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TREES OF HOMOTOPY TYPES OF 2-DIMENSIONAL CW COMPLEXES. II

BY

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ABSTRACT. A π-complex is a finite, connected 2-dimensional CW complex with fundamental group π. The tree $\mathrm{HT}(\pi)$ of homotopy types of π-complexes has width $\leqslant N$ if there is a root Y of the tree such that, for any π-complex X, $X \vee (\bigvee_{i=1}^{N} S_i^2)$ lies on the stalk generated by Y. Let π be a finite abelian group with torsion coefficients $\tau_1, \cdots, \tau_n$. The main theorem of this paper asserts that width $\mathrm{HT}(\pi) \leqslant n(n-1)/2$. This generalizes the results of [4].

1. **Introduction.** Let π be a finitely presentable group. A π-complex is a finite connected 2-dimensional CW complex with fundamental group π. In [4], we gave a complete classification of the homotopy and simple homotopy types of Z_n-complexes, where Z_n is the finite cyclic group of order n. In general, we may describe the set of (simple) homotopy types of π-complexes (S)HT(π) as a directed tree—a directed, connected graph which has no circuits. A vertex of (S)HT(π) is the (simple) homotopy type $[X]$ of a π-complex X. The vertices represented by X and Y are joined by an edge directed from $[X]$ to $[Y]$ if and only if $Y \simeq_{(\mathrm{S})} X \vee S^2$. A π-complex is called a *root* if $[X]$ possesses no predecessor; the *stalk* generated by X is the linearly ordered subgraph of (S)HT(π) determined by the (simple) homotopy types of X, $X \vee S^2$, $X \vee S^2 \vee S^2, \cdots$.

The main theorem of [4] states that (S)HT(Z_n) is a single stalk generated by the pseudo projective plane $P_n = S^1 \cup_n e^2$. We say that the *width of* (S)HT$(\pi) \leqslant n$ if there is a root X such that, for any π-complex Y, $Y \vee (\bigvee_{i=1}^{n} S_1^2)$ is on the stalk generated by X.

It is known by the simple homotopy theory of J. H. C. Whitehead [14] that given any π-complex Y and any root X there is an integer $m(Y)$ such that $Y \vee (\bigvee_{i=1}^{m(Y)} S_i^2)$ is on the stalk generated by X. Width(S)HT$(\pi) \leqslant n$ indicates that there is a root X such that $m(Y)$ can be chosen $\leqslant n$ for any π-complex Y.

THEOREM A. *Let π be a finite abelian group, $n = n(\pi) =$ the number of*

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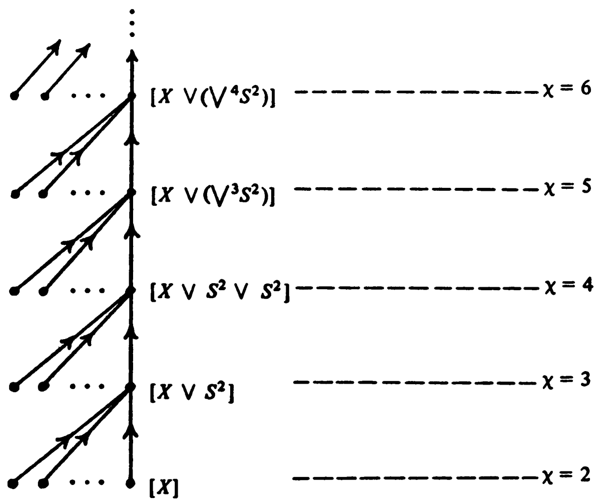
torsion coefficients of π, *and* $k = k(\pi) = n(n-1)/2$. *Then the* width $\mathrm{HT}(\pi) \leqslant k(\pi)$.

