

HOLOMORPHIC FUNCTIONS WITH GROWTH CONDITIONS

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ABSTRACT. Let P be a $p \times q$ matrix of polynomials in n complex variables. If Ω is a domain of holomorphy in \mathbb{C}^n and u is a q -tuple of holomorphic functions we show that the equation $Pv = Pu$ has a solution v which is a holomorphic q -tuple in Ω and which satisfies an L^2 estimate in terms of Pu . Similar results have been obtained by Y.-T. Siu and R. Narasimhan for bounded domains and by L. Hörmander for the case $\Omega = \mathbb{C}^n$.

Let Ω be a domain of holomorphy in \mathbb{C}^n and let $A(\Omega)$ be the space of holomorphic functions in Ω . Let P be a $p \times q$ matrix of polynomials in $\mathbb{C}[z_1, \dots, z_n]$ and let $f = (f_1, \dots, f_p)$ be an element of $A(\Omega)^p$. Our main theorem states that if f satisfies certain growth conditions and if the system of linear equations $Pu = f$ has a solution u in $A(\Omega)^q$ then it has a solution which satisfies certain growth conditions at the boundary of Ω and at infinity. In the case where Ω is a *bounded* domain of holomorphy results of this type have been given by Y.-T. Siu [6] and R. Narasimhan [4], and in the case $\Omega = \mathbb{C}^n$ by L. Hörmander [2, Theorem 7.6.11]. Our method of proof in outline is the same as in [2], and we use the notation of [1] and [2]. As an illustration of the theorem we give a simple application to partial differential equations.

For each $z \in \Omega$ let $d(z) = \min(1, \text{dist}(z, \partial\Omega))$ and define $\psi(z) = -\log d(z)$, $\theta(z) = \log(1 + |z|^2)$. Let φ be a plurisubharmonic function in Ω such that there exists a constant a such that

$$(1) \quad |\varphi(z) - \varphi(w)| \leq a \quad \text{if } z \in \Omega \text{ and } |z - w| \leq \frac{1}{2}d(z).$$

We note that if m and N are nonnegative integers then $\varphi + m\psi + N\theta$ is a plurisubharmonic function in Ω which satisfies (1). If $u = (u_1, \dots, u_q) \in A(\Omega)^q$ we let $|u|^2 = |u_1|^2 + \dots + |u_q|^2$.

THEOREM 1. *If $u \in A(\Omega)^q$ there exists $v \in A(\Omega)^q$ such that $Pu = Pv$ in Ω and*

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$$(2) \quad \int_{\Omega} |v|^2 e^{-\varphi - m\psi - N\theta} dV \leq C \int_{\Omega} |Pu|^2 e^{-\varphi} dV.$$

Here m and N are nonnegative integers which depend only on P , C is a constant which depends only on a and on P , and dV is Lebesgue measure.

For emphasis we note that m , N and C do not depend on Ω . If $z \in \Omega$ and D is the polydisk with center z and radii all equal to $\delta = \frac{1}{2}n^{-1/2}d(z)$, then $|v_j(z)| \leq \pi^{-n} \delta^{-2n} \int_D |v_j| dV$; and hence by the Cauchy-Schwarz inequality and (2) we obtain the estimate

$$(3) \quad |v(z)|^2 \leq C'(1 + |z|^2)^N d(z)^{-m-2n} e^{\varphi(z)}, \quad z \in \Omega,$$

where $C' = \pi^{-n} n^n 2^{2n+N+m} e^a C \int_{\Omega} |Pu|^2 e^{-\varphi} dV$.

COROLLARY. Let Γ be a closed convex salient cone in \mathbb{R}^n with vertex at the origin and let Γ_0^+ be the interior of the dual cone. Let P_1, \dots, P_q be polynomials in $C[z_1, \dots, z_n]$.

Then P_1, \dots, P_q have no common zero with real part in Γ_0^+ if and only if there exist temperate distributions S_1, \dots, S_q with supports in Γ such that

$$(4) \quad P_1(D)S_1 + \dots + P_q(D)S_q = \delta.$$

Here $D_j = \partial/\partial x_j$ and δ is the unit mass at the origin.

PROOF OF COROLLARY. We first note if the temperate distributions S_j with supports in Γ satisfy (4) and if $u_j \in A(\Omega)$ is the Laplace transform of S_j , where $\Omega = \Gamma_0^+ + i\mathbb{R}^n$, then

$$(5) \quad P_1(z)u_1(z) + \dots + P_q(z)u_q(z) = 1, \quad z \in \Omega.$$

Conversely if P_1, \dots, P_q have no common zero in Ω then by Theorem 7.2.9 in [2] we can find $u_j \in A(\Omega)$ so that (5) is satisfied. Moreover by Theorem 1 and (3) we may choose u_j so that

$$|u_j(z)|^2 \leq C'(1 + |z|^2)^{N+n+1} d(z)^{-m-2n}.$$

By Theorem 1 of [5] u_j is the Laplace transform of a temperate distribution with support in Γ .

The proof of Theorem 1 will be postponed until we have established some preliminary results. We first need a theorem on trivial cohomology with bounds for the sheaf of germs of holomorphic functions in Ω .

