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SIMULTANEOUS APPROXIMATION OF ADDITIVE FORMS

BY

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ABSTRACT. Let $X = (x_1, \cdots, x_s)$ be a vector of s real components and $f_i(X) = \Sigma_{j=1}^s \theta_{ij} x_j^k$ $(k = 2, 3, \cdots; i = 1, \cdots, R)$ R additive forms, where θ_{ij} are arbitrary real numbers. The author obtains some results on the simultaneous approximation of $\|f_i(X)\|$, where $\|t\|$ means the distance from t to the nearest integer.

1. **Introduction.** In 1948 Heilbronn [5] improved Vinogradov's result [9] and obtained that for any $\epsilon > 0$ there exists some positive constant $C = C(\epsilon)$ which depends on ϵ only such that for any real number θ and any integer $N \geqslant 1$ there is an integer x satisfying

$$1 \leqslant x \leqslant N \quad \text{and} \quad \|\theta x^2\| < CN^{-1/2+\epsilon}, \tag{1.1}$$

where $\|t\|$ means the distance from t to the nearest integer.

In 1967, see [4], Davenport generalized (1.1) and obtained

$$1 \leqslant x \leqslant N \quad \text{and} \quad \|\theta x^k\| < CN^{-(1/K)+\epsilon}, \tag{1.2}$$

where $K = 2^{k-1}$, $k = 2, 3, \cdots$ and $C = C(\epsilon, k)$ is a positive constant depending on ϵ, k only. Recently, R. J. Cook [2] extended (1.2) to a finite number of θ's. On the other hand, Cook [3] and the author [8] obtained some results on additive forms. These results are as follows:

THEOREM C [2]. *For any $\epsilon > 0$ and any integers $k > 1, R > 0$, there exists a positive constant $C = C(\epsilon, k, R)$ depending on ϵ, k, R only such that for any real numbers $\theta_1, \cdots, \theta_R$ and any integer $N \geqslant 1$ there is an integer x satisfying*

$$1 \leqslant x \leqslant N \quad \textit{and} \quad \max_{1 \leqslant i \leqslant R} (\|\theta_i x^k\|) < CN^{-(1/f(k,R))+\epsilon}, \tag{1.3}$$

where $f(k, R)$ is defined by $K = 2^{k-1}$ and

$$f(k, 1) = K, \qquad f(k, R) = 2f(k, R-1) + KR + 1 \qquad (R \geqslant 2). \tag{1.4}$$

THEOREM L [8]. *For any integers $k \geqslant 2$ and $s \geqslant 1$, put $K = 2^{k-1}$ and $X = (x_1, \cdots, x_s)$, a vector of s real components. Let $f(X), g(X)$ be any two additive forms of degree k in s variables,*

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$$f(X)=\sum_{i=1}^{s}\theta_i x_i^k,\qquad g(X)=\sum_{i=1}^{s}\phi_i x_i^k,$$

where $\theta_1,\cdots,\theta_s;\phi_1,\cdots,\phi_s$ *are arbitrary real numbers. Then for any* $\epsilon>0$ *and any integer* $N\geqslant 1$ *there are integers* $x_1,\cdots,x_s$ *not all zero and some positive constant* $C=C(\epsilon,k,s)$ *which depends on* ϵ, k, s *only such that*

$$1\leqslant\max_{1\leqslant i\leqslant s}|x_i|\leqslant N, \tag{1.5}$$
$$\max(\|f(X)\|,\|g(X)\|)<CN^{-(1/F(k,s))+\epsilon},$$

where

$$F(k,s)=\begin{cases}7 & \text{if } s=1,k=2,\\ 3K+1/k & \text{if } s=1,k\geqslant 3,\\ 2K+1+K/ks & \text{if } s\geqslant 2.\end{cases} \tag{1.6}$$

In this paper we shall extend Theorem L to a finite number of additive forms with some improvements of Theorem C. We shall prove