If p is any positive integer, Theorem A implies that width $\mathrm{HT}(Z_p)$ is zero, which is the result of [4]. If $\pi = Z_p \times Z_q$, where p divides q, then width $\mathrm{HT}(\pi) \leqslant 1$. In this case, the homotopy tree of $Z_p \times Z_q$-complexes looks *at worst* like:



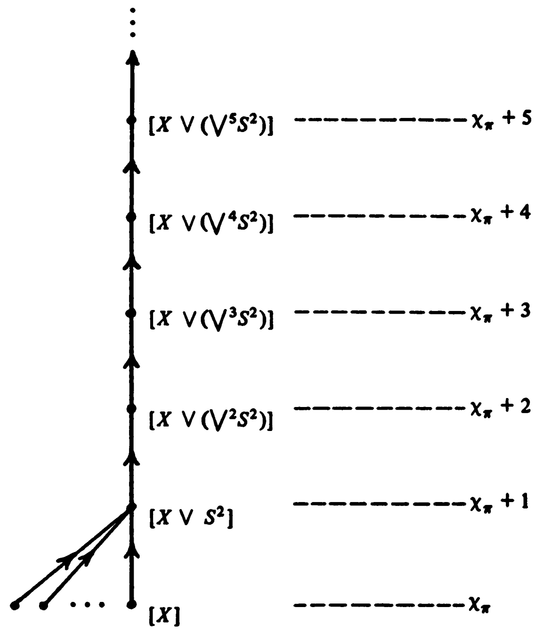
where X is the cellular model [4] of the presentation $(a, b\colon a^p, b^q, aba^{-1}b^{-1})$ and the horizontal levels represent the vertices with common Euler characteristic. At the present time it is unknown whether any of the other "branches" exist. However, at a given level $\chi \geqslant 3$, there are only finitely many branches. See Theorem B.

As a corollary to A, we obtain a theorem on the *cancellation of "large" sums of* 2-*spheres with* π-*complexes*. If π is a finite abelian group and X, Y are π-complexes, then $X \vee (\bigvee_{i=1}^{s} S_i^2) \simeq Y \vee (\bigvee_{i=1}^{t} S_i^2)$ and $s \geqslant t \geqslant k(\pi)$ imply that $\bigvee_{i=1}^{t-k(\pi)} S_i^2$ can be cancelled from each side (up to homotopy type).

For a given finite group π let $\chi_\pi = \min\{\chi(X) \mid X \text{ is a } \pi\text{-complex}\}$, $|\pi|$ be the order of π, and φ be the Euler φ-function.

THEOREM B. *Let* π *be a finite group other than* Z_2. *The number of homotopy types of* π-*complexes with fixed Euler characteristic* $\chi \geqslant \chi_\pi + 1$ *is less than or equal to* $\varphi(|\pi|)/2$.

EXAMPLES. (a) If $\pi = Z_2 \times Z_2$, then Theorems A and B imply that the tree of (simple) homotopy types looks at worst like:



where X is the complex modeled on $(a, b: a^2, b^2, [a, b])$.

(b) If $\pi = \Sigma_3$, the group (of order 6) of permutations on 3 letters, then $\mathrm{HT}(\Sigma_3)$ looks at worst like the above tree, where X is a root of $\mathrm{HT}(\Sigma_3)$ of minimal Euler characteristic. The complex X modeled on the presentation $\{a, b: b^2, bab = a^2\}$ is such a root, since $H_2X = 0$ [16].

2. **The chain functor.** In [4], we associated with each finite presentation $\mathcal{P} = (a_1, \cdots, a_n : r_1, \cdots, r_m)$ of a group π, its cellular model

$$P = \left(\bigvee_{i=1}^{n} S_i^1\right) \cup_r \left(\bigvee_{j=1}^{m} B_j^2\right),$$

which has a single 0-cell, one 1-cell for each generator of $\mathcal{P}$, and one 2-cell for each relator of $\mathcal{P}$. The jth 2-cell is attached to the 1-skeleton $\bigvee_{i=1}^n S_i^1$ according to the instructions provided by the jth relator r_j.

Then we associated with the cellular model P the cellular chain complex $C_*(\widetilde{P})$ of its universal covering $\widetilde{P}$. $C_*(\widetilde{P})$ is a chain complex of free π-modules with preferred bases

$$(*)\quad C\colon C_2(y_1, \cdots, y_m) \xrightarrow{\partial_2} C_1(x_1, \cdots, x_n) \xrightarrow{\partial_1} C_0 = Z[\pi] \xrightarrow{\epsilon} Z \longrightarrow 0$$

in which

(a) ϵ is the augmentation homomorphism $Z[\pi] \longrightarrow Z[1]$ induced by $\pi \longrightarrow 1$.

(b) Exactness holds at C_1, C_0, Z.

(c) $\{y_1, \cdots, y_m\}$ and $\{x_1, \cdots, x_n\}$ are the preferred bases for C_2 and C_1.

