where $V_n = \lim_k V_{n+k,k}$. The pairings of Theorem 1.6(i) now follow by the construction of the previous paragraph, since the Whitney sum pairing $BO_n \times BO_m \longrightarrow BO_{n+m}$ induces a map $\widetilde{BSp}_n \times \widetilde{BSp}_m \longrightarrow \widetilde{BSp}_{n+m}$. Theorem 1.6(ii) follows easily since $\mathrm{gd}(\theta) \leqslant n$ if and only if f lifts to $\widetilde{BSp}_n$. There is a well-known map [9] $P_n \longrightarrow V_n$ which is a $2n$-equivalence. Thus in the stable range the only obstructions to lifting f from E_n^o to $\widetilde{BSp}_n$ are due to the non-bo-primary homotopy of P_n in dimensions $4i - 1$. Note that for n odd all the homotopy of P_n is 2-primary, while for n even and p an odd prime, the p-primary homotopy of P_n cannot cause an obstruction by naturality from the lifting problem for E_{n-1}^o.

THEOREM 2.1. *For $\Delta \leqslant 4$ there are equivalences of skeleta*

$$(E_n^o/E_{n-\Delta}^o)^{(2(n-\Delta)+1)} \longrightarrow \bigvee_I (\Sigma^{4|I|+1}P_{n-\Delta}^{n-1} \wedge bo)^{(2(n-\Delta)+1)},$$

$$(E_n^u/E_{n-\Delta}^u)^{(2(n-\Delta)+1)} \longrightarrow \bigvee_I (\Sigma^{4|I|+1}P_{n-\Delta}^{n-1} \wedge bu)^{(2(n-\Delta)+1)}$$

where I ranges over $\{0\}$ *and all sets of positive integers and $|I|$ is the sum of all elements of I.*

REMARK. This theorem seems to be true for arbitary Δ but we shall just need it for $\Delta \leqslant 4$.

PROOF. As noted by Moore [13], there is a spectral sequence converging to $H^*(\widetilde{BSp}_n, \widetilde{BSp}_{n-\Delta}; \mathbf{Z}_2)$ with $E_2^{p,q} = H^p(BSp; H^q(V_n, V_{n-\Delta}))$, which in the stable range is $H^p(BSp; H^q(\Sigma P_{n-\Delta}^{n-1}))$. The map $(\widetilde{BSp}_n, \widetilde{BSp}_{n-\Delta}) \xrightarrow{f} (BO_n, BO_{n-\Delta})$ induces a map of spectral sequences. Since there are no differentials in the spectral sequence of $(BO_n, BO_{n-\Delta})$ in the stable range, there are none in that of $(\widetilde{BSp}_n, \widetilde{BSp}_{n-\Delta})$. Moreover, it follows from naturality of the external cup product that in the stable range $H^*(\widetilde{BSp}_n, \widetilde{BSp}_{n-\Delta}) \approx H^*(BSp) \otimes H^*(\Sigma P_{n-\Delta}^{n-1})$ as modules over the subalgebra A_1 of the mod 2 Steenrod algebra generated by Sq^1 and Sq^2. Since $\Delta \leqslant 4$ each $H^q(\widetilde{BSp}_n, \widetilde{BSp}_{n-\Delta})$, $q \leqslant 2(n-\Delta)+1$, contains elements of only one filtration, so there is no ambiguity in writing an element as an element of $H^*(BSp) \otimes H^*(\Sigma P_{n-\Delta}^{n-1})$. Then, for example, in $H^*(\widetilde{BSp}_{4m+5}, \widetilde{BSp}_{4m+1})$,

$$\begin{aligned}\mathrm{Sq}^2(e_1 \otimes sa_{4m+3}) &= \mathrm{Sq}^2(f^*(w_4 \cup w_{4m+4})) = f^*((w_6 + w_2w_4) \cup w_{4m+4}) \\ &= \bar{f}^*(w_6 + w_2w_4) \cup f^*(w_{4m+4}) = 0,\end{aligned}$$

where $\bar{f}: \widetilde{BSp}_n \longrightarrow BO_n$.

The E_2-term of the Adams spectral sequence for $\pi_*((\widetilde{BSp}_n/\widetilde{BSp}_{n-\Delta}) \wedge bo)$ is

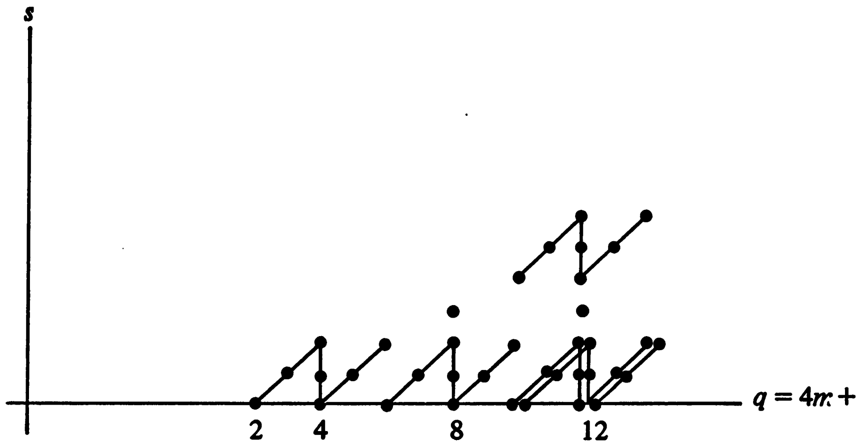
$$\mathrm{Ext}_{A_1}^{s,t}(H^*(\widetilde{BSp}_n, \widetilde{BSp}_{n-\Delta}), \mathbf{Z}_2) \approx \mathrm{Ext}_{A_1}^{s,t}\left(H^*\left(\bigvee \Sigma^{4|I|+1}P_{n-\Delta}^{n-1}\right), \mathbf{Z}_2\right).$$

These groups can be read off from [3]. There can be no differentials by naturality, since there are no possible differentials in the Adams spectral sequence for $\widetilde{BSp}_{n-\epsilon+1}/\widetilde{BSp}_{n-\epsilon} \wedge bo$, $1 \leqslant \epsilon \leqslant \Delta$.

We can use our determination of $\pi_*(\widetilde{BSp}_n/\widetilde{BSp}_{n-\Delta} \wedge bo)$ to map

$$\bigvee \Sigma^{4|I|+1} P^{n-1}_{n-\Delta} \xrightarrow{\vee f_I} \widetilde{BSp}_n/\widetilde{BSp}_{n-\Delta} \wedge bo$$

by using the Barratt-Puppe sequence to extend the map cell-by-cell. For example, consider the case $\widetilde{BSp}_{4m+5}/\widetilde{BSp}_{4m+1}$. $\pi_q(\widetilde{BSp}_{4m+s}/\widetilde{BSp}_{4m+1} \wedge bo)$ $(q \leq 4m+14)$ can be pictured as below, where dots indicate nontrivial classes and vertical lines indicate multiplication by 2. (See [3].)



