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THREE LOCAL CONDITIONS ON A GRADED RING

BY

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ABSTRACT. Let $R = \Sigma_{i \in Z} R_i$ be a commutative graded Noetherian ring with unit and let $A = \Sigma_{i \in Z} A_i$ be a finitely generated graded R module. We show that if we assume that A_M is a Cohen Macaulay R_M module for each maximal graded ideal M of R, then A_P is a Cohen Macaulay R_P module for each prime ideal P of R. With $A = R$ we show that the same is true with Cohen Macaulay replaced by regular and Gorenstein, respectively.

Introduction. The following question was posed by Nagata in [9]: if $R = \Sigma_{i \in Z^+} R_i$ is a commutative graded Noetherian ring with unit such that R_M is Cohen Macaulay for each maximal graded ideal M of R, then is R_P Cohen Macaulay for every prime ideal P of R? Paul Roberts [6], M. Hochster and L. J. Ratliff [4] and the author arrived independently at affirmative answers to the question. An expanded version of the question in the case of a graded ring R, graded by the integers, admitting a finitely generated graded R module A with the property that A_M is Cohen Macaulay for each maximal graded ideal M of R is answered in Theorem 1.1 below.

In §2 the same question is answered with R graded by the integers and Cohen Macaulay replaced by regular. If R is graded by the positive integers and is a regular ring, then R admits a structure theorem in the case that R has a finite number of maximal graded ideals. In Proposition 2.3 we show that R is a direct sum of polynomial rings over semilocal regular domains.

In §3 the question is once again answered when R is graded by the integers, but this time Cohen Macaulay is replaced by Gorenstein. An example is given to show that if $R = \Sigma_{i \in Z^+} R_i$ is a Gorenstein ring, R_0 need not be Gorenstein. In fact in the example R_0 is not even Cohen Macaulay.

Paul Roberts has recently proved these results using methods different from those given below.

The basic references for the ideas that follow are [5], [7] and [8].(1) All rings are commutative Noetherian rings with unit element. $R = \Sigma_{i \in Z} R_i$ will

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(1) The notation is primarily that of [5].

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always be a graded ring and $A = \Sigma_{i \in Z} A_i$ will always be a graded R module. Z will denote the group of integers and Z^+ the set of nonnegative integers. If B is a finitely generated T module and I is an ideal of T, then $G(I, B)$ will denote the length of a maximal R sequence in I on B.

1. **Graded Cohen Macaulay modules.** Let E be a finitely generated module over T, a local ring with maximal ideal M. If $I = \mathrm{ann}_T E = \{r \in T \mid rE = 0\}$, then we denote by $\dim E$ the Krull dimension of T/I. E is a Cohen Macaulay module if either $E = 0$ or if $E \neq 0$ and $\dim E = G(M, E)$.

THEOREM 1.1. *Let A be a finitely generated R module. Suppose A_M is a Cohen Macaulay R_M module for every maximal graded ideal M of R. Then A_P is a Cohen Macaulay R_P module for every prime ideal P of R.*

We say that M is a maximal graded ideal of R if M is maximal among graded ideals of R with respect to avoiding the set of units of R. If R has nonzero homogeneous elements of negative and positive degree, a maximal graded ideal need not be maximal, as simple examples show. Yet such ideals can be characterized easily enough as we show in the lemma below.

LEMMA 1.2. *Let R be a graded ring such that every nonzero homogeneous element is a unit. Then either R is a field or $R = K[u, u^{-1}]$, where K is a field and u is transcendental over K.*

PROOF. If R has no homogeneous elements of nonzero degree, then R is a field. Otherwise, at least R_0 is a field and the set $\Gamma = \{i \in Z \mid R_i \neq 0\}$ is a subgroup of Z. Hence $\Gamma \approx Z^m$ for some positive integer m. Let $0 \neq u \in R_m$. If y is any nonzero homogeneous element of positive degree in R, then $y \in R_{km}$ for some $k > 0$. $yu^{-k} \in R_0 = K$; therefore $y \in K[u]$. Similarly if y is of negative degree, $y \in K[u^{-1}]$. Since any element of R is a sum of homogeneous elements, $R = K[u, u^{-1}]$. It is easily seen that u is transcendental over K.