We can combine these two processes $P \longrightarrow P$ and $P \longrightarrow C_*(\widetilde{P})$ as follows. If $P = (a_1, \cdots, a_n : r_1, \cdots, r_m)$ is a presentation for π, let

$$1 \longrightarrow R_P \longrightarrow F(a_1, \cdots, a_n) \xrightarrow{\varphi_P} \pi \longrightarrow 1$$

be the short exact sequence in which $F = F(a_1, \cdots, a_n)$ is the free group of rank n on generators $\{a_1, \cdots, a_n\}$ and R_P is the normal closure of the relators $\{r_1, \cdots, r_m\}$. The elements $\bar{x}_i = \varphi_P(a_i)$ $(1 \leqslant i \leqslant n)$ serve as a set of generators for π. We associate a chain complex $C_*(P)$ as follows. Let $C_2(P) = C_2(y_1, \cdots, y_m)$ and $C_1(P) = C_1(x_1, \cdots, x_n)$ be free π-modules with preferred bases $\{y_1, \cdots, y_m\}$ and $\{x_1, \cdots, x_n\}$ in 1-1 correspondence with the relators and generators of P, *respectively*. Let $C_0(P)$ be the integral group ring $Z[\pi]$. Then $C_*(P)$ is the chain complex

$$C_*(P)\colon C_2(y_1, \cdots, y_m) \xrightarrow{\partial_2(P)} C_1(x_1, \cdots, x_m) \xrightarrow{\partial_1(P)} Z[\pi] \xrightarrow{\epsilon} Z \longrightarrow 0$$

whose boundary operators have the following matrix representations with respect to the preferred bases:

$$\partial_1(P) = (\bar{x}_1 - 1, \cdots, \bar{x}_n - 1) \quad \text{and} \quad \partial_2(P) = (Z[\varphi_P](\partial r_j/\partial a_i))$$

where $\partial/\partial a_i\colon Z[F] \longrightarrow Z[F]$ is the derivative with respect to a_i in the free calculus of R. H. Fox [5] and $Z[\varphi_P]\colon Z[F] \longrightarrow Z[\pi]$ is induced by $\varphi_P\colon F \longrightarrow \pi$.

For example, let $P = (a_1, a_2\colon a_1 a_2 a_1^{-1} a_2^{-1})$ be a presentation for $\pi = Z \times Z$ under the correspondence $\varphi_P(a_1) = \bar{x}_1 = (1, 0)$ and $\varphi_P(a_2) = \bar{x}_2 = (0, 1)$. Then the associated chain complex $C_*(P)$ takes the form

$$C_2(y_1) \xrightarrow{\begin{pmatrix} 1 - \bar{x}_2 \\ \bar{x}_1 - 1 \end{pmatrix}} C_1(x_1, x_2) \xrightarrow{(\bar{x}_1 - 1, \bar{x}_2 - 1)} Z[Z \times Z] \xrightarrow{\epsilon} Z \longrightarrow 0.$$

DEFINITION. We say that a chain complex C as in $(*)$ above is realized by a presentation P of π if $C_*(P) = C$.

3. The homomorphism ρ. Given a finite presentation $P = (a_1, \cdots, a_n : r_1, \cdots, r_m)$ of π there is a surjective group homomorphism ρ from the relator subgroup R_P onto the free abelian group $\ker \partial_1(P)$ (a π-module also) $\subset C_1(P)$ which has kernel $[R_P, R_P]$.

Following J. H. C. Whitehead in [13] we define the *crossed homomorphism*

$$\bar{\rho}\colon F(a_1, \cdots, a_n) \longrightarrow C_1(P) \equiv C_1(x_1, \cdots, x_n),$$

where F is the free group of rank n, by

(a) $\bar{\rho}(a_i) = x_i$,

(b) $\rho(a_i^{-1}) = -\varphi_P(a_i^{-1}) x_i$,

(c) if W_1, W_2 are any words in F, then $\bar{\rho}(W_1 \cdot W_2) = \bar{\rho}(W_1) + \varphi_P(W_1) \cdot \bar{\rho}(W_2)$.