Thus for each I there is a map

$$S^{4|I|+4m+2} \vee S^{4|I|+4m+4} \xrightarrow{f_I} \widetilde{BSp}_{4m+5}/\widetilde{BSp}_{4m+1} \wedge bo$$

such that $f_I^*(e_I \otimes sa_n \otimes 1) = g_{4|I|+r}$ for $r = 4m+2, 4$. $2f_I|S^{4|I|+4m+2} \simeq 0$ so that f_I extends over $S^{4|I|+4m+2} \cup_2 e^{4|I|+4m+3} \vee S^{4|I|+4m+4}$. The map

$$S^{4|I|+4m+4} \xrightarrow{\bar{\eta}} S^{4|I|+4m+2} \cup_2 e^{4|I|+4m+3}$$

generates $\pi_{4|I|+4m+4}(S^{4|I|+4m+2} \cup_2 e^{4|I|+4m+3}) \approx Z_4$. $f_I\bar{\eta}$ is thus twice the generator of the appropriate summand of $\pi_{4|I|+4m+4}(\widetilde{BSp}_{4m+5}/\widetilde{BSp}_{4m+1} \wedge bo)$ and hence equals $2f_I|S^{4|I|+4m+4}$. Thus f_I extends over

$$S^{4|I|+4m+2} \cup_2 e^{4|I|+4m+3} \vee S^{4|I|+4m+4} \cup_{2,\bar{\eta}} e^{4|I|+4m+5} \approx \Sigma^{4|I|+1} P^{4m+4}_{4m+1}.$$

The composition

$$\bigvee \Sigma^{4|I|+1} P^{4m+4}_{4m+1} \wedge bo \xrightarrow{\vee f_I \wedge I} \widetilde{BSp}_{4m+5}/\widetilde{BSp}_{4m+1} \wedge bo \wedge bo$$

$$\xrightarrow{1 \wedge \mu} \widetilde{BSp}_{4m+5}/\widetilde{BSp}_{4m+1} \wedge bo$$

induces an isomorphism in $\mathbf{Z}_2$-cohomology through the stable range, and since *bo* is localized at 2, it is a $(8m + 3)$-equivalence. But

$$E^o_{4m+5}/E^o_{4m+1} \approx \widetilde{BSp}_{4m+5}/\widetilde{BSp}_{4m+1} \wedge bo.$$

A similar argument works when *bo* is replaced by *bu*.

THEOREM 2.2. (i) *For $\Delta \leq 4$ there are maps $(E^o_n)^{(2(n-\Delta)+1)} \longrightarrow \Sigma P^{n-1}_{n-\Delta} \wedge bo$ whose fiber has the same $2(n-\Delta)$-type as $E^o_{n-\Delta}$, and similarly for E^u_n.*

(ii) *The following diagram is homotopy commutative, where the horizontal maps are induced by complexification $bo \longrightarrow bu$.*

$$\begin{array}{ccc} (E^o_n)^{(2(n-\Delta)+1)} & \longrightarrow & (E^u_n)^{(2(n-\Delta)+1)} \\ \downarrow & & \downarrow \\ \Sigma P^{n-1}_{n-\Delta} \wedge bo & \longrightarrow & \Sigma P^{n-1}_{n-\Delta} \wedge bu \end{array}$$

(iii) *For $\Delta = 1$, the maps $(E^u_n)^{2n-1} \xrightarrow{h_n} S^n \wedge bu$ can be chosen so that the diagrams*

$$\begin{array}{ccc} (E^u_{2n})^{4n-1} \times (E^u_{2m})^{4m-1} & \longrightarrow & (E^u_{2(n+m)})^{4n+4m-1} \\ \downarrow{\scriptstyle h_{2n} \times h_{2m}} & & \downarrow{\scriptstyle h_{2(n+m)}} \\ S^{2n} \wedge bu \wedge S^{2m} \wedge bu & \longrightarrow & S^{2n+2m} \wedge bu \end{array}$$

are homotopy commutative.

PROOF. Let $\mathcal{E}$ be the fiber of the composite

$$E^{o(2(n-\Delta)+1)}_n \longrightarrow E^o_n/E^{o(2(n-\Delta)+1)}_{n-\Delta} \xrightarrow{h} \bigvee \Sigma^{4|I|+1} P^{n-1}_{n-\Delta} \longrightarrow \Sigma P^{n-1}_{n-\Delta} \wedge bo,$$

where h is a homotopy inverse of the map constructed in Theorem 2.1. Through dimension $2(n-\Delta)$ there is a commutative diagram of fibrations:

$$\begin{array}{ccc} P^{n-1}_{n-\Delta} \wedge bo & \longrightarrow & P^{n-1}_{n-\Delta} \wedge bo \\ \downarrow & & \downarrow \\ E^o_{n-\Delta} & \longrightarrow & \mathcal{E} \\ & \searrow \quad \swarrow & \\ & E^o_n & \end{array}$$

In both fibrations the fundamental class $a_{n-\Delta} \otimes 1$ transgresses nontrivially, and since it is the only nonzero cohomology class in its dimension it must be mapped to itself. But all classes in $H^*(P_{n-\Delta}^{n-1} \wedge bo; \mathbf{Z}_2)$ are related to one another by the action of the Steenrod algebra, and hence the map of fibers must be a $2(n-\Delta)$-equivalence. Thus $E_{n-\Delta}^o \longrightarrow E$ is a $2(n-\Delta)$-equivalence.

Part (ii) follows by functorality of complexification; i.e., a map

$$\widetilde{BSp}_n/\widetilde{BSp}_{n-\Delta} \wedge bo \longrightarrow \Sigma P_{n-\Delta}^{n-1} \wedge bo$$

induces a compatible map

$$\widetilde{BSp}_n/\widetilde{BSp}_{n-\Delta} \wedge bu \longrightarrow \Sigma P_{n-\Delta}^{n-1} \wedge bu,$$

which can be thought of as smashing with the identity map of $S^o \cup_\eta e^2$.

(iii) It suffices to find elements $x_{2n} \in K_{\mathrm{U}}^o(\widetilde{BSp}_{2n})$ satisfying

(a) the restriction to $K_{\mathrm{U}}^o(\widetilde{BSp}_{2n-1})$ of x_{2n} is zero,

(b) $m^*(x_{2(n+m)}) = x_{2n} \otimes x_{2m}$, where m is the natural pairing $\widetilde{BSp}_{2n} \times \widetilde{BSp}_{2m} \longrightarrow \widetilde{BSp}_{2n+2m}$,

(c) x_{2n} has filtration $2n$ and is represented by a map $\widetilde{BSp}_{2n} \longrightarrow S^{2n} \wedge bu$ which is nontrivial in $\mathbf{Z}_2$-cohomology.