If $\varphi \in C(\Omega)$ is real-valued we denote by $L_{(p,q)}^2(\Omega, \varphi)$ the Hilbert space of forms of type (p, q) on Ω whose coefficients are square integrable with respect to the measure $e^{-\varphi} dV$. Let $(\Omega_\nu)_{\nu \geq 1}$ be an open covering of Ω . If $\sigma \geq 0$ is an integer we denote by $C^\sigma((\Omega_\nu), Z_{(p,q)}, \varphi)$ the set of all alternating cochains $c = (c_\alpha)$

where $\alpha = (\alpha_0, \dots, \alpha_\sigma)$ is a $(\sigma + 1)$ -tuple of positive integers and $c_\alpha \in L^2_{(p,q)}(\Omega_\alpha, \varphi)$ where $\Omega_\alpha = \Omega_{\alpha_0} \cap \dots \cap \Omega_{\alpha_\sigma}$ and $\bar{\partial}c_\alpha = 0$. We define

$$\|c\|_\varphi^2 = \sum'_\alpha \int_{\Omega_\alpha} |c_\alpha|^2 e^{-\varphi} dV$$

where $|c_\alpha|^2$ is the sum of the squares of the coefficients of the form c_α and where Σ' means we sum only over α such that $\alpha_0 < \dots < \alpha_\sigma$.

We assume that the covering $(\Omega_\nu)_{\nu \geq 1}$ satisfies the following conditions.

(a) *There exist positive constants A and B such that if $z \in \Omega_\nu$ then*

$$B^{-1} \text{diam}(\Omega_\nu) \leq d(z) \leq A \text{diam}(\Omega_\nu).$$

In particular B is an upper bound for the diameters of the sets Ω_ν .

(b) *There exists an integer M such that more than M different sets Ω_ν always have an empty intersection.*

(c) *There exists a partition of unity $(\chi_\nu)_{\nu \geq 1}$ with $\chi_\nu \in C_c^\infty(\Omega_\nu)$ and*

$$\sum_\nu |\bar{\partial}\chi_\nu(z)| \leq A d(z)^{-1}, \quad z \in \Omega.$$

(d) *Each set Ω_ν is a domain of holomorphy.*

Open coverings with these properties may be constructed for any open proper nonempty subset Ω of \mathbb{C}^n with constants A , B and M independent of Ω . Constructions of this type were introduced by Whitney (see E. M. Stein [7, p. 127]). We will construct open coverings with additional properties in Lemma 2 below.

We define the coboundary operator δ in the usual way: if $c \in C^\sigma((\Omega_\nu), Z_{(p,q)}, \varphi)$ then

$$(\delta c)_{\alpha_0 \dots \alpha_{\sigma+1}} = \sum_{j=0}^{\sigma+1} (-1)^j c_{\alpha_0 \dots \hat{\alpha}_j \dots \alpha_{\sigma+1}}.$$

Then hypothesis (b) implies $\delta c \in C^{\sigma+1}((\Omega_\nu), Z_{(p,q)}, \varphi)$ and $\|\delta c\|_\varphi \leq M \|c\|_\varphi$.

THEOREM 2. *Let Ω be a domain of holomorphy in \mathbb{C}^n , let φ be a plurisubharmonic function in Ω and let κ be a continuous function in Ω with $\kappa \leq 0$ such that e^κ is a lower bound for the plurisubharmonicity of φ .*

If $\sigma \geq 1$ then for each $c \in C^\sigma((\Omega_\nu), Z_{(p,q)}, \varphi + \kappa)$ with $\delta c = 0$ we can find a cochain $c' \in C^{\sigma-1}((\Omega_\nu), Z_{(p,q)}, \varphi + 2\psi)$ such that $\delta c' = c$ and $\|c'\|_{\varphi+2\psi} \leq K_\sigma \|c\|_{\varphi+\kappa}$ where $K_\sigma = (1 + 2ABM)^{\sigma-1} (1 + AM)M^{\sigma/2}$.

PROOF. The proof of Theorem 2 is exactly the same as the proof of Theorem 2.4.1 in [1] except that the estimates used there are replaced by the corresponding estimates arising from hypotheses (a) and (c) above in place of (i) and (iii) in [1].

Note by condition (b) we only have to consider $\sigma < M$ and hence we may replace the K_σ by a constant K independent of σ , and by previous remarks independent of Ω . In [2] the proof that Theorem 4.4.1 implies Theorem 4.4.2 is accomplished by replacing φ by $\varphi(z) + 2 \log(1 + |z|^2)$. If we use the same device here we obtain

COROLLARY. *If φ is any plurisubharmonic function in Ω , if $\sigma \geq 1$ and $c \in C^\sigma((\Omega_\psi), Z_{(p,q)}, \varphi)$ and $\delta c = 0$ then there exists $c' \in C^{\sigma-1}((\Omega_\psi), Z_{(p,q)}, \varphi + 2\psi + 2\theta)$ such that $\delta c' = c$ and*

$$(6) \quad \|c'\|_{\varphi+2\psi+2\theta} \leq K \|c\|_\varphi.$$

In order to prove Theorem 1 we first need a semilocal version (Lemma 1). This semilocal result together with Theorem 2 (with $p = q = 0$) then gives a theorem on trivial cohomology with bounds for the sheaf of relations $\mathcal{R}(P)$. This result together with Lemma 1 then gives Theorem 1 easily. In outline this procedure is the same as in [2, §7.6].

