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ITERATED INTEGRALS, FUNDAMENTAL GROUPS AND COVERING SPACES

BY

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ABSTRACT. Differential 1-forms are integrated iteratedly along paths in a differentiable manifold X. The purpose of this article is to consider those iterated integrals whose value along each path depends only on the homotopy class of the path. The totality of such integrals is shown to be dual, in an appropriate sense, to the "maximal" residually torsion free nilpotent quotient of the fundamental group $\pi_1(X)$. Taken as functions on the universal covering space $\widetilde{X}$, these integrals separate points of $\widetilde{X}$ if and only if $\pi_1(X)$ is residually torsion free nilpotent.

Let X be a differentiable manifold, and let $w_1, w_2, \cdots$ be 1-forms on X. Given a piecewise smooth path $\alpha: I \longrightarrow X$, we set $f_i(t) = w_i(\alpha(t), \dot{\alpha}(t))$, and define

$$\left\langle \int w_1 \cdots w_r, \alpha \right\rangle = \int_0^1 \int_0^{t_r} \cdots \int_0^{t_2} f_1(t_1)dt_1 \cdots f_{r-1}(t_{r-1})dt_{r-1} f_r(t_r)dt_r$$

for $r > 0$ and $= 1$ for $r = 0$. Any linear combination $\int u$ of such integrals is called an iterated integral and is taken as a function on the space of all piecewise smooth paths in X. Choose a base point x_0 of X. Denote by $\int_{x_0} u$ (resp. $\oint_{x_0} u$) the restriction of $\int u$ to the space of all piecewise smooth paths from x_0 (resp. piecewise smooth loops from x_0).

The general purpose of this paper is to relate the analysis of such iterated integrals on a differentiable manifold with its fundamental group.

In [1], we study the algebra π^1 of those iterated integrals $\oint_{x_0} u$ whose value on each loop depends only on its homotopy class. Such an iterated integral $\oint_{x_0} u$ is then a function on the fundamental group $\pi_1(X)$ and can be extended to a linear functional on the group algebra $k\pi_1(X)$, k being the real (or complex) number field.

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Taken as a vector space of linear functionals, π^1 annuls a subspace N of $k\pi_1(X)$. Let $\bar{J}$ be the augmentation ideal of $k\pi_1(X)$. Our main theorem is that $N = \bigcap_{s\geq 1} \bar{J}^s$. A corollary asserts that, if and only if $\pi_1(X)$ is residually torsion free nilpotent, then π^1, as a function algebra, separates the elements of $\pi_1(X)$.

An iterated integral $\int_{x_0} u$ whose value on each path depends only on its homotopy class, can be regarded as a function on the covering space $\widetilde{X}$ of X. A further result states that the algebra of all such iterated integrals $\int_{x_0} u$ separates points of $\widetilde{X}$ if and only if $\pi_1(X)$ is residually torsion free nilpotent.

This work brings to a satisfactory conclusion the study of the algebra π^1 in [1]. It turns out in [2] that π^1 is the 0th cohomology group of the cochain complex of all iterated integrals of forms of degree ≥ 1. Therefore the notation π^1 is used only in this introduction and will be appropriately replaced in the text of this article.

It was A. N. Parsin who first investigated the relations between the fundamental group of a Riemann surface and iterated integrals of holomorphic 1-forms [6]. Our results for iterated integrals of C^∞ 1-forms provide some insight into problems arising from his work.

Instead of a differentiable manifold and its exterior algebra, we shall consider a predifferentiable space X and a differential graded subalgebra A of the exterior algebra $\Lambda(X)$. Such a generalized framework is not only designed for wider applications in the future but also is necessary.

In §§1–3, we study a filtered cochain complex A' of iterated integrals constructed from A. In §4, we construct a smooth cubical chain complex $C_*(\Omega(X; x_0))$ for the piecewise smooth loop space $\Omega(X; x_0)$. This chain complex $C_*(\Omega(X; x_0))$ has an associative multiplication and is filtered by the powers of its augmentation ideal. The augmentation filtration gives rise to an ascending filtration $\{B(s)\}$ of the cochain complex $C^*(\Omega(X; x_0); k)$, which results in a subcomplex $B = \bigcup B(s)$. In §§5–6, we prove the main theorem. Our method is essentially a comparison of the cohomology spectral sequences of A' and B on the level of dimensions 0 and 1. For this lower dimensional situation, we choose not to use spectral sequences in a full and formal way. The material in §7 represents a digression into related aspects of group theory. In §§8–9, our result regarding the universal covering space $\widetilde{X}$ is proved.

Our results have been announced in [3] together with an application to nilmanifolds. For other applications of iterated integrals, see [1] and [2]. We also call the reader's attention to a related work of a homotopy theory based on differential forms, which has been recently developed by D. P. Sullivan [8].(2)

(2) After this work was completed, the author learned from D. P. Sullivan that he had also proved similar results in the context of his theory.

Throughout this paper, k will denote the real (or complex) number field. All functions and forms will be k-valued.

1. **Iterated integrals.** By a convex region we mean a closed convex set in R^n for some finite n.

DEFINITION. A predifferentiable space X is a topological space equipped with a family of maps called plots which satisfy the following conditions:

(a) Every plot is a continuous map of the type ϕ: $U \longrightarrow X$ where U is a convex region.

(b) If U' is also a convex region (not necessarily of the same dimension as U) and if θ: $U' \longrightarrow U$ is a C^∞ map, the $\phi\theta$ is also a plot.

(c) Every constant map $U \longrightarrow X$ is a plot.

REMARK. In [2], a predifferentiable space is called a "differentiable space". We propose to amend the definition of a differentiable space by adding the following condition:

(d) Let ϕ: $U \longrightarrow X$ be a continuous map and let $\{\theta_i$: $U_i \longrightarrow U\}$ be a family of C^∞ maps, U, U_i being convex regions, such that a function f on U is C^∞ if and only if each $f \circ \theta_i$ is C^∞ on U_i. If each $\phi \circ \theta_i$ is a plot of X, then ϕ itself is a plot of X.

