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THE RADON-NIKODYM PROPERTY IN CONJUGATE BANACH SPACES

BY

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ABSTRACT. We characterize conjugate Banach spaces X^* having the Radon-Nikodym Property as those spaces such that any separable subspace of X has a separable conjugate. Several applications are given.

Introduction. There are several equivalent formulations of the Radon-Nikodym Property (RNP) in Banach spaces; we give perhaps the earliest definition: a Banach space X has RNP if given any finite measure space (S, Σ, μ) and any X valued measure m on Σ, with m having finite total variation and being absolutely continuous with respect to μ, then m is the indefinite integral with respect to μ of an X valued Bochner integrable function on S. The first study of this property was by Dunford and Pettis [4] and Phillips [11] (see also [5]).

It follows from the work of Dunford and Pettis and Phillips that reflexive Banach spaces and separable conjugate spaces have RNP. More generally, the following is true:

THEOREM A. *If X is a Banach space such that for any separable subspace Y of X, Y^* is separable, then X^* has RNP.*

The above result was observed by Uhl [15] and also can be obtained from a result of Grothendieck (Theorem B below).

The first characterizations of RNP were given by Grothendieck in [6]. Grothendieck's approach, the one we shall use, is that of studying certain classes of operators. An operator $T: X \to Y$ is a continuous linear function T from the Banach space X to the Banach space Y. An operator $T: X \to Y$ is said to be an integral operator if there exist a compact Hausdorff space K, a Radon measure μ on K, and operators R, and S, such that

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$$\begin{array}{ccc} X & \xrightarrow{T} & Y \overset{Q}{\subseteq} Y^{**} \\ R\big\downarrow & & \big\uparrow S \\ C(K) & \xrightarrow{J} & L_1(K, \mu) \end{array}$$

is commutative. The operator Q from Y to Y^{**} is the canonical evaluation operator; the operator J is the canonical operator from $C(K)$, the continuous real (or complex) valued functions on K, to $L_1(K, \mu)$, the equivalence classes of μ-measurable, absolutely summable functions on K. An operator T: $X \rightarrow Y$ is nuclear if there exist sequences $\{x_n^*\} \subseteq X^*$, $\{y_n\} \subseteq Y$ such that $\Sigma_{n=1}^{\infty} \|x_n^*\| \cdot \|y_n^*\| < +\infty$ and $Tx = \Sigma_{n=1}^{\infty} x_n^*(x) y_n$. Let K be a compact Hausdorff space and μ a Radon measure on K. A bounded subset of $L_1(K, \mu)$ is said to be equi-measurable [6, p. 20] (with respect to μ) if for each $\epsilon > 0$ there exists a compact subset K_0 of K such that $|\mu|(K \backslash K_0) < \epsilon$ and $\{f|_{K_0}: f \in S\}$ is a relatively compact subset of $L_\infty(K_0, \mu)$. Grothendieck proved the following [6, Proposition 9, p. 64]:

THEOREM B. *Let X be a Banach space, μ a Radon measure on the compact Hausdorff space K, and T an operator from X to $L_\infty(K, \mu)$; the operator JT is nuclear if and only if $\{JTx: \|x\| \leqslant 1\}$ is an equi-measurable subset of $L_1(K, \mu)$.*

From this theorem the following results can be obtained:

(B.1) X^* has RNP if and only if every integral operator T: $X \rightarrow L_1(S, \Sigma, \mu)$ is nuclear ($L_1(S, \Sigma, \mu)$ any measure space). (This is implicit in [6], but see [3] for a development of this approach.)

(B.2) X has RNP if and only if for any operator T: $L_1(S, \Sigma, \mu) \rightarrow X$ there exist a set Γ and operators S: $l_1(\Gamma) \rightarrow X$, R: $L_1(S, \Sigma, \mu) \rightarrow l_1(\Gamma)$ such that $SR = T$. (This result was perhaps first obtained in [9] where several applications are given.)

We now give a geometrical characterization of RNP. The following definition is due to Rieffel [13] (who also proved a Radon-Nikodym theorem [14]). A subset S of a Banach space will be called dentable if for every $\epsilon > 0$ there is an $x \in S$ such that $x \notin \overline{c}(S \backslash B(x, \epsilon))$. ($B(X, \epsilon)$ is the closed ball about x of radius ϵ and $\overline{c}(M)$ is the closed convex hull of the set M.) Rieffel proved that if X is a Banach space such that every bounded subset of X is dentable then X has RNP [13]. In [10] Maynard made the following definition: a subset S of a Banach space will be called s-dentable if for every $\epsilon > 0$ there is an $x \in S$ such that $x \notin s(S \backslash B(x, \epsilon))$ ($s(M)$, the sequential hull of M, is the set of all converging series $\Sigma_{i=1}^{\infty} \lambda_i x_i$ such that $\lambda_i \geqslant 0$, $\Sigma_{i=1}^{\infty} \lambda_i = 1$, and $x_i \in M$). Maynard proved that if a Banach space X has a bounded, non-s-dentable subset then X fails RNP. Recently, R. Phelps and W. J. Davis [1] have shown that if a Banach space has a bounded, nondentable subset then it has a bounded, non-s-dentable subset. These results may be combined to give the following result:

THEOREM C. *A Banach space X has RNP if and only if every bounded subset of X is dentable.*

This is by no means a comprehensive discussion of the Radon-Nikodym Property. The reader is referred to the papers listed above as well as their bibliographies for more information.

Our purpose here is to prove the converse of Theorem A: if X^* has RNP then for every separable subspace Y of X, Y^* is separable. We prove a more general result (Theorem 1 below) from which the above result follows. We give several applications of this result.

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Results. We begin with the following elementary observation.

LEMMA 1. *Let Y be a nonseparable Banach space. Then for $\epsilon > 0$, there exists for every countable ordinal α, $y_\alpha \in Y$, $y_\alpha^* \in Y^*$ such that $\|y_\alpha\| = 1$, $\|y_\alpha^*\| < 1 + \epsilon$ and*

$$y_\beta^*(y_\alpha) = \begin{cases} 1, & \alpha = \beta, \\ 0, & \alpha < \beta. \end{cases}$$

PROOF. Choose $y_1 \in Y$ and $y_1^* \in Y^*$ such that $\|y_1\| = \|y_1^*\| = y_1^*(y_1) = 1$. Assume we have made the construction for all α, $\alpha < \beta$, where β is a countable ordinal. Since $[y_\alpha]_{\alpha<\beta}$, the closed linear span of $\{y_\alpha\}_{\alpha<\beta}$, is separable there exists a $z_\beta^* \in Y^*$ such that $z_\beta^* \neq 0$, but $z_\beta^*(y_\alpha) = 0$ for all α, $\alpha < \beta$. Let $y_\beta^* = (1 + \epsilon/2)z_\beta^*/\|z_\beta^*\|$. Since $1 < \|y_\beta^*\| < 1 + \epsilon$ there exists y_β, $\|y_\beta\| = 1$ such that $y_\beta^*(y_\beta) = 1$.

If we let Δ denote the Cantor set, by a Haar system on Δ we mean a sequence of functions $\{h_{n,i}\} \subseteq C(\Delta)$, $n = 0, 1, 2, \cdots$, $i = 0, 1, \cdots, 2^n - 1$; $h_{n,i} = \chi_{A_{n,i}}$ (the characteristic function of the set $A_{n,i}$); $A_{0,0} = \Delta$; each $A_{n,i}$ is nonempty, open and closed; for each n, $\bigcup_{i=0}^{2^n-1} A_{n,i} = \Delta$ and $\{A_{n,i}\}$ is pairwise disjoint; $A_{n,i} = A_{n+1,2i} \cup A_{n+1,2i+1}$; and, for each choice of indices i_n, $0 \leqslant i_n \leqslant 2^n - 1$, $\bigcap_{n=0}^{\infty} A_{n,i_n}$ is either empty or a one point set.

THEOREM 1. *If X is a separable Banach space such that X^* is nonseparable, then for $\epsilon > 0$ there exist a subset Δ of the unit sphere of X^* which is weak** *homeomorphic to the Cantor set, a Haar system* $\{h_{n,i}\}$ *for* Δ, *and a sequence* $\{x_{n,i}\} \subseteq X$ *with* $\|x_{n,i}\| < 1 + \epsilon$ *such that if* $T: X \to C(\Delta)$ *is the canonical evaluation operator, then*

$$\sum_{n=0}^{\infty} \sum_{i=0}^{2^n-1} \|Tx_{n,i} - h_{n,i}\| < \epsilon.$$

PROOF. Since X^* is nonseparable, apply Lemma 1 to obtain $\{x_\alpha^*\} \subseteq X^*$, $\{x_\alpha^{**}\} \subseteq X^{**}$, $1 \leqslant \alpha < \omega_1$, ω_1 the first uncountable ordinal, such that $\|x_\alpha^*\| = 1$, $\|x_\alpha^{**}\| < 1 + \epsilon$, and $x_\alpha^{**}(x_\alpha^*) = 1$, $x_\beta^{**}(x_\alpha^*) = 0$ if $\alpha < \beta$. Since $\{x^*: \|x^*\| \leqslant 1\}$, the unit ball of X^*, is a compact metric space in the weak* topology and $\{x_\alpha^*\}$ is an uncountable subset of the unit ball, the set A of condensation points of $\{x_\alpha^*\}$ contains all but an at most countable subset of $\{x_\alpha^*\}$. Thus there exists a countable ordinal γ such that for any $\beta \geqslant \gamma$ and any weak* open set U containing x_β^*, the set $U \cap \{x_\alpha^*\}_{\alpha \geqslant \beta}$ is uncountable.