LEMMA 1. *Let P be a $p \times q$ matrix over $\mathbb{C}[z_1, \dots, z_n]$, let U be the open unit ball centered at the origin in \mathbb{C}^n . Then there exist nonnegative integers m and N and a constant t , $0 < t < 1$, depending on P , and if $B > 0$, a constant $C > 0$ depending on P and B , such that if $z \in \mathbb{C}^n$, $0 < r \leq B$ and $u \in A(z + rU)^q$ then there exists $v \in A(z + trU)^q$ such that $Pu = Pv$ in $z + trU$ and if φ is any continuous real-valued function on $z + r\bar{U}$ then*

$$(7) \quad r^m \int_{z+trU} |u|^2 e^{-\varphi - N\theta} dV \leq C e^a \int_{z+rU} |Pu|^2 e^{-\varphi} dV$$

where $a = \sup_{z+rU} |\varphi(z') - \varphi(z'')|$.

If $0 < t < 1$ satisfies the conclusion of Lemma 1 then we will say that t is good for P . We will postpone the proof of Lemma 1 until we have completed the proof of Theorem 1. The proof of the lemma is the same as the proof of Proposition 7.6.5 in [2] except that it is necessary to pay considerably more attention to the estimates.

The module of relations $M(P)$ of the columns of P is defined to be the kernel of the homomorphism

$$P: \mathbb{C}[z_1, \dots, z_n]^q \rightarrow \mathbb{C}[z_1, \dots, z_n]^p$$

and the sheaf of relations $\mathcal{R}(P)$ of the columns of P is defined to be the kernel of the homomorphism $P: A^q \rightarrow A^p$ where A is the sheaf of germs of holomorphic functions in \mathbb{C}^n . Since $M(P)$ is finitely generated there is a $q \times s_0$ matrix of polynomials Q^0 whose columns generate $M(P)$. Inductively we obtain an $s_{k-1} \times s_k$ matrix of polynomials Q^k whose columns generate $M(Q^{k-1})$, $k \geq 1$. By homo-

logical dimension theory [3] $M(Q^{n-1})$ is a projective module. By adding a sufficient number of columns of zeros to Q^{n-1} we can arrange for $M(Q^{n-1})$ to be a free $\mathbb{C}[z_1, \dots, z_n]$ module. (For this curious fact see R. G. Swan [8, Remark, p. 121].) With the obvious choice of Q^n we now have $Q^k = 0$ for $k > n$. By Lemma 7.6.3 in [2] for each z in \mathbb{C}^n the stalk $R(P)_z$ is generated as an A_z module by the germs at z of the polynomials in $M(P)$. In particular by Theorem 7.2.9 in [2] if Ω is any domain of holomorphy in \mathbb{C}^n we have a long exact sequence

$$(8) \quad 0 \rightarrow A(\Omega)^{s_n} \xrightarrow{Q^n} A(\Omega)^{s_{n-1}} \xrightarrow{Q^{n-1}} \dots \xrightarrow{Q^0} A(\Omega)^q \xrightarrow{P} A(\Omega)^p.$$

Now let $\Omega \neq \mathbb{C}^n$ be a domain of holomorphy in \mathbb{C}^n and assume for each $k = -1, 0, 1, \dots, n+1$ we have an open covering $(\Omega_\nu^k)_{\nu \geq 1}$ of Ω such that the following conditions are satisfied.

(a)' *There exists a positive constant A such that if $z \in \Omega_\nu^k$ then*

$$2 \operatorname{diam}(\Omega_\nu^k) \leq d(z) \leq A \operatorname{diam}(\Omega_\nu^k).$$

(b)' *There exists an integer M such that for each k more than M different sets Ω_ν^k always have an empty intersection.*

(c)' *There exists a partition of unity $(\chi_\nu)_{\nu \geq 1}$ with $\chi_\nu \in C_c^\infty(\Omega_\nu^{n+1})$ such that $\sum_\nu |\bar{\partial} \chi_\nu(z)| \leq A d(z)^{-1}$, $z \in \Omega$.*

(d)' *Each set Ω_ν^k is a domain of holomorphy.*

(e)' *For each ν , $\Omega_\nu^{k+1} \subset \Omega_\nu^k$, $k = -1, 0, \dots, n$, and $\operatorname{dist}(\Omega_\nu^{k+1}, \partial \Omega_\nu^k) \geq t^{-1} \operatorname{diam}(\Omega_\nu^{k+1})$ where $0 < t < 1$ will be specified later.*

The proof that open covers satisfying all of these hypotheses exist will be given below in Lemma 2. M and A may be chosen so as not to depend on Ω . We note that $\Omega_\nu = \Omega_\nu^{n+1}$ gives an open cover which satisfies hypotheses (a)–(d) used earlier.