Two rings of the type described in Lemma 1.2 are those of the form R/M where M is a maximal graded ideal and R_S, where R is a domain and S is the multiplicatively closed set of nonzero homogeneous elements of R. We shall use this latter remark to establish the relationship between a prime ideal P and P^* the prime ideal generated by the homogeneous elements contained in P.

PROPOSITION 1.3. *Let $P \supsetneqq P^*$ be as described above in a graded ring R. If* rank $P = n$ *then* rank $P^* = n - 1, n \geqslant 1$.

PROOF. We proceed by induction on n. If $n = 1$, $P \supsetneqq Q$, where Q is a minimal prime ideal of R. Hence $P^* = Q$, and rank $P^* = 0$. If $n > 1$, then there exists a prime ideal Q such that $P \supsetneqq Q$ and rank $Q = n - 1$. If Q is a graded prime ideal, we are finished since $P^* \supset Q$. So we assume Q is not a graded ideal and $Q \supsetneqq Q^*$. By induction, rank $Q^* = n - 2$. Now $Q^* \subset P^* \subset P$. Hence it will suffice to show that $Q^* \neq P^*$. Otherwise, in $\bar{R} = R/Q^*$, $\bar{P}$ is a prime ideal such that rank $\bar{P} = 2$ and $\bar{P} \cap S = \emptyset$, where S is the set of nonzero homogeneous elements of $\bar{R}$. Hence $\bar{P}$ is a proper prime ideal of $\bar{R}_S$. Yet by Lemma 1.2 the Krull dimension of $\bar{R}_S$ is no larger than 1. This contradiction finishes the proof.

In order to prove Theorem 1.1 we attempt to translate the problem to a suitable polynomial extension of R. As a result we must extend the module A. If X is an indeterminate, the graded $R[X]$ module $A[X] = R[X] \otimes_R A = \Sigma_{i \in Z} B_i$ where $B_k = \Sigma_{i+j=k} A_i X^j$, if we assume deg $X = 1$. In the following proposition we take note of the properties inherited by the $R[X]$ module $A[X]$ from the R module A.

PROPOSITION 1.4. *Let R and A be as in Theorem* 1.1. *Then for each maximal graded ideal M of $R[X]$ the $R[X]_M$ module $A[X]_M$ is Cohen Macaulay.*

PROOF. It is clear that the $R[X]$ module $A[X]$ is finitely generated. Let M be a maximal graded ideal of $R[X]$. $M \cap R = Q$ is a graded prime ideal of R. Now A_Q is a Cohen Macaulay R_Q module and $A[X]_M$ is a localization of $A_Q[X]$. Hence we show that the $R_Q[X]_P$ module $A_Q[X]_P$ is Cohen Macaulay, where $P = M_Q$. Let $u_1, \cdots, u_r$ be a maximal R-sequence in the maximal ideal of R_Q on A_Q. After reducing modulo the ideal generated by this R-sequence, we are in the case where B is a Cohen Macaulay T module, T a local ring, with maximal ideal m, and $G(m, B) = 0 = \dim B$. Now if N is a maximal ideal of $T[X]$ which contains the annihilator of $B[X]$, then $N = (m, f)$ where f is a monic polynomial in X. Since it is monic f cannot be a zero divisor on $B[X]$. Hence $G(N_N, B[X]_N) = 1 = \dim B[X]_N$ and $B[X]_N$ is a Cohen Macaulay $T[X]_N$ module for each maximal ideal N of $T[X]$.

Now suppose $U = R[X_1, \cdots, X_n]$. We shall denote $A[X_1, \cdots, X_n]$ by $A[U]$. We will translate the problem to such a ring U in order to find an R-sequence on $A[U]$ with the right properties. The following proposition shows how to choose the ring U and the appropriate R-sequence.