THEOREM 1. *For any integers* $k\geqslant 2$ *and* $s\geqslant 1$, *put* $K=2^{k-1}$ *and* $X=(x_1,\cdots,x_s)$, *a vector of s real components. For any integer* $R\geqslant 3$ *let*

$$f_i(X)=\sum_{j=1}^{s}\theta_{ij}x_j^k\qquad(i=1,\cdots,R)$$

be any R additive forms of degree k in s variables, where θ_{ij} $(i=1,\cdots,R;j=1,\cdots,s)$ *are arbitrary real numbers. Then for any* $\epsilon>0$ *and any integer* $N\geqslant 1$ *there are integers* $x_1,\cdots,x_s$ *not all zero and some positive constant* $C=C(\epsilon,k,s,R)$ *which depends on* ϵ, k, s, R *only such that*

$$1\leqslant\max_{1\leqslant j\leqslant s}|x_j|\leqslant N, \tag{1.7}$$
$$\max_{1\leqslant i\leqslant R}(\|f_i(X)\|)<CN^{-(1/G(k,s,R))+\epsilon},$$

where $G(k,s,R)$ *is defined by*

$$G(k,s,R)=2g(k,R-1)+(R-1)K/ks+1\qquad(R\geqslant 3) \tag{1.8}$$

and $g(k,R)$ *is defined by*

$$g(k,2)=\begin{cases}7 & \text{if } k=2,\\ 3K+1/k & \text{if } k\geqslant 3.\end{cases} \tag{1.9}$$
$$g(k,R)=2g(k,R-1)+(R-1)K/k+1\qquad(R\geqslant 3).$$

COROLLARY 1. *For any* $\epsilon>0$ *and any integers* $k>1, R>1$ *there exists a positive constant* $C=C(\epsilon,k,R)$ *depending on* ϵ, k, R *only such that for any* *real numbers* $\theta_1, \cdots, \theta_R$ *and any integer* $N \geqslant 1$ *there is an integer* x *satisfying*

$$(1.10) \qquad 1 \leqslant x \leqslant N \quad \text{and} \quad \max_{1 \leqslant i \leqslant R} (\|\theta_i x^k\|) < CN^{-(1/g(k,R))+\epsilon},$$

where $g(k, R)$ *is defined by* (1.9).

Corollary 1 follows from Theorem L ($R = 2$) and Theorem 1 ($R \geqslant 3$).

We shall give the explicit form of $g(k, R)$ in Lemma 1 ((2.1), (2.2)) for the sake of application. For example when $k \geqslant 3$ we have:

R	$g(k, R)$
2	$3K + 1/k$
3	$(6 + 2/k)K + 1 + 2/k$
4	$(12 + 7/k)K + 3 + 4/k$
5	$(24 + 18/k)K + 7 + 8/k$
. . . .	

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2. **Notation and preliminary lemmas.** In what follows k, s, R are the integers given in Theorem 1. We always assume that ϵ is a small enough positive quantity which is not the same ϵ given in Theorem 1 and that N is a sufficiently large positive integer, say $N > N_0 = N_0(\epsilon, k, s, R)$ which is a positive integer depending on ϵ, k, s, R given, such that all the subsequent asymptotic approximations and inequalities in §§3, 4 are satisfied. So it is difficult to define ϵ and N_0 at the beginning or at any particular point. If $y > 0$ we use $x \ll y$ to denote $|x| < Cy$, where C is some positive constant which can depend on ϵ, k, s, R. $[t]$ is the integral part of t. For real t we write $e(t) = \exp(2\pi t i)$. We shall use B to denote some irrelevant numbers which need not be the same from one occurrence to another. For figures x, y, $(x)_y$ indicates the yth formula in (x).

We need several lemmas.

LEMMA 1. *Let* k, R *be positive integers and* $K = 2^{k-1}$. *If*

$$g(k, R) = 2g(k, R-1) + K(R-1)/k + 1 \qquad (R \geqslant 3),$$

where

$$g(k, 2) = \begin{cases} 7 & \text{if } k = 2, \\ 3K + 1/k & \text{if } k \geqslant 3, \end{cases}$$

then

$$g(2, R) = (11)2^{R-2} - (R + 2), \tag{2.1}$$

$$g(k, R) = \left(3K + \frac{3K + k + 1}{k}\right)2^{R-2} - K\left(\frac{R+1}{k} + \frac{1}{K}\right) \quad \text{if } k \geqslant 3. \tag{2.2}$$

PROOF. Since for $R \geqslant 3$, $k \geqslant 2$ we have

$$g(k, R) = 2^{R-2}g(k, 2) + \sum_{i=0}^{R-3} 2^i + \frac{K}{2k}\sum_{i=1}^{R-2}(R - i)(2^i),$$

then (2.1), (2.2) follow by a simple calculation.

LEMMA 2. *Suppose that Δ satisfies $0 < \Delta < \frac{1}{2}$ and r is a positive integer. Then there exists a real valued function $\psi(x)$, periodic with period 1, which satisfies*

$$\psi(x) = 0 \quad \text{if } \|x\| > \Delta, \tag{2.3}$$
$$\psi(x) = \sum_{u=-\infty}^{\infty} c_u e(ux),$$

where c_u are real and

$$c_0 = \Delta, \qquad |c_u| < C(r)\min(\Delta, \Delta^{-r}|u|^{-r-1}) \quad \text{if } u \neq 0, \tag{2.4}$$

where $C(r)$ is some positive constant depending on r only.