We then let h_n be the composite

$$E_{2n}^u \longrightarrow \widetilde{BSp}_{2n} \wedge bu \xrightarrow{x_{2n} \wedge 1} S^{2n} \wedge bu \wedge bu \xrightarrow{1 \wedge \mu} S^{2n} \wedge bu.$$

The elements $y_{2n} \in K_{\mathrm{U}}^o(B\,\mathrm{Spin}_{2n})$ corresponding to the representation $\Delta_{2n}^+ - \Delta_{2n}^-$ [8] under the isomorphism $K_{\mathrm{U}}^o(B\mathrm{Spin}_{2n}) \approx R(\mathrm{Spin}_{2n})^\wedge$ [2] satisfy the analogues of (a), (b), (c), so we choose $x_{2n} = g^* y_{2n}$, where g is the map $\widetilde{BSp}_{2n} \longrightarrow B\mathrm{Spin}_{2n}$ obtained by lifting the map $\widetilde{BSp}_{2n} \longrightarrow BO_{2n}$.

THEOREM 2.3. *For $b = bo$ or bu and any space X,*

$$\left[\bigtimes_{i=1}^{r} QP^{k_i}, X \wedge b\right] \approx \bigoplus_{l_i < k_i} \pi_{4\Sigma l_i}(X \wedge b).$$

PROOF. Let $D_q(\times QP^{k_i})$ be a Spanier-Whitehead dual [16]. Then by [17]

$$\left[\times QP^{k_i}, X \wedge b\right] \approx \pi_q\left(D\left(\times QP^{k_i}\right) \wedge X \wedge b\right).$$

Since $H^q(D(\times QP^{k_i}); \mathbf{Z}_2)$ is nonzero only every fourth dimension, as in the proof of Theorem 2.1, $D(\times QP^{k_i}) \wedge b \simeq \bigvee S^{q-4\Sigma l_i} \wedge b$. Thus

$$\pi_q\left(D\left(\times QP^{k_i}\right) \wedge X \wedge b\right) \approx \pi_q\left(\bigvee S^{q-4\Sigma l_i} \wedge X \wedge b\right) \approx \bigoplus \pi_{4\Sigma l_i}(X \wedge b).$$

THEOREM 2.4. *There exists a generator* $e_L(X, b)$ *of the* $\pi_{4\Sigma l_i}(X \wedge b)$-*component of* $[\times QP^{k_i}, X \wedge b]$ *for* $b = bu$, $X = S^{2n}$, ΣP^{2n}_{2n-1}, *or* ΣP^{2n+1}_{2n-1} *and for* $b = bo$, $X = \Sigma P_n^{n+\Delta}$, $\Delta \leqslant 4$, $n + \Delta \not\equiv 3\ (4)$ *if* $\Delta > 0$ *satisfying*:

(i) *they are natural with respect to inclusions* $\times QP^{k'_i} \longrightarrow \times QP^{k_i}$;

(ii) *they are natural with respect to maps* $X \longrightarrow X'$ *which are inclusions or collapsings and with respect to complexification* $bo \longrightarrow bu$ *in the sense that if* $\pi_{4l}(X \wedge b) \longrightarrow \pi_{4l}(X' \wedge b')$ *sends generator to* 2^e-*generator, then* $[\times QP^{k_i}, X \wedge b] \longrightarrow [\times QP^{k_i}, X' \wedge b']$ *sends* $e_L(X, b)$ *to* $2^e e_L(X', b')$;

(iii) *if* $\Delta: QP^k \longrightarrow (QP^k \times \cdots \times QP^k)^{(4k)}$ *is a skeletal map homotopic to the diagonal, then*

$$\Delta^*: [(QP^k \times \cdots \times QP^k)^{(4k)}, X \wedge bu] \longrightarrow [QP^k, X \wedge bu]$$

satisfies $\Delta^*(e_L) = e_{|L|}$, *where* $|L| = \Sigma l_i$ *if* $L = \langle l_i \rangle$;

(iv) *the natural pairing* $\bigotimes [QP^{k_i}, S^{2n_i} \wedge bu] \longrightarrow [\times QP^{k_i}, S^{\Sigma 2n_i} \wedge bu]$ *sends* $e_{l_1} \otimes \cdots \otimes e_{l_r} \longmapsto e_{l_1, \ldots, l_r}$.

PROOF. As in [1, Lemma 2.5] $k\mathring{o}(\times_{i=1}^{r} QP^{k_i})$ is the truncated polynomial algebra $k\mathring{o}(pt)\,[\alpha_1, \cdots, \alpha_r]/\alpha_i^{k_i+1} = 0$, where $\alpha_i \in k\mathring{o}^4(\times QP^{k_i})$, and similarly for $k\mathring{u}$. Thus

$$\left[\mathop{\times}_{i=1}^{r} QP^{k_i}, S^n \wedge b\right] = k^n\left(\times QP^{k_i}\right) \approx \bigoplus_{l_i \leqslant k_i} \alpha^{l_1} \cdots \alpha^{l_r} \cdot \pi_{4\Sigma l_i}(S^n \wedge b).$$

Let $e_L(S^n, b) \in [\times QP^{k_i}, S^n \wedge b]$ correspond to $\alpha^{l_1} \cdots \alpha^{l_r}$ times the canonical generator of $\pi_{4\Sigma l_i}(S^n \wedge b)$. If

$$\pi_{4|L|}(S^n \wedge b) \xrightarrow{i_\#} \pi_{4|L|}(\Sigma P_{n-1}^{n-1+\Delta} \wedge b)$$

is surjective, let $e_L(\Sigma P_{n-1}^{n-1+\Delta}, b) = i_*(e_L(S^n, b))$. This takes care of all $b = bu$ cases and some bo-cases. For other cases $e_L(\Sigma P_{n-1}^{n-1+\Delta}, bo)$ can be chosen to satisfy (ii) with respect to complexification and an appropriate collapsing map $P_{n-1}^{n-1+\Delta} \longrightarrow P_{n-1+\epsilon}^{n-1+\Delta}$. Thus we have defined e_L which satisfy (ii). They clearly satisfy (i) and (iv). (iii) is clearly true for $e_L(S^{2n}, bu)$ and follows for other $e_L(X, bu)$ by naturality. Indeed,

$$\begin{aligned}\Delta^*(e_L(X, bu)) &= \Delta^* i_*(e_L(S, bu)) = i_*\Delta^*(e_L(S, bu)) \\ &= i_*(e_{|L|}(S, bu)) = e_{|L|}(X, bu).\end{aligned}$$

DEFINITION 2.5. A class $[f] \in [\times QP^{k_i}, X \wedge b]$ can be written uniquely as $\Sigma n_L e_L(X, b)$. We will call $n_{\langle k_i \rangle}$ the *top component* of $[f]$, or the *top obstruction* if $f\colon \times QP^{k_i} \longrightarrow E_n \longrightarrow \Sigma P_{n-\Delta}^{n-1} \wedge b$.