PROPOSITION 1.5. *Let I be a graded ideal of R with $G(I, A) = k$, A a finitely generated R module. There exists a polynomial extension U of R such that $G(IU, A[U]) = k$ and there exist k homogeneous elements in IU forming a maximal R-sequence on $A[U]$.*

PROOF. Choose homogeneous elements $u_1, \cdots, u_r$ in I forming an R-sequence on A. Let J be the ideal generated by these elements. If $I \subset Z(A/JA)$ then $r = k$ and we take $U = R$. If $I \not\subset Z(A/JA)$, assume there exists no homogeneous element in I that is not a zero divisor on A/JA. Let $f = f_{-m} + \cdots + f_n$ be an element of I not in $Z(A/JA)$. Pass to $R[X]$, $\deg X = +1$. Now $f^* = f_{-m}X^{m+n} + \cdots + f_0X^n + \cdots + f_n$ is a homogeneous element of $IR[X]$ and $f^* \notin Z(A[X]/JA[X])$. Otherwise one could find a nonzero $v \in A/JA$ such that $vf_i = 0$ for each $-m \leqslant i \leqslant n$. Hence $u_1, \cdots, u_r, f^*$ are homogeneous elements in $IR[X]$ forming an R-sequence. If we have not obtained a maximal R-sequence at this point, repeat the above procedure. We are assured of finding a maximal R-sequence of homogeneous elements in some finite polynomial extension of R, since for any such extension U, $G(IU, A[U]) = G(I, A) = k$.

We are now in a position to prove a graded version of Theorem 1.1.

LEMMA 1.6. *With the notation and hypotheses of Theorem* 1.1, *if Q is any graded prime ideal of R and $Q \supset I = \operatorname{ann}_R A$ then $G(Q, A) = \operatorname{rank} Q/I$.*

PROOF. Pass to a polynomial extension U of R such that by Proposition 1.5 there exist homogeneous elements $u_1, \cdots, u_k$ forming a maximal R-sequence in QU on $A[U]$. Let J be the ideal of U generated by the u_i. Now QU is an associated prime ideal of the graded module $A[U]/JA[U]$. Hence there exists a homogeneous element $v \in A[U]$, $v \notin JA[U]$ such that $vQU \subset JA[U]$. Let $K = \{r \in U \mid rv \in JA[U]\}$. K is a proper graded ideal of U. There exists a maximal graded ideal M of U such that $M \supset K$. Since $M \supset K \supset QU \supset IU = \operatorname{ann}_U A[U]$, we have that $A[U]_M \neq 0$. Hence $v_M \notin JA[U]_M$ and $v_M QU_M \subset JA[U]_M$. So $G(QU_M, A[U]_M) \leqslant k$. But $G(QU_M, A[U]_M) \geqslant k$ is always true. Now $A[U]_M$ is a Cohen Macaulay U_M module (Proposition 1.4). Hence $k = G(QU, A[U]) = G(QU_M, A[U]_M) = \operatorname{rank} QU_M/IU_M = \operatorname{rank} Q/I$.

We now prove the theorem.

PROOF OF THEOREM 1.1. Let N be a nongraded prime ideal of R. If $N \not\supset I = \operatorname{ann}_R A$, $A_N = 0$. So suppose $N \supset I$. We wish to show that $G(N, A) = \operatorname{rank} N/I$. Now $N \supsetneqq N^*$, and $\operatorname{rank} N = \operatorname{rank} N^* + 1$ (Proposition 1.3). Since A is a graded module, $N^* \supset I$. Hence

$$k = \operatorname{rank} N^*/I = G(N^*, A) \leqslant G(N, A) \leqslant \operatorname{rank} N/I = \operatorname{rank} N^*/I + 1,$$

by the above lemma and Theorem 27 [7, Chapter 6]. It will suffice to show that $G(N^*, A) < G(N, A)$. Otherwise, pass to a polynomial extension U of R, so that there exist homogeneous elements $u_1, \cdots, u_k$ in N^*U forming a maximal R-sequence on $A[U]$. If J is the ideal generated by the u_i then NU is contained in a maximal associated ideal P of the module $A[U]/JA[U]$. P is a graded prime ideal and by the previous lemma $k = G(P, A[U]) = \operatorname{rank} P/IU$. Yet $\operatorname{rank} NU/IU = k + 1$. This contradiction finishes the proof.