PROOF. This is a particular case of Lemma 12 in [10, p. 32] with $\beta = -\alpha = \frac{1}{2}\Delta$.

LEMMA 3. *Let $T = \Sigma_{x=1}^{N} e(tx^k)$, where $k = 2, 3, \cdots$ and t is any real number. For any $\epsilon > 0$ we have*

$$|T|^K \ll N^{K-1} + N^{K-k+\epsilon}\sum_{j=1}^{L}\min(N, 1/\|jt\|), \tag{2.5}$$

where $K = 2^{k-1}$ and $L = k!N^{k-1}$.

PROOF. See Satz 266 in [7, p. 255].

LEMMA 4. *For any real t if there are integers a, q with $q > 0$ such that $(a, q) = 1$ and $|t - a/q| \leqslant q^{-2}$, then for any positive integers P and N,*

$$\sum_{j=P+1}^{P+q}\min(N, 1/\|jt\|) \ll N + q\log q. \tag{2.6}$$

PROOF. Lemma 4 is well known. See, for example, Lemma 3.5 in [6].

LEMMA 5. *For any integer $k \geqslant 2$ put $K = 2^{k-1}$. For any ϵ with $0 < \epsilon < 1$ and any integers $s \geqslant 1$, $R \geqslant 3$ let λ satisfy*

(2.7) $$\lambda\{2g(k, R-1) + (R-1)K/sk + 1\} < 1 - \epsilon$$

and

(2.8) $$\alpha = \frac{2}{K} g(k, R-1) + \frac{(R-1)(1-k)}{sk} + \frac{1}{K},$$

where $g(k, R)$ is defined by (1.9). *Then the following five inequalities hold simultaneously.*

(2.9) $$(\lambda/s)RK < 1 - \epsilon,$$

(2.10) $$(\lambda/s)(sK\alpha + (R-1)K) < 1 - \epsilon,$$

(2.11) $$(\lambda/(k-1)s)(K(1+R^2) + sR - s\alpha K) < 1 - \epsilon,$$

(2.12) $$(\lambda/(k-1))(2kg(k, R-1) - \alpha K + k) < 1 - \epsilon,$$

(2.13) $$\lambda(2g(k, R-1) + 1) < 1 - \epsilon.$$

PROOF. By (1.9) it is easy to see that $g(k, R-1) > KR$ $(R = 3, 4, \cdots)$. It follows that (2.7) ⇒ (2.13) ⇒ (2.9). On the other hand, substituting (2.8) into (2.10) and (2.12) we see that (2.7) = (2.10) = (2.12). So it remains to show that when $R \geqslant 3$

(2.14) $$2kg(k, R-1) + k > K(1+R^2) + R,$$

i.e. (2.12) ⇒ (2.11). Here we only give the arguments for $k \geqslant 3$. By (2.2) we have

$$2kg(k, R-1) + k = 3K(2^{R-2})(k+1) + k(2^{R-2} - 1) + 2^{R-2} - 2KR.$$

Then (2.14) follows since when $k \geqslant 3, R \geqslant 3$ we have

$$3(2^{R-2})(k+1) > (1+R)^2 \quad \text{and} \quad k(2^{R-2} - 1) + 2^{R-2} > R.$$

The proof of Lemma 5 is complete.

LEMMA 6. *Let R, N be any positive integers and $\theta_1, \cdots, \theta_R$ any real numbers. Then there exists an integer n satisfying*

(2.15) $$1 \leqslant n \leqslant N, \qquad \|\theta_i n\| < N^{-1/R} \qquad (i = 1, \cdots, R).$$

PROOF. See Theorem VI [1, p. 13].

3. **Existence of an R-tuple.** We come now to the proof of our theorem.