COROLLARY 2.6. (i) *Let* k, r, *and* $Y = QP^{l_1} \times \cdots \times QP^{l_s}$ *be fixed. Suppose that for all* $K = \langle k_i : i = 1, \ldots, r\rangle$ *such that* $\Sigma k_i = k$ *we have compatible maps*

$$f_K : QP^{k_1} \times \cdots \times QP^{k_r} \times Y \longrightarrow X \wedge bu.$$

These induce a map

$$QP^k \times Y \xrightarrow{\Delta} QP^{k_1} \times \cdots \times QP^{k_r} \times Y \xrightarrow{f} X \wedge bu$$

such that the top component of $[f\Delta]$ *equals the sum of the top components of the* $[f_K]$.

(ii) *If* $f_i : QP^{k_i} \longrightarrow S^{2n_i} \wedge bu$ *and* $f = \times f_i : \times QP^{k_i} \longrightarrow S^{2\Sigma n_i} \wedge bu$, *then the top component of* $[f]$ *is the product of the top components of the* $[f_i]$.

3. Proof of Theorem 1.8. It will be convenient to first prove

THEOREM 3.1. pH_k *lifts to* E^u_M, *where* $M = \max(2k, M(p, k))$ *and* $M(p, k)$ *is defined in Definition* 1.7. *The class in* $\bigoplus_{i=1}^k \pi_{4i}(\Sigma P^M_{M-1} \wedge bu)$ *corresponding to the map* $QP^k \xrightarrow{pH} E^u_{M+1} \longrightarrow \Sigma P^M_{M-1} \wedge bu$ *under the isomorphism of Theorem* 2.3 *and Definition* 2.5 *has a nonzero* π_{4i}*-component if and only if* $\nu(p, i) < \nu(\pi_{4i-1}(P_{M-1} \wedge bu))$.

As outlined in the Introduction, Theorem 3.1 is proved by writing pH_k as a Whitney sum. The combinatorial result which makes the induction work is the following theorem, whose proof is given in §4.

THEOREM 3.2. *If* e_i *are distinct and* $0 \leqslant k_i \leqslant 2^{e_i}$ *and* $\Sigma k_i = k$, *then* $\Sigma \nu(2^{e_i}, k_i) \geqslant \nu(\Sigma 2^{e_i}, k)$. *For fixed* $\langle e_i\rangle$ *and* k *there are an odd number of* $\langle k_i\rangle$ *as above for which equality is obtained.*

Because we can only conclude the liftings of the factors in the stable range, the induction is made somewhat more complicated, requiring the following definitions.

DEFINITION 3.3. A triple $(2^e, k_1, k_2)$ with $0 \leqslant k_i \leqslant 2^e$ is *special* if exactly one of the k_i equals 2^e or exactly one of the k_i satisfies, for all $l \leqslant k_i$, $2l + \nu(l) < e$.

DEFINITION 3.4. Let $S = \{(2^{e_i}, k_i)\}$ be a finite set of ordered pairs such that $0 \leqslant k_i \leqslant 2^{e_i}$. If for two of the elements of S we have $e_\alpha = e_\beta$, let

$$S' = \{(2^{e_\alpha + 1}, k_\alpha + k_\beta)\} \cup \{(2^{e_i}, k_i) \in S : i \neq \alpha, \beta\},$$

and write $S \longmapsto S'$ and let

$$\epsilon(S, S') = \begin{cases} 1 & \text{if } (2^{e_\alpha}, k_\alpha, k_\beta) \text{ is special,} \\ 0 & \text{if not.} \end{cases}$$

There is a (not necessarily unique) sequence $\xi: S = S_0 \mapsto S_1 \mapsto \cdots \mapsto S_n$, where S_n has distinct exponents e_i. Let

$$\gamma'(\xi) = \sum_{i=0}^{n-1} \epsilon(S_i, S_{i+1}) \quad \text{and} \quad \gamma(S) = \min_{\xi}(\gamma'(\xi)),$$

where ξ ranges over all such sequences.

LEMMA 3.5. $\nu(\Sigma 2^{e_i}, \Sigma k_i) \leq \Sigma \nu(2^{e_i}, k_i) + \gamma(\{(2^{e_i}, k_i)\})$.

PROOF. Let $\xi: S_0 \mapsto \cdots \mapsto S_n$ with $S_0 = \{(2^{e_i}, k_i)\}$ and $S_n = \{(2^{e_i'}, k_i')\}$, e_i' distinct, be a sequence having minimal γ'. If $S = \{(2^{e_i''}, k_i'')\}$, let $\nu(S) = \Sigma \nu(2^{e_i''}, k_i'')$. Then

$$\nu(S_i) - \nu(S_{i+1}) = \nu(2^e, k) + \nu(2^e, k') - \nu(2^{e+1}, k + k') \geq - \epsilon(S_i, S_{i+1}).$$

This follows immediately from the observations that $\nu(2^e, k) = e - \nu(k)$ if $k > 0$, and $\nu(k + k') \geq \min(\nu(k), \nu(k'))$. Thus

$$\begin{aligned} \sum \nu(2^{e_i}, k_i) = \nu(S_0) &\geq \nu(S_n) - \sum \epsilon(S_i, S_{i+1}) \\ &= \sum \nu(2^{e_i'}, k_i') - \gamma(S) \geq \nu\left(\sum 2^{e_i'}, \sum k_i'\right) - \gamma(S) \\ &= \nu\left(\sum 2^{e_i}, \sum k_i\right) - \gamma(S). \end{aligned}$$

The last inequality is due to Theorem 3.2.

COROLLARY 3.6. $M(\Sigma 2^{e_i}, \Sigma k_i) \geq \Sigma M(2^{e_i}, k_i) - 2\gamma(\{(2^{e_i}, k_i)\})$.

PROOF. Since $\nu(\pi_{2i-1}(P_{2n} \wedge bu)) = \nu(\pi_{2i-1}(P_{2n+1} \wedge bu)) = i - n$, we have $M(p, k) = \max_{i \leq k,p}(4i - 2\nu(p, i))$. Choose $l_i \leq k_i$ such that $M(2^{e_i}, k_i) = 4l_i - 2\nu(2^{e_i}, l_i)$. Note $l_i = 0$ if and only if for all $l \leq k_i$, $2l + \nu(l) < e$. Then

$$\begin{aligned} \sum M(2^{e_i}, k_i) - 2\gamma(\{(2^{e_i}, k_i)\}) &= 4\sum l_i - 2\left(\sum \nu(2^{e_i}, l_i) + \gamma(\{(2^{e_i}, l_i)\})\right) \\ &\leq 4\sum l_i - 2\nu\left(\sum 2^{e_i}, \sum l_i\right) \leq M\left(\sum 2^{e_i}, \sum k_i\right). \end{aligned}$$

Theorem 3.1 follows immediately from Theorems 3.7(b), 3.8(b) and 2.4(i).

THEOREM 3.7. (a) *$\times 2^{e_i} H_{k_i}$ lifts to E^u_M, where*

$$M = \max\left(2\sum k_i, M\left(\sum 2^{e_i}, \sum k_i\right)\right) + 2\gamma(\{(2^{e_i}, k_i)\}),$$

if the product contains two or more factors.