We define $C^\sigma((\Omega_\nu^k), R(P), \varphi)$ as before except we now require $c_\alpha \in A(\Omega_\alpha^k)^q$ and $Pc_\alpha = 0$ in Ω_α^k . We note $Z_{(0,0)} = A$. If $k < l$ and c is a cochain relative to $(\Omega_\nu^k)_{\nu \geq 1}$ we denote by ρc the restriction of c to $(\Omega_\nu^l)_{\nu \geq 1}$ defined by $(\rho c)_\alpha = c_\alpha|_{\Omega_\alpha^l}$.

THEOREM 3. *Let Ω be a domain of holomorphy in \mathbb{C}^n and let φ be a pluri-subharmonic function in Ω which satisfies (1). Let P and Q^k be as above and assume that t is good for Q^0, \dots, Q^n .*

If $\sigma \geq 1$ and $c \in C^\sigma((\Omega_\nu^0), R(P), \varphi)$ and $\delta c = 0$ then there exists $c' \in C^{\sigma-1}((\Omega_\nu^{n+1}), R(P), \varphi + m\psi + N\theta)$ such that $\delta c' = \rho c$ and $\|c'\|_{\varphi+m\psi+N\theta} \leq K \|c\|_\varphi$. Here $m, N \geq 0$ are integers which depend on P and $K > 0$ is a constant which depends on a and P .

PROOF. In the proof K will denote various constants which may depend on

P , a and Q^k . We denote P by Q^{-1} . We will prove by induction if $-1 \leq k \leq n$ then:

(9) If $\sigma > 0$ and $c \in C^\sigma((\Omega_\nu^{k+1}), R(Q^k), \varphi)$ and $\delta c = 0$ then there exists $c' \in C^{\sigma-1}((\Omega_\nu^{n+1}), R(Q^k), \varphi + m\psi + N\theta)$ such that $\delta c' = \rho c$ and $\|c'\|_{\varphi+m\psi+N\theta} \leq K \|c\|_\varphi$.

When $k = -1$ we have the statement of the theorem. When $k = n$ then $R(Q^n) = 0$ implies $c = 0$ and so (9) holds. Now assume that $n > k \geq -1$ and that we have proved (9) for all larger values of k and for all plurisubharmonic functions φ which satisfy (1).

Let $\alpha = (\alpha_0, \dots, \alpha_\sigma)$ where $\alpha_0 < \alpha_1 < \dots < \alpha_\sigma$ and assume that Ω_α^{k+2} is not empty. Let $r = t^{-1} \min_{\nu \in \alpha} \text{diam}(\Omega_\nu^{k+2})$. Since $d(z) \leq 1$ hypothesis (a)' implies $r \leq (2t)^{-1}$. If $z \in \Omega_\alpha^{k+2}$ then by (e)'

$$\text{dist}(z, \partial\Omega_\alpha^{k+1}) = \min_{\nu \in \alpha} \text{dist}(z, \partial\Omega_\nu^{k+1}) \geq r.$$

We also have

$$\text{diam}(\Omega_\alpha^{k+2}) \leq \min_{\nu \in \alpha} \text{diam}(\Omega_\nu^{k+2}) = tr.$$

It follows that if U is the open ball with center z and radius r and if U' is the open ball with center z and radius tr then $\Omega_\alpha^{k+2} \subseteq U' \subseteq U \subseteq \Omega_\alpha^{k+1}$. By the exactness of (8) with Ω replaced by Ω_α^{k+1} we have $c_\alpha = Q^{k+1} \tilde{g}_\alpha$ where $\tilde{g}_\alpha \in A(\Omega_\alpha^{k+1})^{s_{k+1}}$. Since $t > 0$ is good for Q^{k+1} Lemma 1 with $B = (2t)^{-1}$ implies we have $g_\alpha \in A(\Omega_\alpha^{k+2})^{s_{k+1}}$ such that $Q^{k+1} g_\alpha = c_\alpha$ in Ω_α^{k+2} and

$$(10) \quad r^{m'} \int_{\Omega_\alpha^{k+2}} |g_\alpha|^2 e^{-\varphi - N'\theta} dV \leq C e^a \int_{\Omega_\alpha^{k+1}} |c_\alpha|^2 e^{-\varphi} dV$$

where C depends on t and Q^{k+1} . Here we have used (1) and the fact that if $z, w \in \Omega_\alpha^{k+1}$ then $|z - w| < \text{diam}(\Omega_\alpha^{k+1}) \leq \frac{1}{2}d(z)$. Now if $z \in \Omega_\alpha^{k+2}$ then by (a)' $d(z) \leq A \min_{\nu \in \alpha} \text{diam}(\Omega_\nu^{k+2}) = Art$. Thus we have found $g \in C^\sigma((\Omega_\nu^{k+2}), A^{s_{k+1}}, \varphi + m'\psi + N'\theta)$ such that $Q^{k+1} g = \rho c$ and $\|g\|_{\varphi+m'\psi+N'\theta}^2 \leq C e^a (Art)^{m'} \|c\|_\varphi^2$. Then $Q^{k+1} \delta g = \rho \delta c = 0$ implies that

$$\delta g \in C^{\sigma+1}((\Omega_\nu^{k+2}), R(Q^{k+1}), \varphi + m'\psi + N'\theta).$$