Suppose that for some $\lambda > 0$ there are no integral solutions $X = (x_1, \cdots, x_s)$ of the following inequalities:

(3.1) $$1 \leqslant \max_{1 \leqslant j \leqslant s} |x_j| \leqslant N, \qquad \max_{1 \leqslant i \leqslant R} (\|f_i(X)\|) \leqslant N^{-\lambda},$$

i.e. for each X with integers $x_1, \cdots, x_s$ satisfying $1 \leqslant \max_{1 \leqslant j \leqslant s} |x_j| \leqslant N$, we have some i $(1 \leqslant i \leqslant R)$ such that $\|f_i(X)\| > N^{-\lambda}$. Putting $\Delta = N^{-\lambda}$ in Lemma 2, we have

$$0 = \sum_X \psi(f_1(X)) \cdots \psi(f_R(X))$$

$$(3.2) \qquad = \sum_X \left\{ \sum_{m_1=-\infty}^{\infty} c_{m_1} e(m_1 f_1(X)) \cdots \sum_{m_R=-\infty}^{\infty} c_{m_R} e(m_R f_R(X)) \right\}$$

$$= N^s c_0^R + \sum_X \sum_{\mathrm{m}} c_{m_1} \cdots c_{m_R} e\left(m_1 \sum_{j=1}^{s} \theta_{1j} x_j^k\right) \cdots e\left(m_R \sum_{j=1}^{s} \theta_{Rj} x_j^k\right),$$

where Σ_X is taken over integers x_j with $1 \leqslant j \leqslant s$, $1 \leqslant x_j \leqslant N$, and Σ_{m} is taken over $-\infty < m_1, \cdots, m_R < \infty$ except $\mathrm{m} = (m_1, \cdots, m_R) = (0, \cdots, 0)$. By (2.4) with $c_0 = N^{-\lambda}$ and (3.2) we have

$$(3.3) \qquad N^{s-R\lambda} \leqslant \sum_{\mathrm{m}} |c_{m_1} \cdots c_{m_R}| \prod_{j=1}^{s} |S(\mathrm{m}, j)|,$$

where

$$(3.4) \qquad S(\mathrm{m}, j) = \sum_{x=1}^{N} e\left(x^k \sum_{i=1}^{R} \theta_{ij} m_i\right) \qquad (j = 1, \cdots, s).$$

Write

$$(3.5) \quad \sum_{\mathrm{m}} |c_{m_1} \cdots c_{m_R}| \prod_{j=1}^{s} |S(\mathrm{m}, j)| = \left(\sum\nolimits_1 + \sum\nolimits_2\right) |c_{m_1} \cdots c_{m_R}| \prod_{j=1}^{s} |S(\mathrm{m}, j)|,$$

where Σ_1 is the summation taken over $|m_i| < N^{\lambda+\epsilon}$ $(i = 1, \cdots, R)$ and $\mathrm{m} \neq (0, \cdots, 0)$ while Σ_2 is taken over all remaining terms in Σ_{m}. We are going to show that $\Sigma_2 \ll N^{s-R\lambda-\epsilon}$ if we let r given in Lemma 2 satisfy $r > R(\lambda + \epsilon)/\epsilon$. So in view of (3.3), (3.5) we may neglect Σ_2.

For each $l = 1, \cdots, R$ by $(2.4)_2$ we have

$$\sum_{|m_l| < \infty} |c_{m_l}| = \sum_{|m_l| < N^{\lambda+\epsilon}} |c_{m_l}| + \sum_{|m_l| \geqslant N^{\lambda+\epsilon}} |c_{m_l}|$$

$$(3.6) \qquad \ll N^{-\lambda} \sum_{|m_l| < N^{\lambda+\epsilon}} 1 + N^{r\lambda} \sum_{|m_l| \geqslant N^{\lambda+\epsilon}} |m_l|^{-r-1}$$

$$\ll N^{\epsilon} + N^{r\lambda} N^{-r(\lambda+\epsilon)} \ll N^{\epsilon}.$$

Since by (3.4), $|S(\mathrm{m}, j)| \leqslant N$ we see that

$$\sum\nolimits_2 \ll \sum_{i=1}^{R} \left\{ \sum_{|m_i| \geq N^{\lambda+\epsilon}} |c_{m_1} \cdots c_{m_R}| \prod_{j=1}^{s} |S(\mathbf{m}, j)| \right\}$$

$$\ll N^s \sum_{i=1}^{R} \left\{ \sum_{|m_i| \geq N^{\lambda+\epsilon}} |c_{m_i}| \right\} \left\{ \prod_{i \neq l} \sum_{|m_l| < \infty} |c_{m_l}| \right\}.$$

By (3.6) we have

$$\sum\nolimits_2 \ll N^s \sum_{i=1}^{R} \left\{ \sum_{|m_i| \geq N^{\lambda+\epsilon}} |c_{m_i}| \right\} N^{(R-1)\epsilon}$$

$$\ll N^{s+(R-1)\epsilon} \sum_{i=1}^{R} N^{-r\epsilon} \ll N^{s+(R-1)\epsilon - r\epsilon} \ll N^{s-R\lambda-\epsilon},$$

if $r > R(\lambda + \epsilon)/\epsilon$.