(b) *pH_k lifts to $E^u_{\max(2k, M(p,k))}$.*

PROOF. It suffices to prove:

(i) If (a) is true whenever $\Sigma 2^{e_i} \leqslant p$ and (b) is true for $pH_{k'}$ whenever $k' < k$, then (b) is true.

(ii) If (b) is true whenever $p \leqslant \max\{2^{e_i}\}$ and (a) is true for $\times 2^{e'_i}H_{k'_i}$ whenever $\Sigma 2^{e'_i} = \Sigma 2^{e_i}$ and $\max\{2^{e'_i} + k'_i: M(2^{e'_i}, k'_i) < 2k'_i\} < \max\{2^{e_i} + k_i: M(2^{e_i}, k_i) < 2k_i\}$, then (a) is true.

Proof of (i) *if p is not a power of* 2. Write $p = \Sigma_{i=1}^{r} 2^{e_i}$ with e_i distinct. If $0 \leqslant k_i \leqslant 2^{e_i}$ and $\Sigma k_i = k$, then by hypothesis $\times 2^{e_i}H_{k_i}$ lifts to $E^u_{\max(2k, M(p,k))}$. Thus

$$\bigcup_{\langle k_i \rangle} QP^{k_1} \times \cdots \times QP^{k_r} \xrightarrow{\times 2^{e_i}H} BSp$$

lifts to $E^u_{\max(2k,M(p,k))+1}$ since the difference obstruction for matching the liftings on their intersections is an element of $H^q(\times QP; \pi_q(P_{\max(2k,M(p,k))+1} \wedge bu)) = 0$, and hence the union lifts to $E^u_{\max(2k,M(p,k))}$, since there is no obstruction for lifting a space with cohomology only in even dimensions from E^u_{2M+1} to E^u_{2M}. As noted in the Introduction pH_k is classified by

$$QP^k \xrightarrow{\Delta} \bigcup QP^{k_1} \times \cdots \times QP^{k_r} \longrightarrow BSp$$

and hence it lifts to $E^u_{\max(2k,M(p,k))}$.

Proof of (i) *if* $p = 2^{i+1}$. Let $M = M(p, k)$. By hypothesis $2^iH_l \times 2^iH_{k-l}$ lifts to $E^u_{\max(2k,M+2)}$. Assume $M \geqslant 2k$, for otherwise we are done. As in the proof of the first case the liftings to E^u_{M+3} are compatible, and by Theorem 2.2 there is a map $E^u_{M+3}{}^{(4k)} \longrightarrow \Sigma P^{M+2}_{M+1} \wedge bu$ whose fiber has the $4k$-type of E^u_{M+1}. Thus the lifting to E^u_{M+1} (and hence to E^u_M) exists if and only if the top component of the composite

$$QP^k \xrightarrow{\Delta} \bigcup QP^l \times QP^{k-l} \longrightarrow E^u_{M+3} \longrightarrow \Sigma P^{M+2}_{M+1} \wedge bu$$

is zero, since the lower components are zero by the induction hypothesis. By Corollary 2.6(i) this is zero if and only if the top obstruction is nonzero for an even number of $2^iH_l \times 2^iH_{k-l}$. But the top obstruction for $2^iH_l \times 2^iH_{k-l}$ equals that of $2^iH_{k-l} \times 2^iH_l$, since the map $QP^{k-l} \times QP^l \longrightarrow E^u_{M+3}$ can be chosen to be the composite

$$QP^{k-l} \times QP^l \xrightarrow{T} QP^l \times QP^{k-l} \xrightarrow{g} E^u_{M+3},$$

where g is a lifting for $2^iH_l \times 2^iH_{k-l}$. The only unpaired obstruction is that of $2^iH_{k/2} \times 2^iH_{k/2}$ if k is even, but this is zero by the hypothesis since $\gamma(\{(2^i, k/2), (2^i, k/2)\}) = 0$.

Proof of (ii). If $M(2^{e_i}, k_i) \geq 2k_i$ for all i, then by hypothesis $\times 2^{e_i}H_{k_i}$ lifts to E^u_S, where

$$S = \sum M(2^{e_i}, k_i) \leq M\left(\sum 2^{e_i}, \sum k_i\right) + 2\gamma(\{(2^{e_i}, k_i)\}) = M$$

by Corollary 3.6.

Otherwise, choose the maximal value of $2^{e_i} + k_i$ such that $M(2^{e_i}, k_i) < 2k_i$. After renumbering we may assume that $i = 1$. Then $\times 2^{e_i}H_{k_i}$ is classified by

$$\times QP^{k_i} \xrightarrow{\Delta \times 1} \bigcup_{k_0=0}^{k_1} QP^{k_0} \times QP^{k_1-k_0} \times \mathop{\times}_{i\geq 2} QP^{k_i} \longrightarrow BSp,$$

where the second map classifies $2^{e_1-1}H \times 2^{e_1-1}H \times \times_{i\geq 2} 2^{e_i}H$. By hypothesis $2^{e_1-1}H_{k_0} \times 2^{e_1-1}H_{k_1-k_0} \times \times_{i\geq 2} 2^{e_i}H_{k_i}$ lifts to E^u_M, where

$$\begin{aligned} M = \max\Big(2\sum k_i,\ & M\left(\sum 2^{e_i}, \sum k_i\right) \\ & + 2\gamma(\{(2^{e_1-1}, k_0), (2^{e_1-1}, k_1 - k_0), (2^{e_2}, k_2), \ldots\})\Big) \\ \leq \max\Big(2\sum k_i,\ & M\left(\sum 2^{e_i}, \sum k_i\right) \\ & + 2\gamma(\{(2^{e_1}, k_1), (2^{e_2}, k_2), \ldots\}) + 2\Big) = M'. \end{aligned}$$

As in the proof of (i), all obstructions (except possibly one which has zero obstruction) occur in pairs so that the lifting of $\times 2^{e_i}H_{k_i}$ from $E^u_{M'+1}$ to $E^u_{M'-1}$ exists.

THEOREM 3.8. (a) *Let* $M = M(\Sigma 2^{e_i}, \Sigma k_i) + 2\gamma(\{(2^{e_i}, k_i)\})$. *If* $M > 2\Sigma k_i$. *then the top obstruction for lifting* $\times 2^{e_i}H_{k_i}$ *to* E^u_{M-1} *is nonzero if and only if* $M(\Sigma 2^{e_i}, \Sigma k_i) = \Sigma M(2^{e_i}, k_i) - 2\gamma(\{(2^{e_i}, k_i)\})$ *and for all i*, $M(2^{e_i}, k_i) = 4k_i - 2\nu(2^{e_i}, k_i)$.

(b) *If* $M(p, k) > 2k$, *then the top obstruction for lifting* pH_k *to* $E^u_{M(p,k)-1}$ *is nonzero if and only if* $M(p, k) = 4k - 2\nu(p, k)$.

PROOF. As in Theorem 3.7 it suffices to prove:

(i) If (a) is true whenever $\Sigma 2^{e_i} \leq p$, then (b) is true.

