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PARTITIONS OF UNITY AND A CLOSED EMBEDDING THEOREM FOR (C^p, b^*)-MANIFOLDS

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ABSTRACT. Many manifolds of fiber bundle sections possess a natural atlas $\{(U_\alpha, \phi_\alpha)\}$ such that the transition maps $\phi_\beta\phi_\alpha^{-1}$, in addition to being smooth, are continuous with respect to the bounded weak topology of the model. In this paper we formalize the idea of such manifolds by defining (C^p, b^*)-manifolds, (C^p, b^*)-morphisms, etc. We then show that these manifolds admit (C^p, b^*)-partitions of unity subordinate to certain open covers and that they can be embedded as closed (C^p, b^*)-submanifolds of their model. A corollary of our work is that for any Banach space B, the conjugate space B^* admits smooth partitions of unity subordinate to covers by sets open in the bounded weak-* topology.

0. **Introduction.** In §3 of [10] Richard Palais discusses an additional structure possessed by certain manifolds of fiber bundle sections. (For example, the manifold of L^p_k cross sections of a fiber bundle over a smooth compact n-dimensional manifold M, $p > 1$ and $k > n/p$. See [5] for this and other examples.) This additional structure can be described by saying that the manifold possesses a natural atlas $\{(U_\alpha, \phi_\alpha)\}$ such that the transition maps $\phi_\beta\phi_\alpha^{-1}$, in addition to being smooth, are continuous with respect to the bounded weak topology of the model. (The bounded weak topology on a Banach space B is the finest topology agreeing with the weak topology on bounded sets.) In this paper we formalize the idea of such manifolds by defining (C^p, b^*)-manifolds, (C^p, b^*)-morphisms, etc. We then show that these manifolds admit (C^p, b^*)-partitions of unity subordinate to certain open covers and that they can be embedded as closed (C^p, b^*)-submanifolds of their model.

A corollary of our work is that given any Banach space B, the conjugate space B^* admits smooth partitions of unity subordinate to covers by sets open in the bounded weak-* topology. This is in contrast to the fact that there are conjugate spaces known not to admit smooth functions of bounded support (see §II,

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Theorem II. 2 and the paragraph following it).

The basic definitions and the precise statements of our main results are contained in §II.

Throughout this paper we work with the bounded weak-* topology on the conjugate, B^*, of a Banach space B rather than the bounded weak topology on B. By doing so we are able to avoid the hypothesis of reflexivity by appealing to Alaoglu's theorem. (Alaoglu's theorem [3, p. 424] states that given any Banach space B, the closed unit ball in B^* with its weak-* topology is compact.) We then obtain as corollaries theorems about manifolds which are modelled on reflexive Banach spaces and whose transition maps respect the bounded weak topology. (See the remark at the end of §II.) The reader interested in only this (reflexive) case may well want to read this paper by replacing "B^*" by "B" and "bounded weak-* topology" by "bounded weak topology". Then reflexivity (rather than Alaoglu's theorem) will give the weak compactness of bounded sets. Propositions III. 2 and III.3 may then be omitted, and the rest of our definitions and proofs go through with only minor changes.

The idea of considering manifolds modelled on conjugate spaces and bounded weak-* topologies originates with Richard Graff. Since first submitting this paper for publication the author has received a copy of Graff's thesis [5] which investigates manifolds modelled on conjugate spaces. Graff has independently obtained a strong partition of unity result in the case that B is separable [5, Corollary 4.22, p. 118]. (Related to this see also [5, Theorem 4.21, p. 117] and [5, Example 2.40, p. 64].) Note, however, that our partition of unity result, Theorem II. 1, has no separability restriction. Graff has also obtained, independently, a proof of our Proposition III. 1(e) [5, Proposition 2.7, p. 42].

The author wishes to thank his thesis advisor, David W. Henderson, for suggesting this research and for continually giving help and encouragement. Also, the author wishes to thank David Elworthy, who made several very valuable suggestions resulting in the improvement of this paper. It should be noted that our proof of Theorem II. 3, and especially our introduction of a (τ^*, b^*)-Banach completion in §V, is patterned after the work of N. H. Kuiper and B. Terpstra-Keppler in [7]. Finally, the author wishes to thank the referee for several helpful suggestions. In particular, the referee showed how to simplify the proof of Theorem V. 5.

I. Notation. Throughout this paper B, E, and F will denote real Banach spaces. The norm on any Banach space will be denoted by $\| \ \|$. Given a Banach space B, B^* denotes the conjugate of B. On B^* we consider three topologies: τ_B^*, the metric topology (induced by the norm); w_B^*, the weak-* topology (sometimes called the B-topology of B^*, see [3, p. 420]); and b_B^*, the bounded weak-*

topology (sometimes called the bounded B-topology for B^*, see [3, p. 427]). Recall that b_B^* is the finest topology on B^* that agrees with w_B^* on (norm) bounded sets of B^*. When no confusion will result we suppress the subscripts and write τ^*, w^*, and b^*. By $B^*(\tau^*)$ (resp. $B^*(w^*)$, $B^*(b^*)$), we indicate B^* with its τ^* (resp. w^*, b^*) topology. For any real number $r > 0$, let $B_r^* = \{x^* \in B^* | \ \|x^*\| \leqslant r\}$. We use the notation $B = E \oplus F$ to mean that B is the *topological* direct sum of the closed linear subspaces E and F. As usual R denotes the space of real numbers.