It follows from (3.3), (3.5) $(2.4)_2$ that

$$N^{s-R\lambda} \ll \sum\nolimits_1 |c_{m_1} \cdots c_{m_R}| \prod_{j=1}^{s} |S(\mathbf{m}, j)|$$

$$\ll \sum\nolimits_1 N^{-R\lambda} \prod_{j=1}^{s} |S(\mathbf{m}, j)|.$$

That is

$$N^s \ll \sum\nolimits_1 \prod_{j=1}^{s} |S(\mathbf{m}, j)| \ll \sum_{j=1}^{s} \sum\nolimits_1 |S(\mathbf{m}, j)|^s. \tag{3.7}$$

We see that there exists some j_0 $(1 \leq j_0 \leq s)$ for which we write $\theta_i = \theta_{ij_0}$ and $S(\mathbf{m}) = S(\mathbf{m}, j_0)$, such that

$$N^s \ll \sum\nolimits_1 |S(\mathbf{m})|^s, \tag{3.8}$$

where, by (3.4),

$$S(\mathbf{m}) = \sum_{x=1}^{N} e\left(x^k \sum_{i=1}^{R} m_i \theta_i\right). \tag{3.9}$$

By definition of the notation $\ll$ there is some positive constant B such that we may rewrite (3.8) as

$$BN^s \leq \sum\nolimits_1 |S(\mathbf{m})|^s. \tag{3.8a}$$

We are now going to show that by (3.8a) there exists a ρ

$$0 \leq \rho \leq \lambda + \epsilon \tag{3.10}$$

such that

$$|S(\mathrm{m})| \geqslant (2^{-R-1}B)^{1/s}N^{1-Rp} \qquad (\gg N^{1-Rp}), \tag{3.11}$$
$$p = \rho/s,$$

for at least $[N^{R\rho-\epsilon}] + 1$ R-tuples $(m_1, \cdots, m_R)$ in

$$|m_i| < N^{\lambda+\epsilon} \quad (i = 1, \cdots, R). \tag{3.12}$$

Suppose that such a ρ does not exist. For some integer l with $l\epsilon > 2R(\lambda + \epsilon)$ write

$$\sum_1 |S(\mathrm{m})|^s = \sum_3 |S(\mathrm{m})|^s + \sum_{j=0}^{l-1} T_j,$$

where Σ_3 is taken over $|m_i| < N^{\lambda+\epsilon}$ $(i = 1, \cdots, R)$, $\mathrm{m} \neq (0, \cdots, 0)$ and $|S(\mathrm{m})|^s < 2^{-R-1}BN^{s-R(\lambda+\epsilon)}$ while $T_j = \Sigma |S(\mathrm{m})|^s$ which are taken over $|m_i| < N^{\lambda+\epsilon}$ $(i = 1, \cdots, R)$, $\mathrm{m} \neq (0, \cdots, 0)$ and

$$2^{-R-1}BN^{(s-(j+1)R(\lambda+\epsilon)/l)} \leqslant |S(\mathrm{m})|^s < 2^{-R-1}BN^{(s-jR(\lambda+\epsilon)/l)}.$$

According to our assumption on ρ there are no such terms $|S(\mathrm{m})|^s$ satisfying $2^{-R-1}BN^s \leqslant |S(\mathrm{m})|^s$ even if $2^{-R-1}B < 1$. So we neglect this possibility in the above summation $\Sigma_{j=0}^{l-1} T_j$. Now by our supposition on ρ and $\epsilon/2 > R(\lambda + \epsilon)/l$ we have

$$T_j \leqslant (2^{-R-1}BN^{(s-jR(\lambda+\epsilon)/l)})(N^{((j+1)R(\lambda+\epsilon)/l-\epsilon)})$$
$$= 2^{-R-1}BN^{(s-\epsilon+R(\lambda+\epsilon)/l)} < 2^{-R-1}BN^{s-\epsilon/2} \qquad (j = 0, 1, \cdots, l-1).$$

$$\sum_3 |S(\mathrm{m})|^s < (2^{-R-1}BN^{s-R(\lambda+\epsilon)})(2N^{\lambda+\epsilon})^R = BN^s/2.$$

Hence

$$\sum_1 |S(\mathrm{m})|^s = \sum_3 |S(\mathrm{m})|^s + \sum_{j=0}^{l-1} T_j$$
$$< (BN^s/2) + (2^{-R-1}BlN^{s-\epsilon/2}) < BN^s$$

if N is large. This contradicts (3.8a). So a ρ satisfying (3.10), (3.11) exists.