Recall that $\varphi_P: F \longrightarrow \pi$ is the surjection given by the presentation P. Note that by (c), $\bar{\rho}|_{R_P} \equiv \rho$ is a homomorphism. Also, if $r \in R_P$, then

$$\rho(r) = \sum_{i=1}^{n} \left(Z[\varphi_P] \frac{\partial r}{\partial a_i} \right) x_i.$$

LEMMA. *The following sequence is exact*:

$$1 \longrightarrow [R_P, R_P] \xrightarrow{i} R_P \xrightarrow{\rho} \ker \partial_1(P) \longrightarrow 0.$$

PROOF. This is really a restatement of Theorem 8 of [13]. Part (a) of Theorem 8 says that $\rho(R_P) = \ker \partial_1(P)$. Part (b) says that $\ker \rho = \ker \bar{\rho} =$ image of the commutator subgroup of $\pi_2(P, P^{(1)})$ in $R_P \subset \pi_1(P^{(1)})$ $(= F(a_1, \cdots, a_n))$ under the boundary operator $\partial: \pi_2(P, P^{(1)}) \longrightarrow \pi_1(P^{(1)})$. Since $\operatorname{im} \partial = R_P$, $\ker \rho \subset [R_P, R_P]$. But $\ker \rho \supset [R_P, R_P]$ follows because $\ker \partial_1(P)$ is abelian as a group. □

4. **Proof of Theorem A.** Let $n = n(\pi)$ be the number of torsion coefficients of the finite abelian group π. Let $\{\tau_1, \cdots, \tau_n\}$ be the torsion coefficients of π, where $\tau_i | \tau_{i+1}$ for $i = 1, 2, \cdots, n-1$, and $k = k(\pi) = n(n-1)/2$. Furthermore, let P be the π-complex modeled on the standard presentation

$$P = (a_1, \cdots, a_n : a_1^{\tau_1}, \cdots, a_n^{\tau_n}, \{[a_i, a_j]\ 1 \leqslant i < j \leqslant n\}).$$

Note that $k(\pi)$ is the number of commutators in P and that P is a root of (S)HT(π) (see [15]). We will show that if X is any π-complex, then $X \vee (\bigvee_{i=1}^{k(\pi)} S_i^2)$ is on the stalk generated by P; i.e.,

$$X \vee \left(\bigvee_{i=1}^{k(\pi)} S_i^2 \right) \simeq P \vee \left(\bigvee_{j=1}^{D(X)} S_j^2 \right)$$

where $D(X) = \operatorname{rank} H_2(X)$.

The given π-complex X has the simple homotopy type of a π-complex R modeled on the "pre-abelian" presentation

$$R = (b_1, \cdots, b_l : b_1^{\tau_1} W_1, b_2^{\tau_2} W_2, \cdots, b_n^{\tau_n} W_n, b_{n+1} W_{n+1}, \cdots, b_l W_l, W_{l+1}, \cdots, W_m)$$

where each W_i $(i = 1, \cdots, m)$ has zero exponent sum on each b_j $(j = 1, \cdots, l)$ [4, Proposition 3]. Notice that $R \vee (\bigvee^{k(\pi)} S^2)$ has the simple homotopy type of the π-complex S modeled on the presentation

$$S = (b_1, \cdots, b_l : b_1^{\tau_1} W_1, \cdots, b_n^{\tau_n} W_n; b_{n+1} W_{n+1}, \cdots, b_l W_l; \\ W_{l+1}, \cdots, W_m; \{[b_i, b_j] | 1 \leqslant i < j \leqslant n\}).$$

Observe that in passing from $\mathcal{R} \longrightarrow \mathcal{S}$ we have added *only* those commutators corresponding to the nontrivial generators $b_1, \cdots, b_n$.

Let

$$1 \longrightarrow R_{\mathcal{S}} \longrightarrow F(b_1, \cdots, b_l) \xrightarrow{\varphi_{\mathcal{S}}} \pi \longrightarrow 1$$

be the short exact sequence of groups and homomorphism determined by $\mathcal{S}$. Denote $\varphi_{\mathcal{S}}(b_i)$ by $\bar{x}_i$ $(i = 1, \cdots, n)$ and note that, since π is abelian, $\varphi_{\mathcal{S}}(b_i) = 1$ $(n + 1 \leqslant i \leqslant l)$. The chain complex $C_*(\mathcal{S})$ is given as follows:

$$\begin{array}{ccc} & C_2(\mathcal{S}) & & C_1(\mathcal{S}) & \\ & \| & & \| & \\ C_*(\mathcal{S}): & C_2(y_1, \cdots, y_m; z_{12}, z_{13}, \cdots, z_{n-1,n}) & \xrightarrow{\partial_2(\mathcal{S})} & C_1(x_1, \cdots, x_l) & \\ & & & \xrightarrow[\partial_1(\mathcal{S})]{} Z[\pi] \xrightarrow{\epsilon} Z \longrightarrow 0 & \\ & & & \| & \\ & & & (\bar{x}_1 - 1, \cdots, \bar{x}_n - 1, 0, \cdots, 0) & \end{array}$$

where $\{z_{ij} | 1 \leqslant i < j \leqslant n\}$ corresponds to the set of special relators $\{[b_i, b_j] | 1 \leqslant i < j \leqslant n\}$. Thus

$$\partial_2(z_{ij}) = (1 - \bar{x}_j)x_i + (\bar{x}_i - 1)x_j \qquad (1 \leqslant i < j \leqslant n).$$

Let $\widetilde{Z}$ denote the $n \times k$ matrix of ∂_2 restricted to $\langle z_{12}, z_{13}, \cdots, z_{n-1,n} \rangle$, the submodule of $C_2(\mathcal{S})$ generated by $\{z_{ij} | 1 \leqslant i < j \leqslant n\}$.

By examining the chain complex $C_*(\mathcal{P})$, it follows that $\ker(\partial_1(\mathcal{S})) \cong \ker(\partial_1(\mathcal{P})) \oplus \langle x_{n+1}, \cdots, x_l \rangle$ (we will henceforth identify $\ker \partial_1(\mathcal{P})$ as a submodule of $\langle x_1, \cdots, x_n \rangle \subset C_1(\mathcal{S})$) and that $\ker \partial_1(\mathcal{P})$ is generated by $\{N_i x_i | i = 1, \cdots, n\} \cup \{\partial_2 z_{ij} | 1 \leqslant i < j \leqslant n\}$, where $N_i = \sum_{j=0}^{\tau_i - 1} \bar{x}_i^j \in Z[\pi]$. Note also that, since $\mathcal{R}$ is a presentation of π with the same generators as $\mathcal{S}$, $\{\partial_2 y_i | i = 1, \cdots, m\}$ generates $\ker \partial_1(\mathcal{S}) = \ker \partial_1(\mathcal{R})$.

As in [4, §3], we use H. Jacobinski's theorem on the cancellation of projective π-modules (see [7], [11, Theorem 19.8], or [12, p. 178]) to choose a new basis $\{y'_1, \cdots, y'_m\} \cup \{z_{ij}\}$ for $C_2(\mathcal{S})$ such that the set $\{\partial_2 y'_i | i = 1, \cdots, n, l + 1, \cdots, m\}$ generates $\ker \partial_1(\mathcal{P})$ and $\partial_2 y'_j = x_j$ for $j = n + 1, \cdots, l$. The matrix for $\partial_2(\mathcal{S})$ with respect to the new basis for $C_2(\mathcal{S})$ and the original basis for $C_1(\mathcal{S})$ becomes

$$
\begin{array}{c c}
 & \begin{array}{cccc} \hphantom{?}\quad n & \quad l & \quad m & \quad m+k \end{array} \\
n & \left(\begin{array}{c|c|c|c}
? & 0 & ? & \tilde{Z} \\
\hline
0 & \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & 0 & 0
\end{array}\right)
\end{array}
$$

Let $\psi: \bar{F}(b_1, \cdots, b_n) \longrightarrow \pi$ be the surjection $\varphi_S | \bar{F}(b_1, \cdots, b_n)$ and let $\bar{R} = \ker \psi$. Since the homomorphism $\rho: \bar{R} \longrightarrow \ker \partial_1(P)$ is surjective, we can choose relators $\{r_1, \cdots, r_n, r_{l+1}, \cdots, r_m\} \subset \bar{R}$ such that

$$\rho(r_i) = \sum_{j=1}^{n} \left(Z[\psi]\left(\frac{\partial r_i}{\partial b_j}\right)\right) x_j = \partial_2 y_i \qquad (i = 1, \cdots, n, l+1, \cdots, m).$$

Here it is *crucial* that $\partial_2 y_i \in \langle x_1, \cdots, x_n \rangle$ $(i = 1, \cdots, n, l+1, \cdots, m)$.