By the inductive hypothesis we can find

$$f \in C^\sigma((\Omega_\nu^{n+1}), R(Q^{k+1}), \varphi + m''\psi + N''\theta)$$

where $m'' \geq m'$ and $N'' \geq N'$ such that $\delta f = \rho \delta g$ and

$$\|f\|_{\varphi+m''\psi+N''\theta} \leq K \|\delta g\|_{\varphi+m'\psi+N'\theta}.$$

Then $\rho g - f \in C^\sigma((\Omega_\nu^{n+1}), A^{s_{k+1}}, \varphi + m''\psi + N''\theta)$ and $\delta(\rho g - f) = 0$. Therefore by the corollary to Theorem 2 (with $p = q = 0$) there exists

$$h \in C^{\sigma-1}((\Omega_\nu^{n+1}), A^{s_{k+1}}, \varphi + (m'' + 2)\psi + (N'' + 2)\theta)$$

such that $\delta h = \rho g - f$ and

$$\|h\|_{\varphi + (m'' + 2)\psi + (N'' + 2)\theta} \leq K \|\rho g - f\|_{\varphi + m''\psi + N''\theta}.$$

Now let $c' = Q^{k+1}h$, $m = m'' + 2$ and choose $N > N'' + 2$ such that $N - N'' - 2$ is an upper bound for the degrees of the polynomials which occur in Q^{k+1} .

Then $\delta c' = Q^{k+1}\delta h = \rho Q^{k+1}g = \rho c$ and

$$\|c'\|_{\varphi + m\psi + N\theta} \leq K \|h\|_{\varphi + m\psi + (N'' + 2)\theta}.$$

Since $m'' \geq m'$ and $N'' \geq N'$ we also have

$$\|g\|_{\varphi + m''\psi + N''\theta} \leq \|g\|_{\varphi + m'\psi + N'\theta}$$

and hence the inequality in (9) follows.

Theorem 1 now follows immediately in the same way as Theorem 7.6.11 in [2].

PROOF OF THEOREM 1. We assume that t is good for Q^0, \dots, Q^n and also good for P . For each $\nu \geq 1$ let $h_\nu = Pu|_{\Omega_\nu^{-1}}$. Then $h \in C^0((\Omega_\nu^{-1}), A^p, \varphi)$ and $\|h\|_\varphi^2 \leq M \int_\Omega |Pu|^2 e^{-\varphi} dV$. Since t is good for P the same argument as in the proof of Theorem 3 shows that we can find $g \in C^0((\Omega_\nu^0), A^q, \varphi + m'\psi + N'\theta)$ such that $Pg = \rho h$ and $\|g\|_{\varphi + m'\psi + N'\theta} \leq K \|h\|_\varphi$. Then $P\delta g = \delta Pg = \rho\delta h = 0$ implies that $\delta g \in C^1((\Omega_\nu^0), R(P), \varphi + m'\psi + N'\theta)$. By Theorem 3 we can find $f \in C^0((\Omega_\nu^{n+1}), R(P), \varphi + m\psi + N\theta)$ where $m \geq m'$, $N \geq N'$ such that $\delta f = \rho\delta g$ and $\|f\|_{\varphi + m\psi + N\theta} \leq K \|\delta g\|_{\varphi + m'\psi + N'\theta}$. Then $\delta(\rho g - f) = 0$ implies there exists $v \in A(\Omega)^q$ such that $v = g_\nu - f_\nu$ in Ω_ν^{n+1} . Now $Pv = Pg_\nu = h_\nu = Pu$ in Ω_ν^{n+1} and therefore $Pv = Pu$. Moreover

$$\int_\Omega |v|^2 e^{-\varphi - m\psi - N\theta} dV \leq \|\rho g - f\|_{\varphi + m\psi + N\theta}^2$$

and again because $m \geq m'$ and $N \geq N'$ we obtain (2).

We now turn to the proof of Lemma 1. By an argument similar to the one which precedes the corollary to Theorem 1 it suffices to prove the following result.

LEMMA 1'. Let P be a $p \times q$ matrix over $C[z_1, \dots, z_n]$ and let U be the open unit ball centered at the origin in C^n . Then there exist constants $C > 0$, $0 < t < 1$ and integers $m, N \geq 0$ such that if $z \in C^n$, $r > 0$ and $u \in A(z + rU)^q$ then there exists $v \in A(z + trU)^q$ such that $Pu = Pv$ in $z + trU$ and

$$(11) \quad \sup_{z + trU} |v| \leq C(1 + r^{-m})(1 + |z| + r)^N \sup_{z + rU} |Pu|.$$

PROOF. This lemma is a refinement of Proposition 7.6.5 in [2]. The first part of the proof, induction on p , given there (p. 188) also works in the present case with obvious changes and we omit it. The second part of the proof, induction on the dimension n , is also the same [2, p. 192], but more care in the estimates is required, and therefore we give the proof here.