In what follows we shall confine our attention to the R-tuples $(m_1, \cdots, m_R)$ satisfying (3.10), (3.11), (3.12). From (2.5), (3.9) we have

$$|S(\mathrm{m})|^K \ll N^{K-1} + N^{K-k+\epsilon} \sum_{j=1}^{L} \min(N, 1/\|jt\|), \tag{3.13}$$

where $L = k!N^{k-1}$ and

$$t = \sum_{i=1}^{R} \theta_i m_i. \tag{3.14}$$

Define

(3.15) $$Q = N^{k-1+\alpha\lambda K - Kp} \qquad (p = \rho/s),$$

where

(3.16) $$\alpha = \alpha(k, s, R) = \frac{2}{K} g(k, R-1) + \frac{(R-1)(1-k)}{ks} + \frac{1}{K}$$

and $g(k, R)$ is defined by (1.9). By Dirichlet's theorem for given t and Q ((3.14), (3.15)) there are integers a, q such that

(3.17) $$(a, q) = 1, \quad 1 \leqslant q \leqslant Q, \quad |qt - a| < Q^{-1}.$$

Fix such a q and divide the sum in the right of (3.13) into blocks of q terms. It follows from Lemma 4 and (3.13) that

$$|S(\mathrm{m})|^K \ll N^{K-1} + N^{K-k+\epsilon}(k!N^{k-1}q^{-1} + 1)(N + q \log q).$$

Then by (3.11) we have

(3.18) $$N^{1-RKp-\epsilon} \ll Nq^{-1} + N^{\epsilon} + N^{1-k+\epsilon}q.$$

Suppose that λ satisfies

(3.19) $$\lambda\left\{2g(k, R-1) + \frac{(R-1)K}{sk} + 1\right\} < 1 - A\epsilon \qquad (R \geqslant 3),$$

where $g(k, R)$ is defined by (1.9) and A is a positive constant. The value of A is so defined such that all following inequalities (3.21), (3.22), (3.26), (4.8), (4.10) will be satisfied.

By (3.15), $(3.17)_2$, (3.18) we have

(3.20) $$q^{-1} \gg N^{-RKp-\epsilon}\{1 - N^{2\epsilon+RKp-1} - N^{RKp+2\epsilon-1+\alpha\lambda K-Kp}\}.$$

The last two terms in the curly bracket of (3.20) can be neglected since by (3.10), (3.19) and Lemma 5 ((2.9), (2.10)) we have

(3.21) $$2\epsilon + RKp - 1 < 0$$

and

(3.22) $$RKp + 2\epsilon - 1 + \alpha\lambda K - Kp < 0.$$

Then from (3.20) we have

(3.23) $$(1 \leqslant) q \ll N^{RKp+\epsilon}.$$

By Lemma 6, for given $\theta_1, \cdots, \theta_R$ and integer $[QN^{-\epsilon}]$ there are integers $a_1, \cdots, a_R$ and b such that

(3.24) $$1 \leqslant b \leqslant [QN^{-\epsilon}], \qquad |b\theta_i - a_i| < [QN^{-\epsilon}]^{-1/R} \qquad (i = 1, \cdots, R).$$

It follows from (3.12), (3.14), $(3.17)_3$, (3.23) that

$$\left|q\sum_{i=1}^{R} a_i m_i - ab\right| \leqslant q\sum_{i=1}^{R} |m_i||a_i - \theta_i b| + b\left|q\left(\sum_{i=1}^{R}\theta_i m_i\right) - a\right| \tag{3.25}$$
$$\ll N^{RKp+\epsilon}\sum_{i=1}^{R} N^{\lambda+\epsilon}[QN^{-\epsilon}]^{-1/R} + [QN^{-\epsilon}]Q^{-1}.$$