DEFINITION. A p-form w on a predifferentiable space X is a rule which assigns to each plot ϕ: $U \longrightarrow X$ a p-form w_ϕ on the convex region U such that, if θ: $U' \longrightarrow U$ is given as in (b), then $w_{\phi\theta} = \theta^* w_\phi$.

In an obvious manner, we obtain the exterior algebra $\Lambda(X) = \Sigma \Lambda^p(X)$ of X.

In particular, every C^∞ manifold X with or without boundary is a predifferentiable space, whose family of plots consists of all C^∞ maps from a convex region to X. Every differentiable form w on the manifold X is naturally a differential form on the predifferentiable space X such that $w_\phi = \phi^* w$.

A continuous map α: $I \longrightarrow X$ is called a path (i.e., a piecewise smooth path) if the unit interval I can be subdivided into a finite number of subintervals, on each of which the restriction of α is a plot of X.

Let $\Omega(X)$ denote the space of all paths on X with the compact open topology. Every map σ: $U \longrightarrow \Omega(X)$ gives rise to a map ϕ_σ: $U \times I \longrightarrow X$ given by $(\xi, t) \longmapsto \sigma(\xi)(t)$. A plot of $\Omega(X)$ is defined to be a continuous map σ: $U \longrightarrow \Omega(X)$, U being a convex region, such that (a) all paths $\sigma(\xi)$, $\xi \in U$, have a common initial point and a common endpoint in X, and (b) for some partition $0 = t_0 < \cdots < t_r = 1$ of I, the restriction of ϕ_σ to each $U \times [t_{i-1}, t_i]$ is a plot of X.

Denote by $\Omega(X; x_0)$ (resp. $\Omega(X; x_0, *)$) the subspace of $\Omega(X)$ consisting of all loops (resp. paths) from a fixed base point x_0. The plots of $\Omega(X; x_0)$ (resp. $\Omega(X; x_0, *)$) are those of $\Omega(X)$ whose image lies in $\Omega(X; x_0)$ (resp. $\Omega(X; x_0, *)$).

We point out that the total path space $\Omega(X)$ is different from the predifferentiable space $P(X)$ defined in [2], because every plot of $\Omega(X)$ is required to have a common initial point as well as a common end point.

If $w_1, w_2, \cdots$ are forms of respective positive degree $p_1, p_2, \cdots$ on a predifferentiable space X, we define in [2] a form $\int w_1 \cdots w_r$ of degree $p_1 + \cdots + p_r - r$ on $\Omega(X)$, which is called an iterated integral. We shall use those iterated integrals such that all $w_1, \cdots, w_r$ are of degree 1 with the possible exception of one of them which may be of degree 2. In this case $\int w_1 \cdots w_r$ is a form of degree either 0 or 1 on $\Omega(X)$.

Consider first the case of all $w_1, \cdots, w_r$ being of degree 1. Let $\sigma\colon U \longrightarrow \Omega(X)$ be a plot and let $\phi_\sigma\colon U \times I \longrightarrow X$ be given as before. Then the 1-forms $(w_1)_{\phi_\sigma}, \cdots, (w_r)_{\phi_\sigma}$ are piecewise defined on $U \times I$. Write

$$(w_i)_{\phi_\sigma} = a_i(\xi, t)\,dt + \sum b_{ij}(\xi, t)\,d\xi^j.$$

Then the 0-form (i.e., C^∞ function) $\int w_1 \cdots w_r$ on $\Omega(X)$ is given by

$$\left(\int w_1 \cdots w_r\right)_\sigma = \int_0^1 \cdots \int_0^{t_3}\int_0^{t_2} a_1(\xi, t_1)\,dt_1 a_2(\xi, t_2)\,dt_2 \cdots a_r(\xi, t_r)\,dt_r.$$

Consider now the case where all $w_1, \cdots, w_r$ are of degree 1 except w_λ, $1 \leqslant \lambda \leqslant r$, which is of degree 2. Then

$$(w_\lambda)_{\phi_\sigma} = \sum a_{\lambda j}(\xi, t)\,dt \wedge d\xi^j + \sum b_{\lambda jk}(\xi, t)\,d\xi^j \wedge d\xi^k,$$

and $\int w_1 \cdots w_r$ is given by

$$\left(\int w_1 \cdots w_r\right)_\sigma = \sum_j \int_0^1 \cdots \int_0^{t_{\lambda+1}} \cdots \int_0^{t_2} a_1(\xi, t_1)\,dt_1 \cdots a_{\lambda j}(\xi, t_\lambda)\,dt_\lambda \cdots a_r(\xi, t_r)\,dt_r d\xi^j.$$

Iterated integrals $\int w_1 \cdots w_r$ do not depend on piecewise smooth reparametrization. More precisely, if τ is a piecewise C^∞ orientation preserving homeomorphism of I onto itself and if $\sigma, \nu\colon U \longrightarrow \Omega(X)$ are plots which are related by $\phi_\nu(\xi, t) = \phi_\sigma(\xi, \tau(t))$, then

$$\left(\int w_1 \cdots w_r\right)_\nu = \left(\int w_1 \cdots w_r\right)_\sigma.$$

By restriction, each iterated integral $\int w_1 \cdots w_r$ can be also taken as a form on $\Omega(X; x_0)$ (resp. $\Omega(X; x_0, *)$). The restriction will be denoted by $\oint_{x_0} w_1 \cdots w_r$ (resp. $\int_{x_0} w_1 \cdots w_r$).