(12) *The lemma is true in dimension n for $p = 1$, if it is true in dimension $n - 1$ for all p .*

Since $p = 1$, $P = (P_1, \dots, P_q)$. We may assume P_1 is not identically zero and $\mu = \deg P_1 = \max \deg P_j$. Define

$$\tilde{P}_1(z; r) = \sup_{|w| < r} |P_1(z + w)|.$$

If we apply Lemma 7.6.6 in [2] to the polynomial $w \rightarrow P_1(z + rw)$ we obtain

$$\tilde{P}_1(z; r) \leq C \sup_{\theta \in \Theta} \inf_{|r|=1} |P_1(z + r\theta)|$$

where Θ is a finite subset of $U \sim \{0\}$ and Θ and C may be chosen to depend only on μ and n . We let $E(\theta, r)$ be the set of all z in \mathbb{C}^n such that

$$\tilde{P}_1(z; r) \leq C \inf_{|r|=1} |P_1(z + r\theta)|.$$

It is sufficient to prove that for each $\theta \in \Theta$ the lemma is true when $z \in E(\theta, r)$, since Θ is a finite set and so only a finite number of constants, all independent of r , will occur.

We now assume $\theta \in \Theta$ and $z \in E(\theta, r)$ are fixed. By a unitary transformation of coordinates we may assume $\theta = (0, 0, \dots, s)$ where $0 < s < 1$. Let

$$f = P_1 u_1 + \dots + P_q u_q \in A(z + rU).$$

If we apply Lemmas 7.6.7 and 7.6.8 in [2] to the polynomial $w \rightarrow P_1(z + rw)$ we obtain a polydisk $\Delta = \{w \in \mathbb{C}^n: |w_j| < s', j = 1, \dots, n-1, |w_n| < s\}$ such that $\Delta \subseteq U$ and holomorphic functions $g, h \in A(z + r\Delta)$ such that

$$(13) \quad f = P_1 g + h \quad \text{in } z + r\Delta$$

and h is a polynomial in w_n of degree $< \mu^+$, where μ^+ is the number of roots of $\tau \rightarrow P_1(z', \tau)$ in $|\tau - z_n| < rs$, where $z' = (z_1, \dots, z_{n-1})$. Here s' may be chosen to depend only on s, μ and n , and g and h may be chosen so that

$$(14) \quad \tilde{P}_1(z; r) \sup_{z+r\Delta} |g| + \sup_{z+r\Delta} |h| \leq C \sup_{z+r\Delta} |f|$$

where C is a constant which may be chosen to depend only on μ and n . By the same argument as in [2, p. 193] we obtain

$$(15) \quad P_1 f_1 + \dots + P_q f_q = h$$

where $f_j \in A(z + r\Delta)$ and f_2, \dots, f_q are polynomials in w_n of degree $< \mu^+$. Then $P_1 f_1$ is a polynomial in w_n of degree $< \mu + \mu^+$. Now by Lemma 7.6.8 in [2] applied to the polynomial $w \rightarrow P_1(z + rw)$ we have $P_1 = P^+ P^-$ where P^+ and P^- are bounded functions in $A(z + r\Delta)$ and are polynomials in w_n . Moreover P^+ is monic, has degree μ^+ and if $\Delta' = \{w' \in \mathbb{C}^{n-1} : |w'_j| < s', j = 1, \dots, n-1\}$ then for $w' \in z' + r\Delta'$ all the roots of $\tau \rightarrow P^+(w', \tau)$ lie in $|\tau - z_n| < rs$. Moreover we have

$$(16) \quad r^{-\mu^+} \tilde{P}_1(z; r) \leq C \inf_{z+r\Delta} |P^-|, \quad \sup_{z+r\Delta} |P^-| \leq C r^{-\mu^+} \tilde{P}_1(z; r)$$

where C is a constant which may be chosen to depend only on μ and n . Since $P_1 f_1$ is a polynomial in w_n Lemma 7.6.9 in [2] implies $P^- f_1$ is a polynomial in w_n . Thus if we define $f'_j = P^- f_j, j = 1, \dots, q$ and $h' = P^- h$ then these functions are holomorphic in $z + r\Delta$ and are polynomials in w_n of degree $< 2\mu$, and by (15)

$$(17) \quad P_1 f'_1 + \dots + P_q f'_q = h'.$$

Moreover, from (14) and (16) we have the estimate

$$(18) \quad \sup_{z+r\Delta} |h'| \leq C r^{-\mu^+} \tilde{P}_1(z; r) \sup_{z+rU} |f|$$

where C is a constant which depends only on μ and n . Now f'_j and h' are polynomials in w_n with coefficients which are holomorphic in $z' + r\Delta'$ and therefore (17) is equivalent to a system of 3μ equations in the coefficients of the f'_j where the right-hand sides are the coefficients of h' and zeros. By the inductive hypothesis we can find constants $C > 0, 0 < t' < 1$ and integers $m, N \geq 0$ such that if U' is the open unit ball in \mathbb{C}^{n-1} with center at the origin then we have holomorphic functions $F_{jk} \in A(z' + t'rs'U')$ such that the polynomials $F_j(w) = \sum_k F_{jk}(w') w_n^k$ satisfy the equation $P_1 F_1 + \dots + P_q F_q = h'$ and