By (3.10), (3.19) and Lemma 5 ((2.11)) we see that

$$(RKp + \epsilon) + (\lambda + \epsilon) - \frac{1}{R}(k - 1 + \alpha\lambda K - Kp - \epsilon) \tag{3.26}$$
$$< \frac{(k-1)}{R}\left\{\frac{\lambda}{s(k-1)}(K(1 + R^2) + sR - s\alpha K) + 1 + B\epsilon\right\} < -\epsilon,$$

where B is some number depending on k, s, R only. Then from (3.15), (3.25), (3.26) we have $|q\,\Sigma_{i=1}^{R} a_i m_i - ab| \ll N^{-\epsilon} < 1$, for large N. That is $q\,\Sigma_{i=1}^{R} a_i m_i = ab$. But by $(3.17)_1$ $((a, q) = 1)$ we see that q divides b. Then by (3.15), $(3.24)_1$ the number of possibilities for q is $O(N^\epsilon)$. Since at least $[N^{R\rho-\epsilon}] + 1$ (> 1) R-tuples $(m_1, \cdots, m_R)$ in (3.12) satisfy (3.11) then $\gg N^{R\rho-2\epsilon}$ (or $\geqslant BN^{R\rho-2\epsilon}$ for some positive constant B) of these R-tuples have the same q. Choose a suitable d ($d^R B \geqslant 2^{R+1}$, say) such that the following pigeonhole argument holds. Partition the R-cube, $|m_i| \leqslant N^{\lambda+\epsilon}$ $(i = 1, \cdots, R)$ by

$$|m_i| = ldN^{\lambda+\epsilon-(\rho-(2\epsilon/R))}, \tag{3.27}$$

where $i = 1, \cdots, R$; $l = 0, 1, \cdots$. In all, there are at most

$$\left(\frac{2N^{\lambda+\epsilon}}{dN^{\lambda+\epsilon-(\rho-(2\epsilon/R))}}\right)^R = (2/d)^R N^{R\rho-2\epsilon}$$

R-subcubes in the R-cube, $|m_i| \leqslant N^{\lambda+\epsilon}$ $(i = 1, \cdots, R)$. Now by the pigeonhole argument, there is an R-subcube containing at least two distinct R-tuples $(m_1', \cdots, m_R')$, $(m_1'', \cdots, m_R'')$, say, having the same q. For these two R-tuples we may suppose that for some integer I with $1 \leqslant I \leqslant R$ we have $m_I' > m_I''$. Put

$$m_i = m_i' - m_i'' \qquad (i = 1, \cdots, R). \tag{3.28}$$

In particular, we have

$$m_I \geqslant 1. \tag{3.29}$$

Then by (3.27), (3.28) and $(3.17)_3$ we have

$$|m_i| \ll N^{\lambda+\epsilon(1+(2/R))-\rho} \qquad (i = 1, \cdots, R). \tag{3.30}$$
$$\left\|\left(\sum_{i=1}^{R}\theta_i m_i\right)q\right\| \leqslant \left\|\sum_{i=1}^{R}\theta_i m_i' q\right\| + \left\|\sum_{i=1}^{R}\theta_i m_i'' q\right\| < 2Q^{-1},$$

since R-tuples $(m_1', \cdots, m_R')$, $(m_1'', \cdots, m_R'')$ have the same q in $(3.17)_3$.

4. Completion of the proof. In what follows we shall confine our attention to the new R-tuple $(m_1, \cdots, m_R)$ satisfying (3.28), (3.30). We proceed by induction on R. As usual, our proof consists of two parts. We first show that Theorem 1 is true for R ($\geqslant 4$) if we assume that Theorem 1 is true for $R-1$. Then we can see that in fact Theorem 1 holds for $R = 3$.

Put

$$\phi_i = m_I^{k-1} q^k \theta_i, \tag{4.1}$$

$$M = [N^{(2\lambda + 2\epsilon(1+R^{-1}) - \rho)g/(1-\epsilon g)}], \tag{4.2}$$

where $g = g(k, R-1)$, $i = 1, \cdots, R; R \geqslant 3$. Suppose that Theorem 1 is true for $R-1$ ($\geqslant 3$). Then by Corollary 1, (1.10) is true for M and any $R-1$ ϕ's among $\phi_1, \cdots, \phi_R$. So there is some integer n satisfying

$$1 \leqslant n \leqslant M,$$
$$\max_{1 \leqslant i \leqslant R; i \neq I} (\|\phi_i n^k\|) \leqslant M^{(-1/g(k,R-1))+\epsilon} \leqslant N^{-2\lambda - 2\epsilon(1+R^{-1})+\rho}. \tag{4.3}$$

Let

$$x = nqm_I. \tag{4.4}$$

It follows from (4.1), $(4.3)_2$, $(3.30)_1$ that

$$\|\theta_i x^k\| \leqslant \|(m_I^{k-1} q^k \theta_i) n^k\| m_I = \|\phi_i n^k\| m_I \qquad (i = 1, \cdots, R; R \geqslant 4)$$

and

$$\max_{1 \leqslant i \leqslant R; i \neq I} \|\theta_i x^k\| \leqslant N^{-2\lambda - 2\epsilon(1+R^{-1})+\rho} N^{\lambda + \epsilon(1+(2/R)) - \rho} \leqslant N^{-\lambda - \epsilon}. \tag{4.5}$$