A formula for exterior differentiation of iterated integrals has been given by the corollary to Proposition 4.1.2 of [2]. In particular, if $w_1, \cdots, w_r$ are

1-forms on X, then

$$
(1.1)\quad \begin{aligned} d\int w_1 \cdots w_r = &\sum_{1\leqslant i\leqslant r} (-1)^i \int w_1 \cdots w_{i-1} dw_i w_{i+1} \cdots w_r \\ &- \sum_{1\leqslant i<r} (-1)^i \int w_1 \cdots w_{i-1}(w_i \wedge w_{i+1}) w_{i+2} \cdots w_r. \end{aligned}
$$

Let σ: $U \longrightarrow \Omega(X; x_0)$ and σ': $U' \longrightarrow \Omega(X, x_0, *)$ be plots. Define

$$\sigma \times \sigma' \colon\ U \times U' \longrightarrow \Omega(X; x_0, *)$$

to be the plot given by $(\sigma \times \sigma')(\xi, \xi')(t) = \sigma(\xi)(2t)$ for $0 \leqslant t \leqslant ½$ and $=$ $\sigma'(\xi')(2t - 1)$ for $½ \leqslant t \leqslant 1$.

If U is compact, define

$$\iint_\sigma w_1 \cdots w_r = \int_U \left(\int w_1 \cdots w_r\right)_\sigma$$

which is taken to be 0 when dim U does not agree with the degree of $\int w_1 \cdots w_r$ and to be 1 when dim $U = 0$ and $r = 0$.

If U and U' are both compact, then, according to [2, (1.6.2)], we have

$$(1.2)\qquad \iint_{\sigma\times\sigma'} w_1 \cdots w_r = \sum_{0\leqslant i\leqslant r} \iint_\sigma w_1 \cdots w_i \iint_{\sigma'} w_{i+1} \cdots w_r.$$

If U is of dimension 0 and therefore consists of a single point, then the image of σ is a single path $\alpha \in \Omega(X; x_0)$. Verify that $\int_\sigma \int w_1 \cdots w_r$ agrees with $\int_\alpha w_1 \cdots w_r$ as given in [1].

2. **Filtration.** We begin with a differential graded subalgebra A of the exterior algebra $\Lambda(X)$. The only assumption at this moment is that

$$(2.1)\qquad dA^0 = A^1 \cap d\Lambda^0(X).$$

The reader may keep in mind of the special case of $A = \Lambda(X)$.

For $s \geqslant 0$, let $A_\Omega(s)$ (resp. $A'(s)$, $A''(s)$) the subcomplex of $\Lambda(\Omega(X))$ (resp. $\Lambda(\Omega(X; x_0))$, or $\Lambda(\Omega(X; x_0, *))$) spanned by iterated integrals of the type $\int w_1 \cdots w_r$ (resp. $\oint_{x_0} w_1 \cdots w_r$, or $\int_{x_0} w_1 \cdots w_r$), $0 \leqslant r \leqslant s$, $w_1, \cdots, w_r$ $\in A^+ = \Sigma_{p>0} A^p$. Then

$$(2.2)\qquad k = A'(0) \subset A'(1) \subset \cdots \subset A'(s) \subset \cdots.$$

Set $A'(s) = 0$ for $s < 0$, and define $A' = \bigcup A'(s)$. Similarly we define A'' and A_Ω

We shall first study the filtered cochain complex A' and, in particular, A'^0 and A'^1. The groups A'^p, $p > 1$, will be mentioned mainly for the sake of smoothing the presentation.

For any r, s, define

$$A'(s, s-r) = \{u \in A'(s); du \in A'(s-r)\},$$
$$L'^s_r = A'^0(s, s-r)/A'^0(s-1, s-r),$$
$$M'^s_r = A'^1(s, s-r)/[A'^1(s-1, s-r) + dA'^0(s+r-1, s)].$$

The reader may notice that L'^s_r and M'^s_r are simply terms of degree 0 and 1 of the cohomology spectral sequence of the cochain complex arising from the ascending filtration (2.2).

For $r > s$,

$$L'^s_r = L'^s_s = H^0(A'(s))/H^0(A'(s-1)).$$

Verify the canonical isomorphisms

$$L'^s_1 \approx H^0(A'(s)/A'(s-1)), \qquad M'^s_1 \approx H^1(A'(s)/A'(s-1)).$$

The exterior differentiation induces a homomorphism d_r so that

$$0 \longrightarrow L'^s_{r+1} \longrightarrow L'^s_r \xrightarrow{d_r} M'^{s-r}_r \tag{2.3}$$

is an exact sequence.

In order to calculate L'^s_1 and M'^s_1, we introduce the cochain complex $\bar{A}$ given by $\bar{A}^q = 0$ for $q < 0$, $= A^1/dA^0$ for $q = 0$ and $= A^{q+1}$ for $q > 0$. Then, for $q \geqslant 0$, $H^q(\bar{A}) = H^{q+1}(A)$.

There is a homomorphism

$$\bigotimes^s A^+ \longrightarrow A'(s)/A'(s-1) \tag{2.4}$$

given by $w_1 \otimes \cdots \otimes w_s \longmapsto \oint_{x_0} w_1 \cdots w_s + A'(s-1)$. If $w_i = df$ for some $f \in A^0$, one can show that $\oint_{x_0} w_1 \cdots w_s \in A'(s-1)$. (See [2, (4.1.2)–(4.1.4)].) Thus (2.4) induces a homomorphism

$$\bigotimes^s \bar{A} \longrightarrow A'(s)/A'(s-1) \tag{2.5}$$

which is a map of cochain complexes owing to the formula for exterior differentiation of iterated integrals given by (1.1) and the more general formula in [2].

It has been proved in [2, Lemma 4.3.1] that the map (2.5) is actually an isomorphism provided the predifferentiable space is path connected (by piecewise smooth paths). It follows that there is an induced isomorphism

$$H(\bigotimes^s \bar{A}) \approx H(A'(s)/A'(s-1)). \tag{2.6}$$

Since $H^0(A'(s)/A'(s-1)) = L'^s_1$ and $H^1(A'(s)/A'(s-1)) = M'^s_1$, we obtain

$$L'^s_1 \approx H^0(\bigotimes^s \bar{A}); \tag{2.7}$$

(2.8) $$M'^s_1 \approx H^1(\otimes^s \bar{A}).$$

3. **Relation to de Rham cohomology.** The following algebraic lemma is an immediate consequence of the five lemma:

LEMMA 3.1. *In a commutative diagram of abelian groups*

$$\begin{array}{ccccccc} 0 & \longrightarrow & L_2 & \longrightarrow & L_1 & \longrightarrow & M \\ & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \psi} \\ 0 & \longrightarrow & L'_2 & \longrightarrow & L'_1 & \longrightarrow & M' \end{array}$$

let the rows be exact. If ϕ_1 is an epimorphism and if ψ is a monomorphism, then ϕ_2 is an epimorphism. If, in addition, ϕ_1 is also a monomorphism, then ϕ_2 is an isomorphism.