CLAIM. Each r_i can be written as

$$r_i = b_1^{\beta_{i1}\tau_1} b_2^{\beta_{i2}\tau_2} \cdots b_n^{\beta_{in}\tau_n} W_i \qquad (i = 1, \cdots, n, l+1, \cdots, m)$$

where W_i has zero exponent sum on each b_j, $j = 1, \cdots, n$, and $W_i \in \bar{R} \cap [\bar{F}, \bar{F}]$.

PROOF. Abelianize $\bar{F} = \bar{F}(b_1, \cdots, b_n)$ and obtain the following commutative diagram:

$$
\begin{tikzcd}
\bar{F} \arrow[r, two heads, "\psi"] \arrow[d, "A"'] & \pi \\
\overline{FA}(\bar{b}_1, \cdots, \bar{b}_n) \arrow[ur, two heads, "\psi'"'] &
\end{tikzcd}
$$

where $\overline{FA}(\bar{b}_1, \cdots, \bar{b}_n)$ is the free abelian group of rank n generated by $\bar{b}_1, \cdots, \bar{b}_n$ $(A(b_i) = \bar{b}_i)$. Since $\psi(r_i) = 1 = \psi'(A(r_i)) = \psi'(\bar{b}_1^{\eta_{i1}} \cdots \bar{b}_n^{\eta_{in}}) = \bar{x}_1^{\eta_{i1}} \cdots \bar{x}_n^{\eta_{in}}$ it follows that each η_{ij} is divisible by order $\bar{x}_j = \tau_j$ and $r_i = b_1^{\eta_{i1}} b_n^{\eta_{in}} W_i$, where $W_i \in \ker A = [\bar{F}, \bar{F}]$. Define $\beta_{ij} = \eta_{ij}/\tau_j$.

CLAIM. We may change part of the basis of $C_2(S)$, say to $\{y_1'', \cdots, y_n'', y_{l+1}'', \cdots, y_m''\} \cup \{z_{ij}\} \cup \{y_j' | n+1 \leqslant j \leqslant l\}$, so that we may alter each r_i to $r_i' = \prod_{j=1}^{n} b_j^{\beta_{ij}\tau_j}$ and preserve $\rho(r_i') = \partial_2 y_i''$ for $i = 1, \cdots, n, l+1, \cdots, m$.

PROOF. This follows because $\rho([\bar{F}, \bar{F}]) \subset \ker \partial_1(P) \subset \langle x_1, \cdots, x_n \rangle$ is generated by $\{\partial_2 z_{ij} | 1 \leq i < j \leq n\}$. Consider

$$\rho(r_i) = \rho(b_1^{\eta_{i1}} \cdots b_n^{\eta_{in}} W_i) = \rho(r_i') + \rho(W_i).$$

But

$$\rho(W_i) = \sum_{1 \leq j < k \leq n} \delta_{ijk} \partial_2 z_{jk} \qquad (\delta_{ijk} \in Z[\pi]).$$

Let

$$y_i'' = y_i' - \sum_{1 \leq j < k \leq n} \delta_{ijk} z_{jk} \qquad (i = 1, \cdots, n, l+1, \cdots, m).$$

Clearly $\partial_2 y_i'' = \rho(r_i')$ and $\{y_1'', \cdots, y_n'', y_{l+1}'', \cdots, y_m''\} \cup \{z_{ij}\} \cup \{y_{n+1}', \cdots, y_l'\}$ is a basis for $C_2(S)$.

Thus $\partial_2(y_i'') = \rho(r_i') = \Sigma_{j=1}^n \beta_{ij} N_j x_j$, where $\beta_{ij} \in Z$ $(i = 1, \cdots, n, l+1, \cdots, m)$. Again notice that $\{\partial_2(y_i'') | i = 1, \cdots, n, l+1, \cdots, m\} \cup \{\partial_2 z_{ij} | 1 \leq i < j \leq n\}$ generates $\ker \partial_1(P)$. Thus for each $s = 1, \cdots, n$

$$N_s x_s = \sum_{i=1,l+1}^{n,m} \alpha_{si} \rho(r_i') + \sum_{1 \leq i < j \leq n} r_{sij} \partial_2 z_{ij} \qquad (\alpha_{si}, r_{sij} \in Z[\pi]).$$

Denoting the second term by T_s $(T_s \in \rho([\bar{F}, \bar{F}]))$ we have

$$N_s x_s = \sum_i \alpha_{si} \Big(\sum_j \beta_{ij} N_j x_j\Big) + T_s = \sum_j \Big(\sum_i \alpha_{si} \beta_{ij}\Big) N_j x_j + T_s.$$