$$(19) \quad \sum \sup |F_{jk}| \leq C(1 + r^{-m})(1 + |z'| + r)^N \sum \sup |h_k|$$

where the first supremum is over $z' + t'rs'U'$ and the second is over $z' + rs'U'$ and where $h'(w) = \sum h_k(w') w_n^k$. Now we define

$$(20) \quad v_1 = g + F_1/P^-, \quad v_j = F_j/P^-, \quad j \geq 2.$$

Then $P_1 v_1 + \dots + P_q v_q = P_1 g + h = f = Pu$ and it remains to check the estimate. Note if $D = \{\tau \in \mathbb{C} : |\tau| < s\}$ then the v_j are holomorphic in $z + r((t's'U') \times D)$.

We choose t with $0 < t \leq \min(s, t's')$. Since Θ is finite and s' depends only on s, μ and n only a finite number of values of s and s' have to be considered. Again since Θ is finite only a finite number of unitary transformations (at the beginning of the proof) have to be considered. For each of these P gives rise to a system of 3μ linear equations as above and hence only a finite number of t' arise.

Thus we may choose one value of t which satisfies the above conditions in all cases. Note we have $tU \subseteq (t's'U') \times D$. The same reasoning shows that the constants C , m and N in (19) may be chosen to depend only on P .

Now from (20), (16) and (14) we have

$$(21) \quad \sum \sup_{z+irU} |v_j| \leq \frac{C}{\tilde{P}_1(z; r)} \left\{ \sup_{z+rU} |f| + r^{\mu+} \sum \sup_{z+irU} |F_j| \right\}.$$

We now estimate the right-hand side of (21). For the first term we note if we apply the remark at the top of [2, p. 190] to the polynomial $w \rightarrow P_1(z + rw)$ we obtain $\tilde{P}_1(z; r) \geq Cr^{\mu}$ where the positive constant C depends only on the top order coefficients of P_1 . Since $f = Pu$ we have

$$(22) \quad \frac{1}{\tilde{P}_1(z; r)} \sup_{z+rU} |f| \leq Cr^{-\mu} \sup_{z+rU} |Pu|$$

where C depends only on P . Since $tU \subseteq (t's'U') \times D$ and $s < 1$ we have by (19)

$$\begin{aligned} \sum \sup_{z+irU} |F_j| &\leq C(1 + |z_n| + r)^{2\mu-1} \sum \sup |F_{jk}| \\ &\leq C(1 + r^{-m})(1 + |z| + r)^{N+2\mu-1} \sum \sup |h_k|. \end{aligned}$$

We digress for a moment. We can find a finite subset A of U and polynomials $Q_a \in \mathbb{C}[z_1, \dots, z_n]$ of degree $\leq \mu$ such that if $Q \in \mathbb{C}[z_1, \dots, z_n]$ has degree $\leq \mu$ then $Q(w) = \sum Q(a) Q_a(w)$ ($a \in A$). If we apply this result to the polynomial $w \rightarrow Q(z + rw)$ and then replace w by $r^{-1}(w - z)$ we obtain

$$Q(w) = \sum_{a \in A} Q(z + ra) Q_a(r^{-1}(w - z))$$

and therefore

$$\tilde{Q}(0; 1) \leq C(1 + r^{-\mu})(1 + |z|)^{\mu} \tilde{Q}(z; r)$$

where the constant C depends only on μ and n . The norm $Q \rightarrow \tilde{Q}(0; 1)$ is of course equivalent to any other norm on the space of polynomials of degree $\leq \mu$ and therefore we obtain if $Q(w) = \sum a_{\beta} w^{\beta}$ then

$$(24) \quad \sum |a_{\beta}| \leq C(1 + r^{-\mu})(1 + |z|)^{\mu} \tilde{Q}(z; r)$$

where the constant C depends only on μ and n . If we apply (24) for the case of polynomials in one variable to h' then we obtain

$$\sum_{z+r\Delta} \sup |h_k| \leq C(1 + (rs)^{-2\mu+1})(1 + |z_n|)^{2\mu-1} \sup_{z+r\Delta} |h'|.$$

Since only a finite number of values of s occur we may incorporate s in C . Now by (23), (18) and since $f = Pu$

$$\sum \sup_{z+trU} |F_j| \leq C r^{-\mu+} \tilde{P}_1(z; r) (1 + r^{-m-2\mu+1}) (1 + |z| + r)^{N+4\mu-2} \sup |Pu|.$$

If we combine this inequality with (21) and (22) we obtain

$$\sum \sup_{z+trU} |v_j| \leq C (1 + r^{-m-2\mu+1}) (1 + |z| + r)^{N+4\mu-2} \sup_{z+rU} |Pu|$$

which completes the proof.

Finally to finish the proof of Theorem 1 and to see that Theorems 2 and 3 are not vacuous we need to exhibit open coverings of Ω with the required properties. We formulate the result which we need in terms of open subsets of \mathbf{R}^n .