DEFINITION. Let X and Y be predifferentiable spaces. A continuous map h: $Y \longrightarrow X$ is called a differentiable map if, for every plot ϕ of Y, the map $h \circ \phi$ is a plot of X.

Let A_X and A_Y be differential graded subalgebras of $\Lambda(X)$ and $\Lambda(Y)$ respectively. Suppose that a differentiable map h: $Y \longrightarrow X$ induces a homomorphism h^*: $A_X \longrightarrow A_Y$. Then there is an induced cochain map h': $A'_X \longrightarrow A'_Y$.

We assume that both A_X and A_Y satisfy the condition (2.1).

THEOREM 3.1. *If h^* induces an isomorphism (resp. epimorphism) $H^1(A_X) \longrightarrow H^1(A_Y)$ and a monomorphism $H^2(A_X) \longrightarrow H^2(A_Y)$, then h' induces an isomorphism (resp. epimorphism), $H^0(A'_X) \approx H^0(A'_Y)$.*

PROOF. For A'_X and A'_Y, we have $(L'_X)^s_r$ and $(L'_Y)^s_r$. Consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & (L'_X)^s_{r+1} & \longrightarrow & (L'_X)^s_r & \longrightarrow & (M'_X)^s_r \\ & & \downarrow{\scriptstyle h^s_{r+1}} & & \downarrow{\scriptstyle h^s_r} & & \downarrow{\scriptstyle m^s_r} \\ 0 & \longrightarrow & (L'_Y)^s_{r+1} & \longrightarrow & (L'_Y)^s_r & \longrightarrow & (M'_Y)^s_r \end{array}$$

where the rows are exact, and the vertical arrows are induced by h'. If h^s_r is an isomorphism and if m^s_r is a monomorphism, then, according to the preceding lemma, h^s_{r+1} is an isomorphism. It also follows that m^s_r induces a monomorphism

$$m'^s_r\colon\ (M'_X)^s_r/d_r(L'_X)^s_r \longrightarrow (M'_Y)^s_r/d_r(L'_Y)^s_r.$$

From the definition, we observe that M'^s_{r+1} can be taken as a subgroup of

$M'^s_r/d_rL'^s_r$. Therefore m^s_{r+1} can be taken as a restriction of m'^s_r and is a monomorphism.

Starting from the fact that

$$h^s_1\colon\ (L'_X)^s_1 \approx H^1(\otimes^s \bar{A}_X) \longrightarrow H^1(\otimes^s \bar{A}_Y) \approx (L'_Y)^s_1$$

is an isomorphism and the fact that

$$m^s_1\colon\ (M'_X)^s_1 \approx H^2(\otimes^s \bar{A}_X) \longrightarrow H^2(\otimes^s \bar{A}_Y) \approx (M'_Y)^s_1$$

is a monomorphism, we now conclude that for $r \geqslant 1$, every h^s_r is an isomorphism. In particular

$$h^s_s\colon\ H^0(A'_X(s))/H^0(A'_X(s-1)) \approx H^0(A'_Y(s))/H^0(A'_Y(s-1))$$

is an isomorphism. Hence h induces isomorphisms

$$H^0(A'_X(s)) \approx H^0(A'_Y(s))$$

and

$$H^0(A'_X) = \varinjlim H^0(A'_X(s)) \approx \varinjlim H^0(A'_Y(s)) = H^0(A'_Y).$$

4. **A loop space cohomology.** A singular simplex $\Delta^q \longrightarrow X$ (or a singular cube $I^q \longrightarrow X$) is said to be smooth if it is a plot of the predifferentiable space X. The following condition is assumed to hold for X:

HYPOTHESIS I. The predifferentiable space X is connected by piecewise smooth paths. The piecewise smooth fundamental group $\pi_1(X)$ obtained by using piecewise smooth loops and piecewise smooth homotopies (i.e., smooth paths in the loop space $\Omega(X; x_0)$) coincides with the topological fundamental group.

Two smooth q-cubes $u, u'\colon\ I^q \longrightarrow \Omega(X; x_0)$ are said to be equivalent (by a reparametrization) if there exists a piecewise C^∞ orientation preserving homeomorphism $\tau\colon\ I \longrightarrow I$ such that

$$u'(\xi)(t) = u(\xi)(\tau(t)), \qquad \forall \xi \in I^q,\ t \in I.$$

The equivalence class of u will be denoted by $\bar{u}$.

If u and v are smooth cubes of $\Omega(X; x_0)$, define

$$\bar{u}\,\bar{v} = \overline{u \times v}. \tag{4.1}$$

This multiplication is associative. Moreover, the face operators are well defined for equivalence classes of smooth cubes.

Let $C_*(\Omega(X; x_0))$ be the chain complex having as a basis the totality of equivalence classes of smooth cubes of $\Omega(X; x_0)$ together with an element denoted by 1, which is of degree 0. We equip $C_*(\Omega(X; x_0))$ with an associative multiplication given by (4.1) and

(4.2) $$1\bar{u} = \bar{u}1 = \bar{u}, \qquad 1 \cdot 1 = 1.$$

We shall not distinguish a loop α and the 0-cube of $\Omega(X; x_0)$ it represents.