By augmenting the above equation, and observing that $\epsilon(T_s) = 0$ and $\epsilon(N_j) = \tau_j$, we have $(\Sigma_i \epsilon(\alpha_{si}) \beta_{ij}) \tau_j x_j = \delta_{sj} \tau_s x_s$. Thus we deduce

$$\text{(4.1)} \qquad \sum_{i=1,l+1}^{n,m} \epsilon(\alpha_{si}) \beta_{ij} = \delta_{sj}, \qquad \begin{Bmatrix} s = 1, \cdots, n \\ j = 1, \cdots, n \end{Bmatrix}.$$

The above argument shows we can choose $\alpha_{si} \in Z$ (let α_{si} "=" $\epsilon(\alpha_{si})$) such that

$$\text{(4.2)} \qquad N_s x_s = \sum_{i=1,l+1}^{n,m} \alpha_{si} \rho(r_i') \qquad (s = 1, \cdots, n).$$

Let $p = m + n - l$, the number of basis elements in the set $\{y_i''\}$. Let $(\alpha_{si}) = A$ and $(\beta_{ij}) = B$ denote respectively the $n \times p$ and $p \times n$ matrices with integer coefficients. Relation (4.1) implies that

$$\text{(4.3)} \qquad AB = I_n$$

where I_n is the identity $n \times n$ matrix. Using (4.3), an easy argument on free abelian groups shows that there exists a nonsingular $p \times p$ matrix C with integer coefficients such that

$$\text{(4.4)} \qquad CB = (I_n | 0) \qquad (n \times p \text{ matrix}).$$

Apply the matrix C to the partial basis $\{y_i'' | i = 1, \cdots, n, l+1, \cdots, m\}$

of $C_2(S)$ to obtain a new basis $\{w_i | i = 1, \cdots, n, l+1, \cdots, m\} \cup \{z_{ij}\} \cup \{y'_j | n+1 \leqslant j \leqslant l\}$ for C_2. Then

$$\partial_2(w_i) = \partial_2 \sum_j c_{ij} y''_j = \sum_j c_{ij} \partial_2 y''_j = \sum_s \left(\sum_j c_{ij} \beta_{js} \right) N_s x_s$$

$$= \begin{cases} N_i x_i & \text{if } i = 1, \cdots, n, \\ 0 & \text{if } i = l+1, \cdots, m \end{cases}$$

by (4.4). The matrix of ∂_2 with respect to this new basis for $C_2(S)$ and the old basis for $C_1(S)$ is

$$\begin{array}{cc}
 & \begin{array}{cccc} \qquad\qquad\quad n & \qquad\quad l & \qquad\quad m & \qquad m+k(\pi) \end{array} \\
\begin{array}{c} n \\ \\ l \end{array} &
\left(\begin{array}{c|c|c|c}
\begin{matrix} N_1 & & 0 \\ & \ddots & \\ 0 & & N_n \end{matrix} & 0 & 0 & \tilde{Z} \\
\hline
0 & I_{(l-n)} & 0 & 0
\end{array}\right)
\end{array}$$

The chain complex with this new preferred basis for C_2 can clearly be realized by the presentation

$$V = \Big(b_1, \cdots, b_l : b_1^{\tau 1}, \cdots, b_n^{\tau n}, b_{n+1}, \cdots, b_l, \underbrace{1, \cdots, 1}_{m-l}, \{[b_i, b_j] \mid 1 \leqslant i < j \leqslant n\} \Big).$$

The cellular model V has the same simple homotopy type as $P \vee (\bigvee_{i=1}^{m-l} S_i^2)$. The chain complexes $C_*(S)$ and $C_*(V)$ differ only by a change of basis in C_2. Proposition 4 of [4] shows that there is a homotopy equivalence $f\colon V \longrightarrow S$ which is the identity on the 1-skeleton and such that the matrix of $\tilde{f}_{2\#}\colon C_2(V) \longrightarrow C_2(S)$ is exactly the matrix N recording the basis change in C_2. Furthermore, this matrix N represents the Whitehead torsion $\tau(f) \in \mathrm{Wh}(\pi_1 V)$ of the equivalence f. This completes the proof of Theorem A. □

5. **Proof of Theorem B.** In this section we will show that for a finite group $\pi \neq Z_2$, the number of homotopy types of π-complexes with a given Euler characteristic χ is less than or equal to $\varphi(|\pi|)/2$, provided $\chi \geqslant \chi_\pi + 1$.