Similarly, we have

$$\begin{aligned} \|\theta_I x^k\| &= \|\theta_I n^k q^k m_I^k\| \\ &\leqslant q^{k-1} n^k m_I^{k-1} \left\| q\left(\sum_{i=1}^R \theta_i m_i\right) \right\| + \sum_{i=1; i \neq I}^R |m_i| \|(q^k m_I^{k-1} \theta_i) n^k\| \\ &\leqslant N^{\sigma_1} + N^{-\lambda - \epsilon}, \end{aligned} \tag{4.6}$$

where σ_1 is defined (so as to make the last part of (4.6) valid) by the first part of (4.7). By (3.23), (4.2), (3.30), (3.15) we see that

$$\begin{aligned} \sigma_1 &= (k-1)(RKp + \epsilon) + k(2\lambda + 2\epsilon(1 + R^{-1}) - \rho)g/(1 - \epsilon g) \\ &\quad + (k-1)(\lambda + \epsilon(1 + (2/R)) - \rho) - (k - 1 + \alpha\lambda K - Kp) \\ &= p((k-1)RK - skg - (k-1)s + K) \\ &\quad + \lambda(2kg - \alpha K + k) + B\epsilon - (k-1) - \lambda - \epsilon, \end{aligned} \tag{4.7}$$

where B depends on k, R only. As pointed out in the proof of Lemma 5, $g(k, R-1) = g > KR$ when $R \geqslant 4$, whence we see that $(k-1)RK - skg - (k-1)s + K < 0$. Then together with (4.7), (3.19), and Lemma 5 ((2.12)), we have

$$\text{(4.8)} \quad \sigma_1 < \lambda(2kg(k, R-1) - \alpha K + k) - (k-1) + B\epsilon - \lambda - \epsilon < -\lambda - \epsilon.$$

Hence by (4.5), (4.6), (4.8) we have

$$\text{(4.9)} \qquad \max_{1 \leqslant i \leqslant R} \|\theta_i x^k\| \ll N^{-\lambda-\epsilon}.$$

Next, by (4.2), (3.23), (3.29), (3.30)$_1$ we see that $1 \leqslant nqm_I = x \ll N^{\sigma_2}$, where

$$\begin{aligned}\sigma_2 &= (2\lambda + 2\epsilon(1 + R^{-1}) - \rho)g/(1 - \epsilon g) + (RKp + \epsilon) + (\lambda + \epsilon(1 + (2/R)) - \rho) \\ &= \rho(RK/s - g(k, R-1) - 1) + \lambda(2g(k, R-1) + 1) + B\epsilon \\ &< \lambda(2g(k, R-1) + 1) + B\epsilon,\end{aligned}$$

for some B depending on k, R only. Then by (3.19) and Lemma 5 ((2.13)), we have

$$\text{(4.10)} \qquad \sigma_2 < \lambda(2g(k, R-1) + 1) + B\epsilon < 1 - \epsilon.$$

Hence

$$\text{(4.11)} \qquad 1 \leqslant x \ll N^{1-\epsilon}.$$

(4.9) and (4.11) show that we have obtained an integer x satisfying

$$\text{(4.12)} \qquad 1 \leqslant x \leqslant N \quad \text{and} \quad \max_{1 \leqslant i \leqslant R} (\|\theta_i x^k\|) \leqslant N^{-\lambda}.$$

This contradicts our supposition (3.1) since if we let $x_{j_0} = x$ (for j_0 see statement between (3.7) and (3.8)) and $x_j = 0$ for all $j \neq j_0$ then we have a particular vector $X = (x_1, \cdots, x_j, \cdots, x_s)$ for which

$$f_i(X) = \sum_{j=1}^{s} \theta_{ij} x_j^k = \theta_{ij_0} x_{j_0}^k = \theta_i x^k,$$

where $i = 1, \cdots, R$; $R \geqslant 4$. So by (3.19) with a suitable choice of ϵ Theorem 1 is true for R ($\geqslant 4$), if it is true for $R - 1$.

It remains to see that Theorem 1 is true for $R = 3$. For the case $R = 3$ the proof follows exactly in the same way as that for $R \geqslant 4$ except that now Corollary 1 ((1.10)) is known for $R = 2$ (a special case of Theorem L). This proves Theorem 1.

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