We shall construct for each $n = 0, 1, 2, \cdots$ weak* open sets in the unit ball of X^*, and a sequence $\{x_{n,i}\}_{n=0,i=0}^{\infty\ \ 2^n-1}$ in X such that

(1) weak* diameter $(U_{n,i}) \leqslant 1/(n+2)$ and the weak* closure of $U_{n,i}$, $\bar{U}_{n,i}^*$, is disjoint from $\{x^*: \|x^*\| \leqslant 1/(n+2)\}$;

(2) $U_{n,i} \cap A \neq \varnothing$;

(3) $U_{n+1,2i} \cup U_{n+1,2i+1} \subseteq U_{n,i}$;

(4) $x_{n,i} \in X$, $\|x_{n,i}\| < 1 + \epsilon$ and for each n, $|x^*(x_{n,i}) - \delta_{ij}| < \epsilon/4^{n+1}$ for $x^* \in U_{n,j}$.

For $n = 0$, choose any $x_\beta^* \in A$. Since $x_\beta^{**}(x_\beta^*) = 1$ and $\|x_\beta^{**}\| < 1 + \epsilon$, we know by Helly's Theorem [2, Theorem 3, p. 38] that there exists an $x_{0,0} \in X$, $\|x_{0,0}\| < 1 + \epsilon$, such that $x_\beta^*(x_{0,0}) = 1$. Let $U_{0,0}$ be a weak* open neighborhood of x_β^* of weak* diameter less than ½, $U_{0,0} \subseteq \{x^*: \|x^*\| \leqslant 1$ and $|x^*(x_{0,0}) - 1| < \epsilon/4\}$, and $\bar{U}_{0,0}^* \cap \{x^*: \|x^*\| \leqslant ½\} = \varnothing$.

Assume we have made the construction up to n. Choose $x_{\beta_{n,i}}^* \in U_{n,i} \cap A$ with $\beta_{n,0} < \beta_{n,1} < \cdots < \beta_{n,2^n-1}$. Choose $x_{\beta_{n+1,0}}^* \in U_{n,0} \cap A$ with $\beta_{n+1,0} > \beta_{n,2^n-1}$. Since $x_{\beta_{n+1,0}}^{**}(x_{\beta_{n+1,0}}^*) = 1$ and $x_{\beta_{n+1,0}}^{**}(x_\alpha^*) = 0$ for all $\alpha < \beta_{n+1,0}$, there exists by Helly's theorem an $x_{n+1,0} \in X$, $\|x_{n+1,0}\| < 1 + \epsilon$, such that $x_{\beta_{n,i}}^*(x_{n+1,0}) = 0$ for $0 \leqslant i < 2^n$ and $x_{\beta_{n+1,0}}^*(x_{n+1,0}) = 1$. Choose $x_{\beta_{n+1,2}}^* \in U_{n,1} \cap A$ such that $\beta_{n+1,2} > \beta_{n+1,0}$ and $|x_{\beta_{n+1,2}}^*(x_{n+1,0})| < \epsilon/4^{n+2}$. (This happens because $x_{n+1,0}$ vanishes at some point of $U_{n,1} \cap A$ so $x_{n+1,0}$ is less than $\epsilon/4^{n+2}$ on some open, hence uncountable, subset of $U_{n,1} \cap A$; in this uncountable set there must be a point of index larger than $\beta_{n+1,0}$.) Now choose $x_{n+1,2} \in X$, $\|x_{n+1,2}\| < 1 + \epsilon$, $x_{\beta_{n,i}}^*(x_{n+1,2}) = 0$, $0 \leqslant i < 2^n$, $x_{\beta_{n+1,0}}^*(x_{n+1,2}) = 0$ and $x_{\beta_{n+1,2}}^*(x_{n+1,2}) = 1$. In general, for $0 \leqslant k < 2^n$, choose $x_{\beta_{n+1,2k}}^* \in$

$U_{n,k} \cap A$, $\beta_{n+1,2k} < \beta_{n+1,2(k+1)}$, $x_{n+1,2k} \in X$, $\|x_{n+1,2k}\| < 1 + \epsilon$ such that

(i) $x^*_{\beta_{n,i}}(x_{n+1,2k}) = 0, 0 \leqslant i, k < 2^n$;

(ii) $x^*_{\beta_{n+1,2l}}(x_{n+1,2k}) = 0, 0 \leqslant l < k < 2^n$;

(iii) $x^*_{\beta_{n+1,2l}}(x_{n+1,2k}) = 1, 0 \leqslant l = k < 2^n$;

(iv) $|x^*_{\beta_{n+1,2l}}(x_{n+1,2k})| < \epsilon/4^{n+2}, 0 \leqslant k < l < 2^n$.