If u is a smooth 1-cube of $\Omega(X, x_0)$, then $\partial\bar{u}$ is of the type $\bar{\beta} - \bar{\alpha}$, where α and β are homotopic. Therefore $H_0(\Omega(X)) = Z \oplus Z\pi_1(X)$ where $Z\pi_1(X)$ denotes the integral group ring of $\pi_1(X)$. This slight deviation from the usual 0th loop space homology group (which is $Z\pi_1(X)$) stems from the introduction of the element $1 \in C_0(\Omega(X; x_0))$.

We remark that, for this work, the correctness of $H_q(\Omega(X; x_0))$, $q > 0$, is not our concern. For this reason, the chain complex $C_*(\Omega(X; x_0))$ is not normalized.

The augmentation ϵ: $C_*(\Omega(X; x_0)) \longrightarrow Z$ is given by $\epsilon 1 = 1$ and $\epsilon\bar{u} = 0$ or 1 according as u is a smooth cube of degree > 0 or $= 0$. The ideal $J = \operatorname{Ker} \epsilon$ is called the augmentation ideal of $C_*(\Omega(X; x_0))$. Take note that $J_q = C_q(\Omega(X; x_0))$ for $q > 0$ and that J_0 has a basis consisting of all $\bar{\alpha} - 1$, α being a loop at x_0.

Observe that $(J^s)_0 = (J_0)^s$ is spanned by all elements of the type

$$(\bar{\alpha}_1 - 1) \cdots (\bar{\alpha}_s - 1)$$

where $\alpha_1, \cdots, \alpha_s$ are loops at x_0.

Let $\bar{x}_0$ denote the equivalence class of the 0-cube at the constant loop at the base point x_0. Then $1 - \bar{x}_0 \in J$. In fact, for any $s \geqslant 1$, $1 - \bar{x}_0 = (1 - \bar{x}_0)^s \in J^s$.

For $s \geqslant 0$, define a subcomplex $B(s)$ of $C^*(\Omega(X; x_0); k)$ such that

$$B^q(s) = \{y \in C^q(\Omega(X; x_0); k); \langle y, J^{s+1} \rangle = 0\}.$$

Then $C^*(\Omega(X; x_0); k)$ has an ascending filtration

$$k = B(0) \subset B(1) \subset \cdots \subset B(s) \subset \cdots.$$

For $s < 0$, define $B(s) = 0$. Set $B = \bigcup B(s)$ which is a subcomplex of $C^*(\Omega(X; x_0); k)$.

We are going to characterize $H^0(B(s))$ which consists of the 0-cocycles of $C^0(\Omega(X; x_0); k)$ that annul J^{s+1}.

Since $H^0(\Omega(X; x_0); k) = \operatorname{Hom}_Z(Z \oplus Z\pi_1(X); k)$, there is a homomorphism $H^0(\Omega(X; x_0); k) \longrightarrow \operatorname{Hom}_Z(Z\pi_1(X); k)$. Let $\bar{J}$ be the augmentation ideal of the group ring $Z\pi_1(X)$. Then $\bar{J}^{s+1}$ is spanned by all $([\alpha_1] - [x_0]) \cdots ([\alpha_{s+1}] - [x_0])$, where $\alpha_1, \cdots, \alpha_{s+1}$ are loops at x_0, $[\alpha_i]$ denotes the element of $\pi_1(X)$ represented by α_i, and $[x_0]$ denotes the unit element of the fundamental group $\pi_1(X)$. We shall take $\operatorname{Hom}_Z(Z\pi_1(X)/\bar{J}^{s+1}; k)$ as a subgroup of $\operatorname{Hom}_Z(Z\pi_1(X); k)$.

The composite homomorphism

$$(4.3) \qquad H^0(B(s)) \twoheadrightarrow H^0(\Omega(X; x_0); k) \longrightarrow \mathrm{Hom}_Z(Z\pi_1(X); k)$$

sends every 0-cocycle $f \in H^0(B(s))$ to f' such that $f'([\alpha]) = \langle f, \bar{\alpha}\rangle$. Since $1 - \bar{x}_0 \in J^{s+1}$, we have $\langle f, 1\rangle = \langle f, \bar{x}_0\rangle = f'([x_0])$. It follows that the map (4.3) is injective. We assert that the image of the map (4.3) is precisely $\mathrm{Hom}_Z(Z\pi_1(X)/\bar{J}^{s+1}; k)$ and hence gives rise to an isomorphism

$$(4.4) \qquad H^0(B(s)) \approx \mathrm{Hom}_Z(Z\pi_1(X)/\bar{J}^{s+1}, k).$$

In order to verify this assertion, we first observe that J^{s+1} is also spanned by all elements of the types $(\bar{\alpha}_1 - \bar{x}_0) \cdots (\bar{\alpha}_{s+1} - \bar{x}_0)$, $\bar{\alpha}\,\bar{x}_0 - \bar{\alpha}$, $\bar{x}_0\bar{\alpha} - \bar{\alpha}$, $\bar{x}_0 - 1$ where $\alpha, \alpha_1, \cdots, \alpha_{s+1}$ are loops.

If $f \in H^0(B(s))$, then f annuls J^{s+1}, and its image f' in $\mathrm{Hom}_Z(Z\pi_1(X); k)$ annuls all elements of the type $([\alpha_1] - [x_0]) \cdots ([\alpha_{s+1}] - [x_0])$. Consequently $f' \in \mathrm{Hom}_Z(Z\pi_1(X)/\bar{J}^{s+1})$.

On the other hand, let f' be an arbitrary element of $\mathrm{Hom}_Z(Z\pi_1(X)/\bar{J}^{s+1}; k)$. Define $f \in C^0(\Omega(X; x_0); k)$ such that $\langle f, \bar{\alpha}\rangle = f'([\alpha])$ for every loop α, and $\langle f, 1\rangle = f'([x_0])$. Then f is a 0-cocycle and annuls every element listed in (4.5). Hence the isomorphism (4.4) is established.