LEMMA 2. *Let Ω be a proper nonempty open subset of \mathbf{R}^n . For each $x \in \Omega$ let $d(x) = \min(1, \text{dist}(x, \partial\Omega))$. If $t > 0$ is a real number and $m \geq 0$ is an integer then there exist $m + 1$ open coverings $(\Omega_\nu^k)_{\nu \geq 1}$ of Ω , $k = 0, 1, \dots, m$, such that the following properties are all satisfied.*

(a) *There exists a constant A , depending on n, m and t , such that if $x \in \Omega_\nu^k$ then $2 \text{diam}(\Omega_\nu^k) \leq d(x) \leq A \text{diam}(\Omega_\nu^k)$.*

(b) *There exists an integer M such that for each k more than M different sets Ω_ν^k always have an empty intersection. M depends on n, m and t .*

(c) *There exists a partition of unity $(\chi_\nu)_{\nu \geq 1}$ with $\chi_\nu \in C_c^\infty(\Omega_\nu^0)$ such that $\sum_\nu |D^\beta \chi_\nu(x)| \leq A_\beta d(x)^{-|\beta|}$ for certain constants A_β depending on β, m, n and t .*

(d) *Each Ω_ν^k is a cube with edges parallel to the coordinate axes.*

(e) *For each ν , $\Omega_\nu^k \subseteq \Omega_\nu^{k+1}$, $k = 0, 1, \dots, m-1$, and $\text{dist}(\Omega_\nu^k, \partial\Omega_\nu^{k+1}) \geq t^{-1} \text{diam}(\Omega_\nu^k)$.*

PROOF. Let M_0 be the mesh of all closed cubes in \mathbf{R}^n with edges of unit length and with vertices in the lattice \mathbf{Z}^n . Assuming we have already defined M_{k-1} ($k \geq 1$), let M_k be the mesh of all closed cubes in \mathbf{R}^n obtained by bisecting the edges of each cube in M_{k-1} . Let $s > 1$ be a real number to be specified later and for each integer $k \geq 1$ define

$$U_k = \{x \in \Omega: sn^{1/2}2^{-k} < d(x) \leq sn^{1/2}2^{-k+1}\}$$

and let

$$F_0 = \bigcup_{k \geq 1} \{Q \in M_k: Q \cap U_k \neq \emptyset\}.$$

Let F be the set of all maximal cubes in F_0 . By arguments similar to those in [7, p. 167] Ω is the union of the cubes in F and these cubes have disjoint interiors. Moreover if $Q \in F$ and $x \in Q$ then

$$(s-1) \text{diam}(Q) < d(x) \leq (2s+1) \text{diam}(Q).$$

If $s \geq 5/2$ and if Q, Q' are cubes in F which intersect then

$$\frac{1}{2} \text{diam}(Q) \leq \text{diam}(Q') \leq 2 \text{diam}(Q).$$

In particular there are at most $4^n + 3^n - 2$ cubes in F which meet a given cube in F .

If $\lambda \geq 1$ we let $Q(\lambda)$ be the cube obtained by enlarging the cube Q by a homothetic transformation from its center in the ratio λ to 1. A computation shows if $1 \leq \lambda \leq (1/5)(2s - 1)$ then for each $Q \in F$ we have $Q(\lambda) \subseteq \Omega$ and

$$(25) \quad 2 \text{diam } Q(\lambda) < d(x) < 3s \text{diam } Q(\lambda), \quad x \in Q(\lambda).$$

Let $Q \in F$ and suppose that $Q(\lambda)$ meets $Q'(\lambda)$ for some other $Q' \in F$. Then (25) implies that $2 \text{diam } Q(\lambda) \leq 3s \text{diam } Q'(\lambda)$ and therefore

$$Q(\lambda) \subseteq Q'((3s + 1)\lambda).$$

If $|Q|$ is the volume of Q we then have $|Q'| \geq (3s + 1)^{-n} |Q|$. By symmetry we also have $Q' \subseteq Q'(\lambda) \subseteq Q((3s + 1)\lambda)$. Since the cubes Q' in F have disjoint interiors we conclude that if $1 \leq \lambda \leq (1/5)(2s - 1)$ and $Q \in F$ then there are at most $5^{-n}(3s + 1)^{2n}(2s - 1)^n$ cubes Q' in F such that $Q(\lambda) \cap Q'(\lambda) \neq \emptyset$.

Now choose s so large that $s \geq 1 + 5(1 + 2t^{-1}n^{1/2})^m$ and let $\lambda_k = 2(1 + 2t^{-1}n^{1/2})^k$ and define $\Omega_\nu^k = \text{interior of } Q_\nu(\lambda_k)$, where $F = (Q_\nu)_{\nu \geq 1}$. Then conditions (a), (b) and (d) are certainly satisfied. Moreover $\Omega_\nu^{k+1} = \Omega_\nu^k(1 + 2t^{-1}n^{1/2})$ and so (e) is satisfied. Finally to obtain the partition of unity which satisfies (c) we argue as in [7].

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