Choose $x^*_{\beta_{n+1,1}} \in U_{n,0} \cap A$ such that $|x^*_{\beta_{n+1,1}}(x_{n+1,2k})| < \epsilon/4^{n+2}$ for $0 \leqslant k < 2^n$ and $\beta_{n+1,1} > \beta_{n+1,2^{n+1}-2}$. Choose $x_{n+1,1} \in X$, $\|x_{n+1,1}\| < 1 + \epsilon$ such that $x^*_{\beta_{n,i}}(x_{n+1,1}) = 0$ for $0 \leqslant i < 2^n$, $x^*_{\beta_{n+1,2k}}(x_{n+1,1}) = 0$ for $0 \leqslant k < 2^n$ and $x^*_{\beta_{n+1,1}}(x_{n+1,1}) = 0$. In general, for $0 \leqslant k < 2^n$, choose $x^*_{\beta_{n+1,2k+1}} \in U_{n,k} \cap A$, $\beta_{n+1,2k+3} > \beta_{n+1,2k+1}$, and $x_{n+1,2k+1} \in X$, $\|x_{n+1,2k+1}\| < 1 + \epsilon$, such that

(v) $x^*_{\beta_{n,i}}(x_{n+1,2k+1}) = 0, 0 \leqslant i < 2^n$;

(vi) $x^*_{\beta_{n+1,2l}}(x_{n+1,2k+1}) = 0, 0 \leqslant k, l < 2^n$;

(vii) $x^*_{\beta_{n+1,2l+1}}(x_{n+1,2k+1}) = 0, 0 \leqslant l < k < 2^n$;

(viii) $x^*_{\beta_{n+1,2l+1}}(x_{n+1,2k+1}) = 1, 0 \leqslant l = k < 2^n$;

(ix) $|x^*_{\beta_{n+1,2l+1}}(x_{n+1,2k+1})| < \epsilon/4^{n+2}, 0 \leqslant k < l < 2^n$.

Define, for $0 \leqslant j < 2^{n+1}$,

$$U'_{n+1,j} = \{x^* \in U_{n,[j/2]}: |x^*(x_{n+1,k}) - \delta_{kj}| < \epsilon/4^{n+2} \text{ for all } k, 0 \leqslant k < 2^{n+1}\}.$$

We have that $x^*_{\beta_{n+1,j}} \in U'_{n+1,j}$ for $0 \leqslant j < 2^{n+1}$. Choose $U_{n+1,j}$ a weak* neighborhood of $x^*_{\beta_{n+1,j}}$ with weak* diameter less than $1/(n+3)$, weak* closure of $U_{n+1,j}$ is disjoint from $\{x^*: \|x^*\| \leqslant 1/(n+3)\}$, and $U_{n+1,j} \subseteq U'_{n+1,j}$.

This completes the construction.

Let $\Delta = \bigcap_{n=1}^{\infty} [\bigcup_{i=0}^{2^n-1} \bar{U}^*_{n,i}]$. As is well known [2, p. 93], Δ is homeomorphic to the Cantor set and we have that $\Delta \subseteq \{x^*: \|x^*\| = 1\}$. If we let $h_{n,i} = \chi_{A_{n,i}}$, $A_{n,i} = \Delta \cap \bar{U}^*_{n,i}$, then $\{h_{n,i}\}$ is a Haar system and

$$\sup\{|x^*(x_{n,i}) - h_{n,i}(x^*)|: x^* \in \Delta\}$$
$$\leqslant \sup\{|x^*(x_{n,i}) - h_{n,i}(x^*)|: x^* \in \bigcup U_{n,j}, 0 \leqslant j < 2^n\} \leqslant \epsilon/4^{n+1}.$$

Let $T: X \to C(\Delta)$ be the canonical evaluation operator $(Tx)(x^*) = x^*(x)$; then we have that

$$\sum_{b=0}^{\infty} \sum_{i=0}^{2^n-1} \|Tx_{n,i} - h_{n,i}\| \leqslant \sum_{n=0}^{\infty} \sum_{i=0}^{2^n-1} \frac{\epsilon}{4^{n+1}} = \sum_{n=0}^{\infty} 2^n \frac{\epsilon}{4^{n+1}} = \frac{\epsilon}{2}. \quad \text{Q.E.D.}$$