5. **The main theorem.** For any s, r, define $B(s, s-r)$ such that

$$B^q(s, s-r) = \{y \in B^q(s); \delta y \in B^{q+1}(s-r)\}.$$

Define

$$L_r^s = B^0(s, s-r)/B^0(s-1, s-r),$$

$$M_r^s = B^1(s, s-r)/[B^1(s-1, s-r) + \delta B^0(s+r-1, s)].$$

Verify that

$$L_1^s \approx H^0(B(s)/B(s-1)), \quad M_1^s \approx H^1(B(s)/B(s-1)).$$

As in the case of L'^s_r and M'^s_r, the reader may keep in mind that L_r^s and M_r^s are terms of degree 0 and degree 1 of the cohomology spectral sequence of B arising from the filtration by $\{B(s)\}$.

As before, the boundary operator induces a homomorphism δ_r: $L_r^s \longrightarrow M_r^s$ so that

$$0 \longrightarrow L_{r+1}^s \longrightarrow L_r^s \xrightarrow{\delta_r} M_r^s$$

is an exact sequence.

There is a pairing

$$(5.1) \qquad A' \times C_*(\Omega(X; x_0)) \longrightarrow k$$

such that $\langle \oint_{x_0} w_1 \cdots w_r, 1 \rangle = 1$ or 0 according as $r = 0$ or 1 and

$$\left\langle \oint_{x_0} w_1 \cdots w_r, \bar{u} \right\rangle = \iint_u w_1 \cdots w_r$$

for every smooth q-cube u of $\Omega(X; x_0)$, $q = 0, 1$. If $r = 0$, then the r.h.s. is equal to 1 or 0 according as the degree of u is 0 or > 0.

LEMMA 5.1. *Under the pairing* (5.1), $\oint_{x_0} w_1 \cdots w_s$ *is orthogonal to* J^{s+1}.

PROOF. Since J_0 is spanned by elements of the type $\bar{\alpha} - 1$, the case of $s = 0$ is trivial. For $s > 0$, we use the formula (1.2) and thus prove the lemma by induction on s.

Noticing that (5.1) represents a pairing of chain complex with a cochain complex, we have the next assertion.

COROLLARY. *The pairing* (5.1) *gives rise to a filtration preserving cochain map*

$$A' \longrightarrow B. \tag{5.2}$$

Let $C_*(X)$ be the subcomplex of the singular chain complex of X having as a basis the totality of smooth simplices of X whose vertices are at the base point x_0. Let $H^*(X; k)$ be the corresponding cohomology over k. We shall require that the following three conditions hold for X in addition to Hypothesis I given in §4:

HYPOTHESIS II. For any choice of the base point x_0, the canonical homomorphism $H^q(A) \longrightarrow H^q(X; k)$ is an isomorphism for $q = 1$ and a monomorphism for $q = 2$.

HYPOTHESIS III. The condition (2.1) holds.

HYPOTHESIS IV. For any choice of the base point, every element of $\pi_1(X)$ can be represented by a smooth loop.

If X is a connected C^∞ manifold, then all four hypotheses are implied by the conditions: (a) $H^1(A)$ is isomorphic with $H^1(X; k)$; and (b) $H^2(A) \longrightarrow H^2(X; k)$ is monic.

Hypothesis IV does not seem to be necessary. However, the author sees no way of avoiding it without going into considerable complications.

LEMMA 5.2. *The cochain map* (5.2) *induces an isomorphism*

$$\phi_1^s\colon\ L'^s_1 \xrightarrow{\approx} L^s_1$$

and a monomorphism

$$\psi_1^s\colon\ M'^s_1 \longrightarrow M^s_1.$$

We shall first assume the validity of this lemma, which will be proved in the next two sections.

Consider now the commutative diagram

$$
(5.3)\qquad
\begin{array}{ccccccccc}
0 & \longrightarrow & L'^{s}_{r+1} & \longrightarrow & L'^{s}_{r} & \xrightarrow{d_r} & M'^{s-r}_{r} \\
 & & \Big\downarrow \phi^{s}_{r+1} & & \Big\downarrow \phi^{s}_{r} & & \Big\downarrow \psi^{s}_{r} \\
0 & \longrightarrow & L^{s}_{r+1} & \longrightarrow & L^{s}_{r} & \xrightarrow{r} & M^{s-r}_{r}
\end{array}
$$

where the rows are exact, and the vertical arrows are induced by the map (5.2).

THEOREM 5.3. *The cochain map* (5.2) *induces an isomorphism*

$$H^0(A'(s)) \approx H^0(B(s)) \approx \mathrm{Hom}_Z(Z\pi_1(X)/\bar{J}^{s+1}; k),$$

and

$$H^0(A') \approx H^0(B).$$

PROOF. We show by induction on $r \geqslant 1$ that ϕ^s_r is an isomorphism and that ψ^s_r is a monomorphism. Lemma 5.2 asserts the case $r = 1$. The rest of our argument is essentially the same as that for Theorem 3.1.

6. **Proof of Lemma 5.2.** Denote by $\bar{C}$ the chain complex such that $\bar{C}_0 = C_1(X)$, $\bar{C}_1 = C_2(X)$ and $\bar{C}_q = 0$ otherwise. The boundary operator of $\bar{C}$ coincides with that of $C_*(X)$ on the group $\bar{C}_2$. Then

$$(6.1)\qquad H^0(\bar{C}; k) = H^1(X; k);$$

and there is a monomorphism

$$(6.2)\qquad H^2(X; k) \longrightarrow H^1(\bar{C}; k).$$

Recall the augmentation ideal J of $C_*(\Omega(X; x_0))$. We are going to construct a chain map

$$(6.3)\qquad h\colon\ \bar{C} \longrightarrow J/J^2.$$

Every smooth 1-simplex α of X is a smooth loop at x_0, whose equivalence class $\bar{\alpha}$ is an element of $C_0(\Omega(X; x_0))$. Define h_0 such that $h_0\alpha = \bar{\alpha} - 1 + J^2$.