2. Graded regular rings. In this section we prove a theorem analogous to the one proved in the previous section.

THEOREM 2.1. *Let R be such that R_M is a regular local ring for each maximal graded ideal M of R. Then R_N is a regular local ring for each maximal ideal N of R.*

Before beginning the proof we take note of localizations at multiplicative sets of homogeneous elements. If T is a graded ring and S a multiplicative set of homogeneous elements then T_S has an obvious graded structure, $T_S = \Sigma_{i \in Z} T_i$ where $T_i = \{a/b \mid a, b$ are homogeneous elements of T, $b \in S$ and $\deg a - \deg b = i\}$. If S is the set of homogeneous elements that do not belong to a graded prime ideal P of T then P_S is the unique maximal graded ideal of T_S. It is easy to see that if T_P is a domain and P and S are as above then T_S is a domain. We now prove the theorem.

PROOF. In order to show that R_N is a regular local ring, we will show that T_P is a regular local ring, where $T = R_S$, $P = N_S$ and S is the set of homogeneous elements that do not belong to N^*. If we let $M = N_S^*$, then M is the unique maximal graded ideal of T. What we will actually show is that T_U is a regular local ring for each maximal ideal U of T, and we shall proceed by induction on the rank of M in T. Now T_M is a regular local ring, since it is a localization of R_Q for some maximal graded ideal Q of R. So if rank $M = 0$ then T_M is a field. It follows that T is a ring as described in Lemma 1.2. Hence T_U is a regular local ring for each maximal ideal U of T. So suppose rank $M = n > 0$ and Q is any maximal ideal of T. Now $Q^* \subset M$, and if this inclusion is proper, then by localizing at the homogeneous elements that do not belong to Q^* in T we can reduce to a graded ring with unique maximal graded ideal which has rank less than n. Since T_Q is a localization of that graded ring, T_Q is a regular local ring by induction. So suppose that $Q^* = M$. Now choose an $x \in M - M^2$, $x \neq 0$, such that x is homogeneous. Let $\overline{T} = T/(x)$, $\overline{M} = M/(x)$ and $\overline{Q} = Q/(x)$. $\overline{T}_{\overline{M}}$ is a regular local ring and rank $\overline{M} = n - 1$. By induction $\overline{T}_{\overline{Q}}$ is a regular local ring. Hence T_Q is a regular local ring.

Examples of rings with the properties described in the theorem, graded regular rings, are polynomial rings over regular rings and symmetric algebras with a natural grading of the form $S_T(I)$, where T is a regular ring and I is a nonprincipal invertible ideal of T.

We conclude this section by showing that a class of graded regular rings are polynomial rings. D. K. Harrison [3] has obtained this result in the case that R_0 is a field.

PROPOSITION 2.2. *Let $R = \Sigma_{i \in Z^+} R_i$ be a graded regular ring with a unique maximal graded ideal M. Then R_0 is a regular local ring and $R = R_0[u_1, \cdots, u_k]$, where the u_i are homogeneous elements of R and independent indeterminates over R_0.*