COROLLARY 1. *If X is separable and S is a nonseparable subset of X^* in the norm topology and is a weak* G_δ set, then for $\epsilon > 0$ there exist a subset $\Delta \subseteq S$ which is weak* homeomorphic to the Cantor set, a Haar system $\{h_{n,i}\}$ on Δ, a sequence $\{x_{n,i}\} \subseteq X$, and a constant $C > 0$ such that $\|x_{n,i}\| \leqslant C$ and $\Sigma_{n=0}^\infty \Sigma_{i=0}^{2^n-1} \|Tx_{n,i} - h_{n,i}\| < \epsilon$ where $T: X \to C(\Delta)$ is the canonical evaluation operator.*

PROOF. The proof is essentially the same as that of Theorem 1 with the additional restriction that each $\bar{U}^*_{n,i} \subseteq V_n$ where V_n are weak* open sets such that $\bigcap_{n=0}^\infty V_n = S$.

THEOREM 2. *Suppose X^* has RNP. Then for every separable subspace Y of X, Y^* is separable.*

PROOF. Assume there exists a separable subspace Y of X such that Y^* is not separable. By Theorem 1, there exist a Haar system $\{h_{n,i}\}$ on the Cantor set and an operator $T: Y \to C(\Delta)$, $\|T\| \leqslant 1$, $\Sigma_{n=0}^\infty \Sigma_{i=0}^{2^n-1} \|Ty_{n,i} - h_{n,i}\| < \epsilon$, $\|y_{m,i}\| < 1 + \epsilon$. For ν any measure on Δ let $L_\infty(\Delta, \nu)$ denote the equivalence classes of ν essentially bounded functions on Δ. We shall consider T as an operator from Y to $L_\infty(\Delta, \nu)$. We may extend T to an operator $\widetilde{T}: X \to L_\infty(\Delta, \nu)$ since $L_\infty(\Delta, \nu)$ is an injective space [2, pp. 94–95]. We shall complete the proof in two different ways:

(1) Suppose ν is not purely atomic. Let K_0 be a compact subset of Δ, K_0 has no atoms, $\nu(K_0) > 0$. It is easy to see that $\{Ty_{n,i}|K_0: n = 0, 1, \cdots; 0 \leqslant i < 2^n\}$ is not relatively compact in $L_\infty(K_0, \nu)$. By Theorem B, JT is not nuclear ($J: L_\infty(\Delta, \nu) \to L_1(\Delta, \nu)$ the canonical operator). Thus by Theorem (B.1) X^* does not have RNP.

(2) Suppose ν is the measure on Δ such that $\int h_{n,i} d\nu = 2^{-n}$.

Regarding $2^n h_{n,i}$ as elements of $L_\infty(\Delta, \nu)^*$, let $S = \{\widetilde{T}^*(2^n h_{n,i})\}$. Suppose $n \geqslant m$ and $i \neq j$ if $n = m$; then

$$\|\widetilde{T}^*(2^n h_{n,i}) - \widetilde{T}^*(2^m h_{m,j})\| \geqslant \frac{1}{1+\epsilon}\left|2^n \int_{A_{n,i}} x^*(y_{n,i})d\nu - 2^m \int_{A_{m,j}} x^*(y_{n,i})d\nu\right|$$

$$= \frac{1}{1+\epsilon}\left|2^n \int_{A_{n,i}} h_{n,i}(x^*)d\nu - 2^m \int_{A_{m,j}} h_{n,i}(x^*)d\nu \right.$$

$$\left. + 2^n \int_{A_{n,i}} [x^*(y_{n,i}) - h_{n,i}(x^*)]\,d\nu - 2^m \int_{A_{m,j}} [x^*(y_{n,i}) - h_{n,i}]\,d\nu\right|$$

$$\geqslant \frac{1}{1+\epsilon}\left(1 - \frac{1}{2} - \frac{\epsilon}{4^{n+1}} - \frac{\epsilon}{4^{n+1}}\right) = \frac{1}{1+\epsilon}\left(\frac{1}{2} - \frac{2\epsilon}{4^{n+1}}\right).$$

By choosing $0 < \epsilon < 1/4$, then the distance between any two distinct points of S is greater than $1/4$; but $\widetilde{T}^*(2^n h_{n,i}) = \frac{1}{2}(\widetilde{T}^*(2^{n+1}h_{n+1,2i}) + T^*(2^{n+1}h_{n+1,2i+1}))$. Thus S is clearly not dentable (not even s-dentable); by Theorem C, X^* does not have RNP.