Let v_0, v_1, v_2 be the vertices of the standard simplex Δ^2. Let β be the edge path (v_0, v_2), and γ the edge path (v_0, v_1, v_2). Thus,

$$\beta(t) = (1 - t)v_0 + tv_2, \qquad 0 \leqslant t \leqslant 1,$$

$$\gamma(t) = \begin{cases} (1 - 2t)v_0 + 2tv_1, & 0 \leqslant t \leqslant \tfrac{1}{2}, \\ (2 - 2t)v_1 + (2t - 1)v_2, & \tfrac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

If $\sigma\colon\ \Delta^2 \longrightarrow X$ is a smooth 2-simplex, define $\bar{\sigma}\colon\ I \longrightarrow \Omega(X; x_0)$ to be the 1-cube given by

$$\bar{\sigma}(\xi)(t) = \sigma((1-\xi)\beta(t) + \xi\gamma(t)); \qquad \xi \in I, \ t \in I.$$

Let $\bar{\sigma}$ also denote the equivalence class of the 1-cube, which belongs to $C_2(\Omega(X; x_0))$. Define h_1 such that $h_1\sigma = \bar{\sigma} + J^2$, and verify that, if $\sigma^{(0)}$, $\sigma^{(1)}$, $\sigma^{(2)}$ are the faces of σ, then

$$\begin{aligned} \partial h_1 \sigma &= -\overline{\sigma^{(1)}} + \overline{\sigma^{(2)}}\ \overline{\sigma^{(0)}} + J^2 \\ &= \sum (-1)^i(\overline{\sigma^{(i)}} - 1) + (\overline{\sigma^{(2)}} - 1)(\overline{\sigma^{(0)}} - 1) + J^2 \\ &= h_0 \partial\sigma. \end{aligned}$$

Therefore h is a well-defined chain map.

For $s \geqslant 1$, there is a chain map $\bigotimes^s(J/J^2) \longrightarrow J^s/J^{s+1}$ given by $(c_1 + J^2) \cdots (c_s + J^2) \longmapsto c_1 \cdots c_s + J^{s+1}$. Composing with the chain map $\bigotimes^s h$: $\bigotimes^s \bar{C} \longrightarrow \bigotimes^s(J/J^2)$, we obtain a chain map

$$\bigotimes^s \bar{C} \longrightarrow J^s/J^{s+1}. \tag{6.4}$$

There is a canonical isomorphism

$$B(s)/B(s-1) \approx \mathrm{Hom}_Z(J^s/J^{s+1}; k).$$

Therefore (6.4) induces a cochain map

$$B(s)/B(s-1) \longrightarrow \mathrm{Hom}_Z(\textstyle\bigotimes^s \bar{C}; k). \tag{6.5}$$

Since the cochain map $A' \longrightarrow B$ given in (5.2) preserves filtration, we obtain a composite cochain map g_s:

$$\textstyle\bigotimes^s \bar{A} \approx A'(s)/A'(s-1) \longrightarrow B(s)/B(s-1) \longrightarrow \mathrm{Hom}_Z(\bigotimes^s \bar{C}; k).$$

If $\bar{w}_i \in \bar{A}$ denotes the image of $w_i \in A$, then $g_s(\bar{w}_1 \otimes \cdots \otimes \bar{w}_s)$: $\bigotimes^s \bar{C} \longrightarrow k$ is given by

$$\sigma_1 \otimes \cdots \otimes \sigma_s \longmapsto \left(\int_{\sigma_1} w_1\right) \cdots \left(\int_{\sigma_s} w_s\right)$$

where $\sigma_1, \cdots, \sigma_s$ are smooth 1- or 2-simplices of X.

By (6.1) and (6.2), the induced cohomology map g_s^* has the following factorization:

$$H(A'(s)/A'(s-1)) \approx \textstyle\bigotimes^s H(\bar{A}) \longrightarrow \bigotimes^s H^*(\bar{C}; k) \longrightarrow H^*(\bigotimes^s \bar{C}; k),$$

where the last arrow is monic. Owing to (6.1), (6.2) and Hypotheses II and III, $(g_s^*)^q$ is a monomorphism for $q = 0, 1$. Since g_s^* factors through $H(B(s)/B(s-1))$, the homomorphism

$$H^q(A'(s)/A'(s-1)) \longrightarrow H^q(B(s)/B(s-1))$$

is monic for $q = 0, 1$. This means that ϕ_1^s and ψ_1^s are monic.

On the other hand, Hypothesis IV implies that (6.4) induces an epimorphism $H_0(\otimes^s \bar{C}) \longrightarrow H_0(J^s/J^{s+1})$. Consequently

$$H^0(B(s)/B(s-1)) \approx H^0(J^s/J^{s+1}; k) \longrightarrow H^0(\otimes^s \bar{C}; k)$$

is a monomorphism. Hence ϕ_1^s is indeed an isomorphism.

7. **Residually torsion free nilpotent groups.** Let Γ be a group, and $\bar{J}$ the augmentation ideal of the group ring $Z\Gamma$. Write, for $s \geqslant 0$, $F_s(\Gamma) = \mathrm{Hom}_Z(Z\Gamma/\bar{J}^{s+1}; k)$ which is canonically embedded in $\mathrm{Hom}_Z(Z\Gamma; k)$, which can be identified with the algebra of functions on Γ. Set $F(\Gamma) = \bigcup F_s(\Gamma)$.

Recall that the lower central series of Γ consists of commutator subgroups $\{\Gamma_s\}_{s \geqslant 1}$ such that $\Gamma_1 = \Gamma$, and, for $s > 1$, $\Gamma_s = [\Gamma_{s-1}, \Gamma]$. A result of Quillen [7, Corollary 4.2] implies that, if each quotient group Γ_s/Γ_{s+1}, $s \geqslant 1$, is torsion free, then

$$\Gamma_s = \{g \in \Gamma : f(g-1) = 0, \ \forall f \in F_{s+1}(\Gamma)\}.$$

Definition. A group Γ is said to be torsion free residually nilpotent if every nontrivial element of Γ does not belong to the kernel of some epimorphism from Γ to a torsion free nilpotent group.

The next assertion is essentially a known result.