PROOF. The proof is by induction on $n = \operatorname{rank} M = \text{Krull dim } R$. (That $\operatorname{rank} M = \text{Krull dim } R$ is an easy deduction from Proposition 1.3 and the fact that in this case M is a maximal ideal of R.) If $n = 0$, $R = R_0$ is a field. If $n > 0$, choose a nonzero homogeneous element x of positive degree in $M - M^2$ (if no such x exists one can show that $R = R_0$). Let $\bar{R} = R/(x)$ and $\bar{M} = M/(x)$. Now $\bar{R}$ is a graded regular ring, since $\bar{R}_{\bar{M}}$ is a regular local ring by choice of x. Now $n - 1 = \operatorname{rank} \bar{M} = \text{Krull dim } \bar{R}$; hence by induction $\bar{R}_0$ is a regular local ring and $\bar{R} = \bar{R}_0[\bar{u}_1, \cdots, \bar{u}_k]$. Now $\bar{R}_0 = R_0$, and if Krull dim $R_0 = d$ then $d + k = n - 1$. Since the mapping of R to $\bar{R}$ is a graded map choose homogeneous pre-images u_i of the $\bar{u}_i$. Letting $u_{k+1} = x$ then it is easy to see that $(u_1, \cdots, u_{k+1}) = \Sigma_{i>0} R_i$. Hence $R = R_0[u_1, \cdots, u_{k+1}]$. Lastly let $T = R_0[X_1, \cdots, X_{k+1}]$ be mapped to R in the obvious fashion, preserving degrees, where the X_i are indeterminates. Now since $\operatorname{rank} M = \operatorname{rank} \bar{M} + 1$, Krull dim $R = k + d + 1$. Since this is also the Krull dimension of T, the above mapping is an isomorphism and this finishes the proof.

Using the above proposition we show that a graded regular ring with a finite number of maximal graded ideals is a direct sum of polynomial rings.

PROPOSITION 2.3. *Let $R = \Sigma_{i \in Z^+} R_i$ be a graded regular ring with a finite number of maximal graded ideals. Then R is the direct sum of a finite number of graded regular domains, each with a finite number of maximal graded ideals. If $T = \Sigma_{i \in Z^+} T_i$ is a graded regular domain with a finite number of maximal graded ideals, then $T = T_0[u_1, \cdots, u_k]$ where the u_i are homogeneous elements of T and independent indeterminates over T_0.*

PROOF. Since R is a regular ring, R is the direct sum of a finite number of regular domains. Since the idempotents are homogeneous elements of degree zero in R, the first assertion follows. As for the second assertion we proceed by induction on Krull dim $T = n$. If $n = 0$, $T = T_0$ is a field. If $n > 0$ and $T \neq T_0$ let T_j be the nonzero T_0 module of least positive degree in T. Since by the previous proposition T_j is a locally free, finitely generated T_0 module, T_j is a free T_0 module. Let x be a free generator of T_j. Now x is in the square of no maximal graded ideal of T. Hence $\bar{T} = T/(x)$ is a graded regular ring of Krull dimension $n - 1$. $\bar{T}$ is a domain since $\bar{T}_0 = T_0$ and T_0 has no idempotents. By induction $\bar{T} = \bar{T}_0[\bar{u}_1, \cdots, \bar{u}_k]$, where if the Krull dim $T_0 = d$ then $d + k = n - 1$.

Letting u_i be the homogeneous preimages of the $\bar{u}_i$ in T and $u_{k+1} = x$, then it is easy to see that $T = T_0[u_1, \cdots, u_{k+1}]$. Let $U = T_0[X_1, \cdots, X_{k+1}]$ and map U to T in the obvious way preserving degrees, where the X_i are independent indeterminates over T_0. Then since Krull dim $T = d + k + 1 =$ Krull dim U, the above mapping is an isomorphism and this finishes the proof.

That in the above propositions the graded rings considered are Z^+ graded is essential as the following example shows. If $R = K[x, y, z, w]$ where K is a field and x, y, z, w are independent indeterminates over K, then R is a Z graded ring by letting $\deg x = \deg y = +1$ and $\deg z = \deg w = -1$. R is clearly a regular ring, yet $R_0 = K[xz, xw, yz, yw]$ is not a regular ring and R cannot be a polynomial ring over R_0. Notice that $m = (x, y, z, w)$ contains all homogeneous elements in R of degree different from zero. Hence, if we localize R at the set of homogeneous elements that do not belong to m and call this new ring T, then the subring of homogeneous elements of T of degree zero, T_0, is just R_0 localized at the ideal (xz, xw, yz, yw). T is a regular Z graded ring with a unique maximal graded ideal, which cannot be a polynomial ring over T_0.