COROLLARY 2. *Let X be a Banach space such that there exists a separable subspace Y of X such that Y^* is nonseparable. (Equivalently, X^* does not have RNP.) Then there exists a separable subspace Z of X^* such that Z is not isomorphic to a subspace of a separable conjugate space.*

PROOF. It is proved in [15] that if every separable subspace Z of X^* is isomorphic to a subspace of a separable conjugate space, then X^* has RNP.

It is not difficult to see that the arguments of Lemma 1 and Theorem 1 may be repeated for higher ordinals. In particular, this argument will give a proof of the complex version of a theorem proved by Leach and Whitfield [8] in the real case:

THEOREM 3. *Let X be a Banach space such that* $\dim X < \dim X^*$. (dim X *is the smallest cardinal m such that there exists a set S of cardinality m such that $[S] = X$.) Then there exists a separable subspace Y of X such that Y^* is nonseparable.*

Let $\{h_{n,i}\}$ be a Haar system for the Cantor set and let μ be the measure on Δ such that $\int h_{n,i} d_\mu = \mu(A_{n,i}) = 2^{-n}$, $h_{n,i} = \chi_{A_{n,i}}$. Let $l_1 = \{(t_{n,i})$: $n = 0$, $i = 0$, $n = 1, 2, \cdots, 0 \leq i < 2^{n-1}, t_{n,i}$ real (or complex), $|t_{0,0}| + \sum_{n=1}^{\infty} \sum_{i=0}^{2^n-1} |t_{n,i}| < +\infty\}$. Let $b_{0,0} = h_{0,0}$; $b_{n,i} = h_{n,2i} - h_{n,2i+1}$, $n = 1, 2, \cdots, 0 \leq i < 2^{n-1}$; and define H: $l_1 \to L_\infty(\Delta, \mu)$ by $H((t_{n,i})) = t_{0,0}b_{0,0} + \sum_{n=1}^{\infty} \sum_{i=0}^{2^{n-1}-1} t_{n,i}b_{n,i}$.

THEOREM 4. *Let X be a separable Banach space and Y a Banach space and T: $X \to Y$ such that $T^*(Y^*)$ is nonseparable. Then there exist operators R: $l_1 \to X$ and S: $Y \to L_\infty(\Delta, \mu)$ such that $STR = H$, where H is the operator given above.*

PROOF. Since $T^*(Y^*)$ is nonseparable, $\{T^*y^*\colon \|y^*\| \leq 1\}$ is a non-norm-separable, weak* compact subset of X^*. By Corollary 1, there exist a subset $\Delta \subseteq \{T^*y^*\colon \|y^*\| \leq 1\}$, Δ weak* homeomorphic to the Cantor set, a Haar system $\{h_{n,i}\}$ on Δ, a sequence $\{x_{n,i}\} \subseteq X$, a constant $C > 0$ such that $\|x_{n,i}\| \leq C$, and $\sum_{n=0}^{\infty} \sum_{i=0}^{2^n-1} \|Ux_{n,i} - h_{n,i}\| < \epsilon < 1$ where U is the canonical evaluation operator. Define $\{b_{n,i}\}$ as above; $\{b_{n,i}\}$ is a Schauder basis for $C(\Delta)$. Define $g_{0,0} = Ux_{0,0}$, $g_{n,i} = U(x_{n,2i} - x_{n,2i+1})$, $n = 1, 2, \cdots, 0 \leq i < 2^{n-1}$. Since

$$\|g_{0,0} - b_{0,0}\| + \sum_{n=1}^{\infty} \sum_{i=0}^{2^{n-1}-1} \|g_{n,i} - b_{n,i}\| < \epsilon < 1$$

the Paley-Wiener stability theorem [12] shows the existence of an operator $A\colon C(\Delta) \to C(\Delta)$, such that A is an onto isomorphism, $\|A\| < 1 + \epsilon$, $\|A^{-1}\| < 1/(1 - \epsilon)$, and $Ag_{n,i} = b_{n,i}$. We have the following relations:

$$\{A^*\eta\colon \|\eta\| \leqslant 1\} \subseteq \{\nu\colon \|\nu\| \leqslant (1 + \epsilon)/(1 - \epsilon)\}$$

and

$$\{U^*A^*\eta\colon \|\eta\| \leqslant 1\} \subseteq \{U^*\nu\colon \|\nu\| \leqslant (1 + \epsilon)/(1 - \epsilon)\} \subseteq \{T^*y^*\colon \|y^*\| \leqslant (1 + \epsilon)/(1 - \epsilon)\}.$$