(ii) If (b) is true whenever $p \leq \max\{2^{e_i}\}$ and (a) is true for $\times 2^{e'_i}H_{k'_i}$ whenever $\Sigma 2^{e'_i} = \Sigma 2^{e_i}$ and $\max\{2^{e'_i} + k'_i : M(2^{e'_i}, k'_i) \leq 2k'_i\} < \max\{2^{e_i} + k_i : M(2^{e_i}, k_i) \leq 2k_i\}$, then (a) is true.

Proof of (i) *if p is not a power of* 2. Write $p = \Sigma 2^{e_i}$ with e_i distinct. The top obstruction for pH_k is nonzero if and only if there is an odd number of $\langle k_i \rangle$ such that $\times 2^{e_i}H_{k_i}$ has nonzero top obstruction (for lifting to $E^u_{M(p,k)-1}$). The top obstruction is certainly zero for those $\langle k_i \rangle$ for which $\Sigma M(2^{e_i}, k_i) < M(\Sigma 2^{e_i}, \Sigma k_i)$, so we need consider only those $\langle k_i \rangle$ for which $\Sigma M(2^{e_i}, k_i) = M(\Sigma 2^{e_i}, \Sigma k_i)$. Then

$$\sum M(2^{e_i}, k_i) = M(p, k) \geqslant 4k - 2\nu(p, k) \geqslant \sum (4k_i - 2\nu(2^{e_i}, k_i)).$$

If $M(p, k) > 4k - 2\nu(p, k)$, then $\Sigma(M(2^{e_i}, k_i) - (4k_i - 2\nu(2^{e_i}, k_i))) > 0$, so for some $i, M(2^{e_i}, k_i) > 4k_i - 2\nu(2^{e_i}, k_i)$ and hence by hypothesis $\times\, 2^{e_i}H_{k_i}$ has top obstruction zero. Since this is true for all $\langle k_i \rangle$, pH_k has top obstruction zero by Corollary 2.6(i).

If $M(p, k) = 4k - 2\nu(p, k)$, then $\times\, 2^{e_i}H_{k_i}$ has nonzero top obstruction if and only if for all $i, M(2^{e_i}, k_i) = 4k_i - 2\nu(2^{e_i}, k_i)$ if and only if $\nu(p, k) = \Sigma\nu(2^{e_i}, k_i)$, and by Theorem 3.2 this happens for an odd number of $\langle k_i \rangle$. (Note that $\{\langle k_i \rangle: \nu(p, k) = \Sigma\nu(2^{e_i}, k_i)\} \subset \{\langle k_i \rangle: M(p, k) = \Sigma M(2^{e_i}, k_i)\}$.) Thus $\times\, 2^{e_i}H_{k_i}$ has nonzero top obstruction.

Proof of (i) *if* $p = 2^{i+1}$. Let $M = M(p, k)$. There is a commutative diagram

$$\begin{array}{ccccccc} & & & & E^u_{M+1} & \longrightarrow & \Sigma P^M_{M-1} \wedge bu \\ & \nearrow & & & \downarrow & & \downarrow \\ & & & & & \nearrow & \Sigma P^{M+2}_{M-1} \wedge bu \\ & & & & & & \downarrow \\ QP^k & \xrightarrow{\Delta} & \bigcup QP^l \times QP^{k-l} & \xrightarrow{2^iH \times 2^iH} & E^u_{M+3} & \longrightarrow & \Sigma P^{M+2}_{M+1} \wedge bu. \end{array}$$

By Theorem 2.4(ii) and Corollary 2.6(i), $2^{i+1}H_k$ has nonzero top obstruction in $\pi_{4k}(\Sigma P^M_{M-1} \wedge bu)$ if and only if it has nonzero top obstruction in $\pi_{4k}(\Sigma P^{M+2}_{M-1} \wedge bu)$ if and only if the sum of the top obstructions for the $2^iH_l \times 2^iH_{k-l}$ in $\pi_{4k}(\Sigma P^{M+2}_{M-1} \wedge bu)$ is nonzero. If $(2^i, l, k-l)$ is not special, then $2^iH_l \times 2^iH_{k-l}$ lifts to E^u_M and hence its top obstruction in $\pi_{4k}(\Sigma P^{M+2}_{M-1} \wedge bu)$ is divisible by 2, so that the sum of the top obstructions for $2^iH_l \times 2^iH_{k-l}$ and $2^iH_{k-l} \times 2^iH_l$ is zero. (If k is even, $2^iH_{k/2} \times 2^iH_{k/2}$ lifts to $E^u_{M(p,k)-1}$.) If $(2^i, l, k-l)$ is special but $\{l, k-l\} \cap \{0, 2^i\}$ is empty, then by the induction hypothesis $2^iH_l \times 2^iH_{k-l}$ has zero top obstruction for lifting to E^u_{M+1}, so that the sum of the obstructions in $\pi_{4k}(\Sigma P^{M+2}_{M-1} \wedge bu)$ for $2^iH_l \times 2^iH_{k-l}$ and $2^iH_{k-l} \times 2^iH_l$ is zero. The only case remaining is $2^iH_{\epsilon 2^i} \times 2^iH_{k-\epsilon 2^i}$, where $\epsilon = 0$ if $k \leqslant 2^i$ and $\epsilon = 1$ if $k > 2^i$. The sum of the obstructions in $\pi_{4k}(\Sigma P^{M+2}_{M-1} \wedge bu)$ from this and its transpose is zero if and only if the top obstruction for lifting $2^iH_{\epsilon 2^i} \times 2^iH_{k-\epsilon 2^i}$ to E^u_{M+1} is zero if and only if $M(2^i, k-\epsilon 2^i) > 4(k-\epsilon 2^i) - 2\nu(2^i, k-\epsilon 2^i)$ if and only if $M(2^{i+1}, k) > 4k - 2\nu(2^{i+1}, k)$.

Proof of (ii). If $M(2^{e_i}, k_i) > 2k_i$ for all i, this follows from Corollary 2.6(ii). Otherwise, the proof follows by combining the methods of Theorem 3.6(ii) and the case $p = 2^{i+1}$ above.

In order to deduce Theorem 1.8, we need the following lemma, whose proof is similar to that which will be used in Theorem 1.8.