3. **Graded Gorenstein rings.** A local ring T is said to be Gorenstein if T as a module over itself has finite injective dimension. As a result T is Cohen Macaulay and any system of parameters of T generates an irreducible ideal. These conditions are also sufficient for T to be Gorenstein. The basic references for this material are [5, Chapter 4, §5] and Bass's *Ubiquity* paper [1].

THEOREM 3.1. *Let R be such that R_M is Gorenstein at every maximal graded ideal M of R. Then R_N is Gorenstein at each maximal ideal N of R.*

The plan of proof is much the same as that of the proof of the theorem in the first section. We shall translate the problem to a suitable polynomial extension of R and make use of Proposition 1.5.

LEMMA 3.2. *Let R be a graded ring with the property given in Theorem 3.1. Then $R[X]$ also has this property, X an independent indeterminate of positive degree over R.*

PROOF. If N is a maximal graded ideal of $R[X]$ and $N \cap R = Q$, then $R[X]_N$ is a localization of $R_Q[X]$. But R_Q is Gorenstein and any localization of $R_Q[X]$ is also Gorenstein.

PROPOSITION 3.3. *Let R be as in the above theorem. Suppose $I = (y_1, \cdots, y_r)$ is an ideal of* rank *r in R, generated by r homogeneous elements.*

Then I is unmixed and each primary component of I is irreducible.

PROOF. If R has the property stated in the above theorem then R_N is Cohen Macaulay at each maximal ideal N of R. Hence I is unmixed. Let P be a prime ideal minimal over I. P is a graded ideal and the P-primary component q belonging to I is $IR_P \cap R$. Now letting $\bar{R} = R/I$ and $\bar{P} = P/I$, $\bar{R}_{\bar{P}} = R_P/IR_P$ has zero Krull dimension and is Gorenstein. Hence the zero ideal of $\bar{R}_{\bar{P}}$ is irreducible. Then since IR_P is irreducible in R_P, q is irreducible in R.

We now sketch some technical lemmas concerning homogenization in general graded rings. Let R be a graded ring and define the operation ^{h} on the elements of R: if $f = f_{-m} + \cdots + f_n \in R$ then $f^h = f_{-m}X^{m+n} + \cdots + f_0X^n + \cdots + f_n \in R[X]$, deg $X = 1$. The corresponding operation ^{a} on the elements of $R[X]$ is as follows: if $u = u_{-k}X^{k+j} + \cdots + u_0X^j + \cdots + u_j$ is a homogeneous element of degree j in $R[X]$ then $u^a = u_{-k} + \cdots + u_0 + \cdots + u_j$. If u and v are homogeneous elements of $R[X]$, then it is easy to see that $(uv)^a = u^av^a$, and if u and v have the same degree then $(u + v)^a = u^a + v^a$. If $f \in R$ then $(f^h)^a = f$, and if u is a homogeneous element of $R[X]$ and $k = \deg u - \deg(u^a)^h$ then $(u^a)^hX^k = u$. If f and g are elements of R define $d(f)$ and $d(g)$ to be the degrees of the constituent homogeneous elements of highest degree in a representation of f and g respectively. If $k = d(f) + d(g) - d(fg)$ then $X^k(fg)^h = f^hg^h$. If $k = \max(d(f), d(g)) - d(f + g)$ then $X^k(f + g)^h = f^h + X^lg^h$, where if $d(f) > d(g)$ then $l = d(f) - d(g)$.