Since AU has dense range, U^*A^* is one-to-one. Let $\Delta_1 = \{U^*A^*\delta_k\colon \delta_k$ a positive point mass in $C(\Delta)^*\}$ and let $K = \{y^* \in Y^*\colon \|y^*\| \leqslant (1 + \epsilon)/(1 - \epsilon)\}$. Choose T^*y^* in Δ_1. Since K is a weak* compact subset of Y^* the canonical evaluation operator $V\colon Y \to C(K)$ is well defined. Let $Q\colon C(\Delta) \to C(K)$ be the operator such that $(Qf)y^* = f(k)$ where $T^*y^* = U^*A^*(\delta_k)$; Q is an isometry of $C(\Delta)$ into $C(K)$ and $QAU = VT$. Let $I\colon C(\Delta) \to L_\infty(\Delta, \mu)$ denote the canonical operator. Since Q is an isometry and $L_\infty(\Delta, \mu)$ is injective [2, pp. 94–95] there exists an operator $\widetilde{I}\colon C(K) \to L_\infty(\Delta, \mu)$ such that $\widetilde{I}Q = I$. Define $R\colon l_1 \to X$ by

$$R((t_{n,i})) = t_{0,0}x_{0,0} + \sum_{n=1}^{\infty} \sum_{i=0}^{2^{n-1}-1} t_{n,i}(x_{n,2i} - x_{n,2i+1}).$$

Combining the facts above we have that $\|R\| \leqslant 2C$, $IAUR = H$, and $(\widetilde{I}V)TR = \widetilde{I}(VT)R = \widetilde{I}(QAU)R = IAUR = H$. If we let $S = \widetilde{I}V$ then we have the desired result.

COROLLARY 3. *Let X be a separable Banach space such that X^* is not separable. Then X has a bounded biorthogonal set of the cardinality of the continuum.*

PROOF. By Theorem 1, for $\epsilon > 0$, there exist a subset Δ of the unit sphere of X^*, Δ weak* homeomorphic to the Cantor set, a Haar system $\{h_{n,i}\}$ on the Cantor set, and a sequence $\{x_{n,i}\} \subseteq X$, $\|x_{n,i}\| < 1 + \epsilon$ and

$$\sum_{n=0}^{\infty} \sum_{i=0}^{2^n-1} \|Tx_{n,i} - h_{n,i}\| < \epsilon$$

where T is the canonical evaluation operator. Let $x^* \in \Delta$. Choose the unique sequence (n, i_n) such that $h_{n,i_n}(x^*) = 1$, $n = 0, 1, \cdots$. Let x^{**} be any weak* cluster point in X^{**} of the sequence $\{x_{n,i_n}\}$, $\|x^{**}\| \leqslant 1 + \epsilon$. Since

$$\sum_{n=0}^{\infty} |x^*(x_{n,i_n}) - 1| < \epsilon,$$

the sequence $x^*(x_{n,i_n})$ converges to 1, but also clusters at $x^{**}(x^*)$ so $x^{**}(x^*) = 1$. For any $y^* \in \Delta, y^* \neq x^*$,

$$\sum_{n=0}^{\infty} |y^*(x_{n,i_n}) - h_{n,i_n}(y^*)| < \epsilon,$$

but there exists a positive integer N such that for $n \geqslant N$, $h_{n,i_n}(y^*) = 0$, so $\Sigma_{n=N}^{\infty} |y^*(x_{n,i_n})| < \epsilon$. Thus $y^*(x_{n,i_n})$ converges to 0, but also clusters at $x^{**}(y^*)$ so $x^{**}(y^*) = 0$. Thus for each $x^* \in \Delta$, there exists an $x^{**} \in X^{**}$, $\|x^{**}\| \leqslant 1 + \epsilon$, such that $x^{**}(x^*) = 1$ and $x^{**}(y^*) = 0$ for all $y^* \in \Delta,\ y^* \neq x^*$. Thus $\{(x^*, x^{**})\colon\ x^* \in \Delta\}$ is a biorthogonal system of the cardinality of the continuum.

COROLLARY 4. *Let X be a separable Banach space. A necessary and sufficient condition that X^* be nonseparable is that there exists a bounded biorthogonal sequence $\{(x_i^*, x_i^{**})\}$ in X^* such that $\{x_i^*\}$ is dense in itself in the weak* topology.*

PROOF. If X^* is nonseparable, by Corollary 3, there exists a bounded biorthogonal system $\{(x^*, x^{**})\}$ such that $\{x^*\}$ is weak* homeomorphic to the Cantor set. Thus we have only to choose a sequence $\{x_i^*\}$ in $\{x^*\colon\ x^* \in \Delta\}$ that is weak* dense in Δ.