LEMMA 3.9. *For M even, if a map $QP \longrightarrow BSp$ lifts to E^o_M and to E^u_{M-4}, then it lifts to E^o_{M-2}.*

PROOF. In the remainder of this section we shall denote $O^n_k = \Sigma P^n_k \wedge bo$ and $U^n_k = \Sigma P^n_k \wedge bu$. Suppose $M = 4m$. Consider the following commutative diagram:

$$\begin{array}{ccccc} QP^k & \xrightarrow{l} & E^o_{M+1} & \longrightarrow & E^u_{M+1} \\ & & \downarrow{\scriptstyle i_1} & & \\ & & O^M_{M-3} & \xrightarrow{i_3} & U^M_{M-3} \\ & & \downarrow{\scriptstyle i_2} & & \\ & & O^M_{M-2} & & \end{array}$$

The maps of homotopy groups

$$\pi_{4i}(O^M_{M-2}) \longleftarrow \pi_{4i}(O^M_{M-3}) \longrightarrow \pi_{4i}(U^M_{M-3})$$

are $\mathbf{Z}_4 \xleftarrow{g} \mathbf{Z}_8 \xrightarrow{g} \mathbf{Z}_4$ if $i - m$ is even, and $\mathbf{Z}_2 \xleftarrow{g} \mathbf{Z}_2 \xrightarrow{2g} \mathbf{Z}_4$ if $i - m$ is odd. $[i_3 i_1 l] = 0 \in \bigoplus_{i=m}^k \pi_{4i}(U^M_{M-3})$. Thus by Theorem 2.4(ii) $[i_1 l]$ is divisible by 4 in $\bigoplus_{i=m}^k \pi_{4i}(O^M_{M-3})$ and hence $[i_2 i_1 l] = 0$. Thus the lifting to E^o_{M-2} exists. A similar proof works if $M = 4m + 2$.

LEMMA 3.10. (i) *If $M(p, k) = 4m - 2$, then $N(p, k) = 4m$ if there is an $i \leqslant k$ such that $i - m$ is odd and $\nu(p, i) < \nu(\pi_{4i}(\Sigma P_{4m-3} \wedge bu))$, and otherwise $N(p, k) = 4m - 1$.*

(ii) *If $M(p, k) = 4m$, then $N(p, k) = 4m + 2$ if there is an $i \leqslant k$ such that $i - m$ is odd and $\nu(p, i) < \nu(\pi_{4i}(\Sigma P_{4m-1} \wedge bu))$, and otherwise $N(p, k) = 4m$.*

PROOF. If $0 \leqslant \epsilon \leqslant 3$, then

$$\pi_{4i}(\Sigma P_{4m+\epsilon} \wedge bo) = 2(i - m) + \begin{cases} 1 & \text{if } \epsilon = 0 \text{ or } 1 \text{ and } i - m \text{ is odd}, \\ -1 & \text{if } \epsilon = 3 \text{ and } i - m \text{ is odd}, \\ 0 & \text{otherwise}. \end{cases}$$

Thus $\nu(\pi_{4i}(\Sigma P_{4m-2} \wedge bu)) \geqslant (\pi_{4i}(\Sigma P_{4m} \wedge bo))$ and

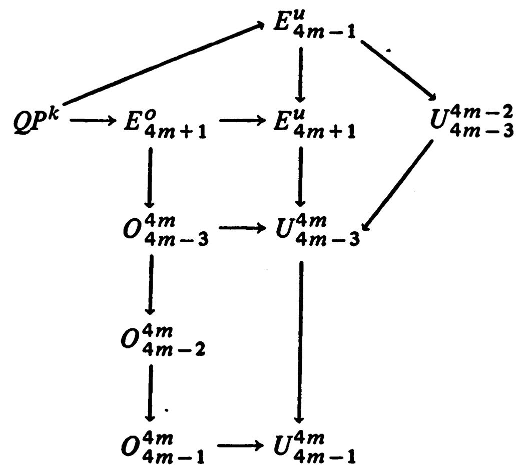
$$\begin{aligned} \nu(\pi_{4i}(\Sigma P_{4m-3} \wedge bu)) &= \nu(\pi_{4i}(\Sigma P_{4m-2} \wedge bo)) \\ &= \nu(\pi_{4i}(\Sigma P_{4m-1} \wedge bo)) + \begin{cases} 1 & \text{if } i - m \text{ even}, \\ 0 & \text{if } i - m \text{ odd}. \end{cases} \end{aligned}$$

(i) follows from this, and (ii) is proved similarly.

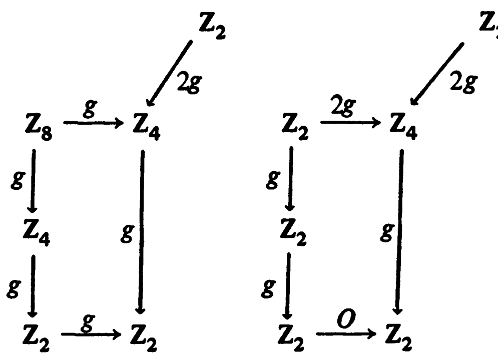
PROOF OF THEOREM 1.8. pH_k lifts to $\widetilde{BSp}_{4k}$ and hence to E^o_{4k}. Thus by using Lemma 3.9, it lifts to $E^o_{M(p,k)+2}$.

If $M(p, k) = 4m - 2$, the obstruction for lifting pH_k from E^u_{4m-1} to E^u_{4m-3} is an element of $\bigoplus_{i=m}^{k} \pi_{4i}(\Sigma P^{4m-2}_{4m-3} \wedge bu)$. We will show below that if this class has a nonzero component for $i - m$ odd, then pH_k does not lift to E^o_{4m-1}, and if its π_{4i}-component is zero for all i such that $i - m$ is odd, then pH_k lifts to E^o_{4m-1} but not to E^o_{4m-2}. Combining this with Theorem 3.1 and Lemma 3.10 proves Theorem 1.8 when $M(p, k) = 4m - 2$.

The claim follows from the commutative diagram:



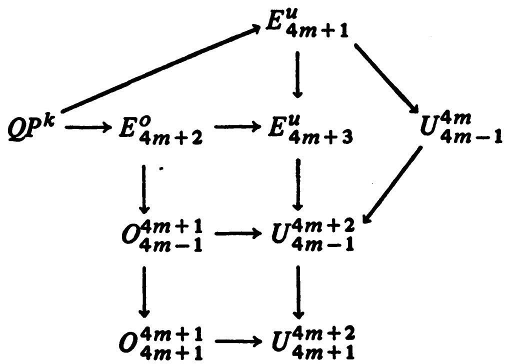
The induced maps in π_{4i} of the O- and U-spaces are



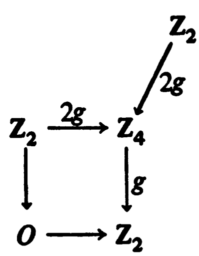
$i - m$ even, $\qquad$ $i - m$ odd.

Since pH_k lifts to E^u_{4m-1}, the class in $\pi_{4i}(U^{4m}_{4m-1})$ is zero. Thus the class in $\pi_{4i}(O^{4m}_{4m-1})$ is zero if $i - m$ is even, while for $i - m$ odd the class in $\pi_{4i}(O^{4m}_{4m-1})$ is zero if and only if the class in $\pi_{4i}(U^{4m-2}_{4m-3})$ is zero. The claims about lifting to E^o_{4m-1} follow, since pH_k lifts to E^o_{4m-1} if and only if the class in $\bigoplus \pi_{4i}(O^{4m}_{4m-1})$ is zero. Since pH_k does not lift to E^u_{4m-1}, the class in $\pi_{4i}(U^{4m-2}_{4m-3})$ is nonzero for some i, and for either parity this shows the class in $\pi_{4i}(O^{4m}_{4m-2})$ is nonzero. Hence pH_k does not lift to E^o_{4m-2}.