The operation ^{h} can be extended to ideals so that if I is an ideal of R then I^h is a graded ideal of $R[X]$ and each homogeneous element of I^h is of the form X^mf^h where $m > 0$ and $f \in I$. The usual propositions (like those listed in [10, Chapter VII, §5, Theorem 17]) are true in this more general situation. We just note the following: If I and J are ideals of R and if $I \supsetneqq J$ then $I^h \supsetneqq J^h$; if I is a prime ideal and J is I-primary then I^h is prime and J^h is I^h-primary: if $I = \bigcap_{i=1}^n J_i$ is an irredundant primary representation of I then $I^h = \bigcap_{i=1}^n J_i^h$ is an irredundant primary representation of I^h. One final note: if (f) is a principal ideal of R then $(f)^h$ need not be a principal ideal of $R[X]$ if R has zero divisors. But $(f)_S^h$ is a principal ideal generated by f^h in $R[X]_S$ where $S = \{1, X, X^2, \cdots\}$.

We now prove Theorem 3.1.

PROOF. Let N be any nongraded maximal ideal of R. We shall ultimately show that R_N is Gorenstein. Let $N \supsetneqq N^*$. By Proposition 1.5 there exists a polynomial extension T of R such that there exist homogeneous elements $u_1, \cdots, u_k$ in T that form a maximal R-sequence in N^*T, where $k = G(N^*T, T) =$ rank N^*T. Let I be the ideal generated by these elements in T. In $U = T/I$, let $Q = NT/I$ and $Q^* = N^*T/I$. U has the local Gorenstein property at its maximal graded ideals and since rank $N =$ rank $N^* + 1$, rank $Q = 1$ and rank $Q^* = 0$. Since U is locally Cohen Macaulay, $G(Q) = 1$. Hence let $y = y_{-m} + \cdots + y_n$ be a nonzero divisor in Q. In $U[X]$, deg $X = 1$, y^h is a homogeneous nonzero divisor. In $U[X]_S$, $(y)_S^h = (y^h)_S$, $S = \{1, X, X^2, \cdots\}$. By Proposition 3.3 every primary component of $(y^h)_S$ is irreducible. It follows that every primary component of (y) is irreducible. Now the zero ideal of the zero Krull dimensional local ring U_Q/yU_Q is irreducible. Hence U_Q/yU_Q is Gorenstein. Then T_{NT} is Gorenstein. Since T_{NT} is a faithfully flat R_N module it follows that R_N is Gorenstein.

We conclude with an example of a graded ring which is locally Gorenstein such that the subring of homogeneous elements of degree 0 is not Cohen Macaulay.

EXAMPLE 3.4. Let $T = k[X, Y]/(X^2, XY)$, where k is any field. If $q_1 = (X^2, Y)$ and $q_2 = (X)$, then in T the image of $q_1 \cap q_2$ is the zero ideal. Let W, V be indeterminates, each assigned degree one. Let $S = T[W]$, $Q_1 = q_1S$, $Q_2 = q_2S + WS$ and $I = Q_1 \cap Q_2$. Let $U = S/I$. Letting small letters denote the images in U, our example is $R = U[V]/(xV + yV, V^2)$. Notice that R is a graded ring of Krull dimension one, and its subring of degree zero $R_0 = T$ is not Cohen Macaulay. We claim that R_N is Gorenstein, N the unique maximal graded ideal of R. Now $R \approx k[A, B, C, D]/J$, where $J = (A^2, AB, BC, AD + BD, D^2)$, under the mapping X, Y, W, V to A, B, C, D respectively. Now $B + C$ is not a zero divisor on J. Otherwise, if $g(B + C) \in J$, one can show that $g = \beta BD$ for some β. But if $g(B + C) \in J$ then $gB = \beta B^2D \in J$. Now if $\beta \notin J$ then β is a power of B. But now $B^sD \notin J$ for any s. Finally $k[A, B, C, D]/(J, B + C)$ is a zero dimensional Gorenstein local ring, and hence R_N is Gorenstein.

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