If $\{(x_i^*, x_i^{**})\}$ is such a biorthogonal system then the construction in Theorem 1 can be repeated with slight modifications to construct an operator $T\colon\ X \to C(\Delta)$ such that $T^*(C(\Delta)^*)$ has nonseparable range.

COROLLARY 5. *Let X be a Banach space such that for any bounded sequence $\{x_i^*\}$ in X^*, the weak* closure of $\{x_i^*\}$ is norm separable. Then X^* has RNP.*

PROOF. If X^* does not have RNP then from Theorem 1, we know there exist an operator $S\colon\ L_1(\Delta, \nu) \to X^*$ (ν some nonatomic measure on the Cantor set Δ) and a Haar system $\{h_{n,i}\}$ on Δ, $\int h_{n,i}\,d\nu = 2^{-n}$ such that $S(2^n h_{n,i})$ does not have separable weak* closure.

We point out that the converse of Corollary 5 is false. Precisely, there exists a compact Hausdorff space K such that K is separable, uncountable, and has no perfect subsets. Since $C(K)^*$ is isometric to $l_1(K)$, $C(K)^*$ has RNP [4] but if $\{k_i\}$ is a dense sequence in K, then the weak* closure of $\{\delta_{k_i}\}$ in $C(K)^*$ contains all δ_k which is not a norm separable set.

To obtain such a space K, let $(n, i) = k_{n,i}$ be the sequence of pairs of integers for $n = 0, 1, 2, \cdots, 0 \leqslant i < 2^n$ and let k be any sequence of the form (n, i_n), $n = 0, 1, \cdots$, with $2i_n \leqslant i_{n+1} < 2(i_n + 1)$. The set of $\{k\}$ is uncountable. Define the topology on $\{k_{n,i}, k\}$ to be the following:

(1) each $\{k_{n,i}\}$ is an open set;

(2) a neighborhood basis of each $k = \{(n, i_n)\}$ is given by $U_N = \{k, k_{n,i_n}: n \geqslant N\}$ for each $N = 0, 1, 2, \cdots$.

It is easy to see that $\{k, k_{n,i}\}$ is a locally compact Hausdorff space, so we let K be the one-point compactification of this space.

For reference we state the following result.

COROLLARY 6. (1) *If X^* has RNP and Y is isomorphic to a subspace of a quotient space of X, then Y^* has RNP.*

(2) *If there is a subspace Y of X such that Y^* and $[X/Y]^*$ have RNP then X^* has RNP.*

PROOF. Since (1) is obvious we shall prove only (2). Suppose $Q: X \rightarrow X/Y$ is the canonical quotient operator. Let Z be a separable subspace of X. Since Q is onto there exists a separable subspace W of X, $Z \subseteq W$ and $Q(W)$ is closed in X/Y. Let $T: W \rightarrow Q(W)$, $T = Q|_W$. The kernel of the operator T is $W \cap Y$. Both $Q(W)$ and $W \cap Y$ are separable and their duals have RNP so their duals are separable. From this it is clear that W^* is separable so Z^* is separable. Thus X^* has RNP.

Finally, we state a tensor product formulation of Theorem 2 (see [6]).

COROLLARY 7. *Let X be a Banach space. For X^* to have RNP it is necessary and sufficient that for every Banach space Y, the natural operator from $X^* \hat{\otimes} Y^*$ to $[X \hat{\hat{\otimes}} Y]^*$ is onto.*

Questions. Related to Theorem C and our Theorem 2 is the following question: if X does not have RNP do there exist a bounded sequence $\{x_i\}$ in X and a $\delta > 0$ such that $\|x_i - x_j\| \geqslant \delta$ for all i, j with $i \neq j$ and for each i there exists $j, k, j \neq i \neq k$, such that $x_i = \frac{1}{2}(x_j + x_k)$? By Theorem C, if such a sequence exists then X does not have RNP. Our Theorem 2 shows there is such a sequence in conjugate spaces not having RNP.

Related to Corollary 5 is the following question: if the set of extreme points of the unit ball of X^* is a norm separable set, is X^* separable?

If X has RNP does every separable subspace of X embed in a separable conjugate space? This is a problem posed by Uhl [15].

Probably the best known question about a separable Banach space X with X^* nonseparable is the following: Does X have a subspace isomorphic to l_1 (the space of all absolutely summing sequences)? Since the preparation of this paper R. C. James [7] has shown that the answer to this question is negative. In fact, James' example seems to indicate that the construction in our Theorem 1 is the best possible.

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