Proposition 7.1. *Taken as a vector space of functions on a finitely generated group Γ, $F(\Gamma)$ separates the elements of Γ if and only if Γ is residually torsion free nilpotent.*

Theorem 5.3 together with the above proposition yields the following corollary under the assumption of finite generatedness of $\pi_1(X)$.

Theorem 7.2. *The fundamental group $\pi_1(X)$ is residually torsion free nilpotent if and only if $H^0(A')$ separates elements of $\pi_1(X)$.*

For an arbitrary group Γ, $F(\Gamma)$ is a Hopf algebra. If Γ is finitely generated nilpotent, then $F(\Gamma)$ is an affine Hopf algebra and gives rise to a simply connected nilpotent Lie group G. If Γ is finitely generated torsion free nilpotent, then G is the Malcev completion of Γ. (See [4].)

8. **The groups $H^0(A'')$ and $H^0(A_\Omega)$.** Recall the cochain complexes A_Ω and A'' of iterated integrals on the total path space $\Omega(X)$ and the space $\Omega(X; x_0, *)$ of paths from the base point x_0. The restriction maps induce homomorphisms

$$H^0(A_\Omega) \longrightarrow H^0(A'') \longrightarrow H^0(A').$$

THEOREM 8.1. *The homomorphism $H^0(A'') \longrightarrow H^0(A')$ is epic.*

PROOF. We are going to show that $H^0(A_\Omega) \longrightarrow H^0(A')$ is an epimorphism. For this purpose, construct a direct summand $\hat{A}$ of A such that $\hat{A}^0 = k$, $\hat{A}^q = A^q$ for $q > 1$, and $A^1 = \hat{A}^1 \oplus dA^0$. Then A is a differential graded subalgebra of A, and the inclusion induces isomorphisms $H^q(\hat{A}) \approx H^q(A)$, $q > 0$.

Construct from $\hat{A}$ the filtered cochain complex $\hat{A}_\Omega$ of iterated integrals. Using a similar argument that leads to the isomorphism (2.6), we obtain

$$H(\otimes^s \bar{A}) \approx H(\hat{A}_\Omega(s)/\hat{A}_\Omega(s-1)).$$

Therefore the restriction map $\hat{A}_\Omega \longrightarrow A'$ induces isomorphisms

$$H^q(\hat{A}_\Omega(s)/\hat{A}_\Omega(s-1)) \approx H^q(A'(s)/A'(s-1)),$$

in particular, for $q = 0, 1$.

A slightly modified version of Theorem 3.1 leads to the isomorphism $H^0(\hat{A}_\Omega) \approx H^0(A')$. This isomorphism factors through $H^0(A_\Omega)$. The theorem hence follows.

9. **Covering spaces.** Let X be a connected C^∞ manifold with a base point x_0. Let $\tilde{X}$ be the universal covering space of X with a base point $\tilde{x}_0$ over x_0. Let $\bar{J}$ be the augmentation ideal of the group ring $Z\pi_1(X)$. Let A be a differential graded subalgebra of the exterior algebra $\Lambda(X)$ such that the induced homomorphism $H^q(A) \longrightarrow H^q(\Lambda(X))$ is an isomorphism for $q = 1$ and a monomorphism for $q = 2$.

Observe that there is an embedding $A^0 \subset H^0(A'')$ given by $f \longmapsto f(x_0) + \int_{x_0} df$. Since the value of each $\int_{x_0} u \in H^0(A'')$ depends only on the homotopy class of a path from x_0, we may regard $\int_{x_0} u$ as a function on $\tilde{X}$ which vanishes at $\tilde{x}_0$.

PROPOSITION 9.1. *The algebra of functions on $\tilde{X}$, which results from $H^0(A'')$, does not depend on the choice of the base points x_0 and $\tilde{x}_0$.*

PROOF. In order to indicate the base point, write $A'' = A''_{\tilde{x}_0}$. Let $\tilde{x}_1 \in \tilde{X}$ be over $x_1 \in X$, and let γ be a path from x_0 to x_1 which lifts to a path from $\tilde{x}_0$ to $\tilde{x}_1$ in $\tilde{X}$. Then, for every path α from x_1,

$$\left\langle \int w_1 \cdots w_r, \gamma\alpha \right\rangle = \sum_{0 < i < r} \left\langle \int w_1 \cdots w_i, \gamma \right\rangle \left\langle \int w_{i+1} \cdots w_r, \alpha \right\rangle$$

so that there is a homomorphism $A''_{x_0} \longrightarrow A''_{x_1}$ given by

$$\int_{x_0} w_1 \cdots w_r \longmapsto \sum \left\langle \int w_1 \cdots w_i, \gamma \right\rangle \int_{x_1} w_{i+1} \cdots w_r.$$

By making use of the path γ^{-1}, the induced homomorphism

$$H^0(A''_{x_0}) \to H^0(A''_{x_1}) \tag{8.1}$$

has an inverse. By embedding both $H^0(A''_{x_0})$ and $H^0(A''_{x_1})$ in the function algebra on $\widetilde{X}$, (8.1) becomes an automorphism. Hence the proof is completed.

THEOREM 9.2. *Assume that A^0 separates points of the manifold X. Then, $H^0(A'')$, as a function algebra on the universal covering space $\widetilde{X}$, separates points of $\widetilde{X}$ if and only if $\pi_1(X)$ is residually torsion free nilpotent.*

PROOF. The necessity follows from Theorems 7.2 and 8.1. Suppose that $\pi_1(X)$ is residually torsion free nilpotent. Let $\tilde{x}$ and $\tilde{x}'$ be two distinct points of $\widetilde{X}$. We may assume they are over a common point in X. Otherwise they can be separated by A^0. Since $H^0(A'')$, as a function algebra on $\widetilde{X}$, does not depend on the choice of the base point, we may further assume that $\tilde{x}$ and $\tilde{x}'$ are over x_0. It again follows from Theorems 7.2 and 8.1 that $H^0(A'')$ separates $\tilde{x}$ and $\tilde{x}'$. Hence the theorem is proved.

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