DEFINITION. Let M be a finitely generated π-module. M has the *cancellation property* if for any finitely generated π-module N such that $M \oplus (Z\pi)^i \cong N \oplus (Z\pi)^j$ $(i \geqslant j)$, we have $N \cong M \oplus (Z\pi)^{i-j}$.

The following lemma was shown to us by R. G. Swan [12].

LEMMA. *Let X be any π-complex. Then $\pi_2(X) \oplus Z\pi$ has the cancellation property.*

PROOF. We will show that $\pi_2(X) \oplus Z\pi$ satisfies the Eichler condition. That $\pi_2(X) \oplus Z\pi$ has the cancellation property follows from the theorem of H. Jacobinski ([7], [12, p. 178]). A finitely generated, torsion free π-module M satisfies the Eichler condition $\Longleftrightarrow$ the algebra $\mathrm{End}_{Q\pi}(Q \otimes M)$ has no totally definite quaternion algebra as a direct summand (see [7] for a definition).

Consider the cellular chain complex $C_*(\widetilde{X})$ of the universal cover $\widetilde{X}$ of X. This gives an exact sequence of π-modules

$$0 \longrightarrow \pi_2(X) \longrightarrow (Z\pi)^r \longrightarrow (Z\pi)^s \longrightarrow Z\pi \xrightarrow{\epsilon} Z \longrightarrow 0.$$

Tensoring with Q, the rationals. The resulting sequence splits and gives

$$\pi_2(X) \otimes \mathrm{Q} \cong (\mathrm{Q}I)^{n+1} \oplus Q^n$$

where $n = r - s$ and I is the augmentation ideal. Therefore $\mathrm{Q} \otimes (\pi_2(X) \oplus Z\pi) \cong (\mathrm{Q}I)^{n+2} \oplus \mathrm{Q}^{n+1}$ and

$$\mathrm{End}_{\mathrm{Q}\pi}(\mathrm{Q} \otimes (\pi_2(X) \oplus Z\pi)) \cong M_{n+2}(\mathrm{End}_{\mathrm{Q}\pi}\mathrm{Q}I) \times M_{n+1}(\mathrm{Q}).$$

Since $n \geqslant 0$, no totally definite quaternion algebras occur. $\square$

We appeal to the theory of 2-types (see [10]) and the cancellation theorem above. Let X be any π-complex with $\chi(X) \geqslant \chi_\pi + 1$. By a theorem of J. H. C. Whitehead [13],

$$\pi_2(X) \oplus (Z[\pi])^m \cong \pi_2(Y) \oplus (Z[\pi])^n$$

where Y is a π-complex with $\chi(Y) = \chi_\pi$, and $n \geqslant m + 1$. The cancellation theorem above guarantees that

$$\pi_2(X) \cong \pi_2(Y) \oplus (Z[\pi])^{n-m}$$

where $n - m = \chi(X) - \chi_\pi$. Thus π-complexes with fixed Euler characteristic $\chi \geqslant \chi_\pi + 1$ have the same second homotopy module

$$\pi_2 \cong \pi_2(Y) \oplus (Z[\pi])^{\chi-\chi_\pi}.$$

We conclude that their algebraic 2-types (π, π_2, k) differ only by the obstruction invariant $k \in H^3(\pi, \pi_2) \cong Z_{|\pi|}$. But each $k \in H^3(\pi, \pi_2)$ which is the obstruction invariant for a π-complex must be a generator of $H^3(\pi, \pi_2)$ (see [3]). There are exactly $\varphi(|\pi|)$ such generators. The sign changing automorphism

$$\lambda\colon \pi_2 \longrightarrow \pi_2 \qquad (\lambda(x) = -x,\ x \in \pi_2)$$

together with $\mathrm{id}\colon \pi \longrightarrow \pi$ gives an isomorphism of the 2-types

$$\bar{\lambda}: (\pi, \pi_2, k) \longrightarrow (\pi, \pi_2, -k)$$

and shows that the number of k-invariants representing distinct homotopy types of π-complexes with Euler characteristic χ is less than or equal to $\varphi(|\pi|)/2$, since $\pi \neq Z_2$.

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