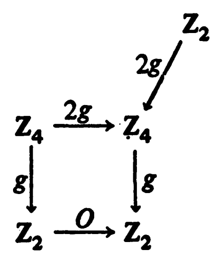
If $M(p, k) = 4m$, the theorem is proved similarly by showing that pH_k lifts exactly to E^o_{4m+2} if the obstruction in $\pi_{4i}(U^{4m}_{4m-1})$ is nonzero for some i such that $i - m$ is odd, and otherwise it lifts exactly to E^o_{4m}. This follows as before from the diagram



where the homomorphisms of π_{4i} are



and



$i - m$ even, $\qquad$ $i - m$ odd.

4. Proof of Theorems 1.3(b) 1.4, and 3.2. The following proposition is easily verified.

PROPOSITION 4.1. (i) $\nu(m, n) = \alpha(n) + \alpha(m - n) - \alpha(m)$;
(ii) $\alpha(2^i - n) = i - \alpha(n - 1)$;
(iii) $\alpha(n - \Delta) = \alpha(n) - \Delta + \Sigma_{j=0}^{\Delta-1}\nu(n - j)$.

THEOREM 4.2. *If $p < 2^i$, $0 \leq l \leq 2^i$, and $0 \leq k - l \leq p$, then $\nu(2^i, l) + \nu(p, k - l) \geq \nu(2^i + p, k)$. For fixed i, p, and k, there is an odd number of l for which equality is achieved.*

PROOF (M. G. BARRATT). By using Proposition 4.1(i) it is immediately verified that

$$\nu(2^i, l) + \nu(p, k - l) = \nu(2^i + p, k) + \nu(k, l) + \nu(2^i + p - k, 2^i - l)$$
$$\geq \nu(2^i + p, k).$$

By comparing coefficients of x^k in $(1 + x)^{2^i}(1 + x)^p = (1 + x)^{2^i+p}$, we see that

$$\sum_l \binom{2^i}{l}\binom{p}{k-l} = \binom{2^i+p}{k},$$

and hence

$$\nu\left(\sum_l \binom{2^i}{l}\binom{p}{k-l}\right) = \nu(2^i + p, k).$$

The result then follows from the observation that $\nu(\Sigma A_l) \geq \min(\nu(A_l))$ with equality if and only if the minimum is attained for an odd number of l.

Theorem 3.2 follows immediately from Theorem 4.2 by induction on the number of e_i.

PROPOSITION 4.3 [15]. *Let ξ_k denote the Hopf bundle over* $\mathbf{R}P^k$. *Then*
(i) $\text{gd}(4p\xi_{4k+3}) \leq \text{gd}(mH_k)$;
(ii) $\mathbf{R}P^k \subseteq \mathbf{R}^{k+l}$ *if and only if* $\text{gd}((2^I - k - 1)\xi_k) \leq l$ *for large I.*

PROOF OF THEOREM 1.4. Let $K = 8k + 7$. Then it suffices to show $\text{gd}((2^I - 2k - 2)H_{2k+1}) \leq 8k + 7 - D$. $\nu(\pi_{4i-1}(P_n \wedge bo))$ is easily seen to be [3, 7] given by the table

n \ i	$2k-2$	$2k-1$	$2k$	$2k+1$
$8k-3$	0	0	3	4
$8k-5$	0	1	4	5
$8k-6$	0	2	4	6
$8k-7$	0	3	4	7
$8k-10$	2	4	6	8

Proposition 4.1 easily implies

$$\nu(2^I - 2k - 2, i) = \begin{cases} \alpha(k) - 1 & \text{if } i = 2k-2, \\ \alpha(k) + \nu(k) + 1 & \text{if } i = 2k-1, \\ \alpha(k) & \text{if } i = 2k, \\ \alpha(k) + 1 & \text{if } i = 2k+1. \end{cases}$$

One can now verify that

$$\nu(2^I - 2k - 2, i) \geq \nu(\pi_{4i-1}(P_{8k+7-D} \wedge bo)) \quad \text{for all } i \leq 2k+1.$$

The condition of Theorem 1.3(b) is easily verified. A similar argument can be made when $K = 8k + 3$.

PROOF OF THEOREM 1.3(b). By observing the tables of [11] and [7], we see that the condition of Theorem 1.1(b) is satisfied for $m \geq k - \{^3_4$ unless m

even, $\epsilon = 1$ or m odd, $\epsilon = 2$ or 3. In these cases we note there is a class h_2^2 at height 2 in dimension

$$n + \begin{cases} 6 \\ 9 \\ 8 \end{cases} \text{if } n \equiv \begin{cases} 1\ (8) \\ 6\ (8), \\ 7\ (8) \end{cases}$$

which is not bo-primary. (Here we use n to correspond to k in [11] and to our $4m + \epsilon$.) This class is not present in the homotopy of $P_{n-\delta}$, where

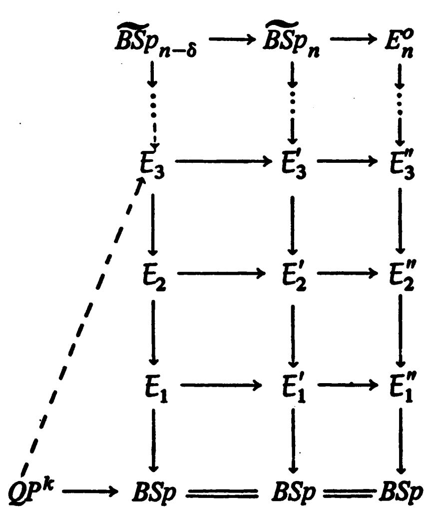
$$\delta = \begin{cases} 4 \\ 1 \\ 2 \end{cases} \text{if } n \equiv \begin{cases} 1 \\ 6. \\ 7 \end{cases}$$

We consider the maps of modified Postnikov towers [6] induced by the maps of fibrations

$$\begin{array}{ccccc} V_{n-\delta} & \longrightarrow & V_n & \longrightarrow & V_n \wedge bo \\ \searrow & & \searrow & & \searrow \\ \widetilde{BSp}_{n-\delta} & \longrightarrow & \widetilde{BSp}_n & \longrightarrow & E_n^o \\ \downarrow & & \downarrow & & \downarrow \\ BSp & = & BSp & = & BSp. \end{array}$$

Let E_i, E_i', E_i'' indicate the spaces obtained in modified Postnikov towers through the dimension $4k$ of QP^k; this will be $\leqslant$

$$n + \begin{cases} 11 \\ 14 \\ 13 \end{cases} \text{if } n \equiv \begin{matrix} 1 \\ 6. \\ 7 \end{matrix}$$



By the assumptions on $\nu(p, i)$, QP^k lifts to E_3 and to E_n^o. But all k-invariants of E_i', $i \geq 3$, come from those of E_i'' and hence map to zero in QP^k. Thus QP^k lifts to $\widetilde{BSp}_n$.

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