II. Basic definitions and summary of results. Let M be a set, $-1 \leqslant p \leqslant \infty$. In order to have the following definitions cover the cases $p = -1$ and $p = 0$, we regard a C^{-1}-diffeomorphism as a set bijection, a C^0-diffeomorphism as a homeomorphism, etc. (Thus, our (C^{-1}, b^*)-manifolds will be topological manifolds carrying only a b^* topology, and our (C^0, b^*)-manifolds will be topological manifolds carrying both a b^* and a metric topology.) A *(C^p, b^*)-atlas* for M is a collection of pairs $\{(U_i, \phi_i) | i \in I\}$, I some indexing set, such that (1) each $U_i \subset M$, (2) $M = \bigcup_{i \in I} U_i$, (3) for every $i, j \in I$ such that $U_i \cap U_j \neq \varnothing$, $\phi_i(U_i \cap U_j)$ is open in $B_i^*(b^*)$ for some Banach space B_i, and (4) for every i, j with $U_i \cap U_j \neq \varnothing$, $\phi_j \phi_i^{-1}$: $\phi_i(U_i \cap U_j) \longrightarrow \phi_j(U_i \cap U_j)$ is a C^p-diffeomorphism w.r.t. the metric topologies and a homeomorphism w.r.t. the b^* topologies of B_i^* and B_j^*. Any such pair (U_i, ϕ_i) will be called a *(C^p, b^*)-chart* on M. As usual, two (C^p, b^*)-atlases will be called compatible if their union is again a (C^p, b^*)-atlas for M. "Compatibility" is then an equivalence relation on the family of (C^p, b^*)-atlases for M. An equivalence class of (C^p, b^*)-atlases for M will be called a *(C^p, b^*)-manifold structure* on M, and M equipped with a (C^p, b^*)-manifold structure will be called a *(C^p, b^*)-manifold*. Note that a (C^p, b^*)-atlas determines two topologies on M. One is defined by requiring each U_i to be open and each ϕ_i: $U_i \longrightarrow \phi_i(U_i) \subset B_i^*(\tau^*)$ to be a homeomorphism for each i. The other is defined by requiring each U_i to be open and each ϕ_i: $U_i \longrightarrow \phi_i(U_i) \subset B_i^*(b^*)$ to be a homeomorphism. When regarding M with the first (resp. second) of these topologies we write $M(\tau^*)$ (resp. $M(b^*)$). It is clear that equivalent atlases induce the same topologies, so that a (C^p, b^*)-manifold has well defined τ^* and b^* topologies.

If a (C^p, b^*)-manifold M has a (C^p, b^*)-atlas $\{(U_i, \phi_i)\}$ such that all the $\phi_i(U_i)$ lie in the same conjugate space, say B^*, then we say M is a *(C^p, b^*)-manifold modelled on B^**. Note that B^* is naturally a (C^p, b^*)-manifold modelled on B^*, $-1 \leqslant p \leqslant \infty$.

Let M, N be two (C^p, b^*)-manifolds. A map f: $M \longrightarrow N$ is a *(C^p, b^*)-morphism* if $\forall x \in M$ there is a (C^p, b^*)-chart (U, ϕ) about x and a (C^p, b^*)-chart (V, ψ) about $f(x)$ such that $f(U) \subset V$ and $\psi f \phi^{-1}$ is a C^p-morphism in the

τ^*-topologies and a continuous map in the b^*-topologies. We obtain a category of (C^p, b^*)-manifolds and (C^p, b^*)-morphisms. If $f: M \longrightarrow N$ has an inverse in this category it will be called a *(C^p, b^*)-isomorphism.*

Let M be a (C^p, b^*)-manifold, and let $A \subset M$ be such that for every $x \in A$ there is a (C^p, b^*)-chart (U, ϕ) such that $\phi(U) = U_i \oplus U_2 \subset B^*$ and $\phi(U \cap A) = U_1$ where U_i is open in E_i^*, $i = 1, 2$, and $B = E_1 \oplus E_2$. (Here we have identified B^* with $E_1^* \oplus E_2^*$. We use the hypothesis $B = E_1 \oplus E_2$ to insure that the natural map $E_1^* \oplus E_2^* \longrightarrow E_1^* \times E_2^*$ is a homeomorphism in the b^*, as well as τ^*, topologies. See Proposition III. 3.) Regarding $\phi/U \cap A$ as a map into E_1^*, the charts $(U \cap A, \phi/U \cap A)$ induce a (C^p, b^*)-manifold structure on A. We call A with this structure a *(C^p, b^*)-submanifold* of M. If A is also closed in $M(b^*)$ we call A a *closed (C^p, b^*)-submanifold* of M.

If $f: M \longrightarrow N$ is a (C^p, b^*)-morphism and $x \in M$ we say f is a *split (C^p, b^*)-embedding at x* if there is a neighborhood W of x in $M(b^*)$ such that $f/W: W \longrightarrow f(W)$ is a (C^p, b^*)-isomorphism onto a (C^p, b^*)-submanifold of N. (Note that our definition of (C^p, b^*)-submanifold incorporates a strong splitting requirement. See [4, p. 767].) We say f is a *local split (C^p, b^*)-embedding* if it is a split (C^p, b^*)-embedding at each point of M. If a local split (C^p, b^*)-embedding gives a (C^p, b^*)-isomorphism onto a [closed] (C^p, b^*)-submanifold of N we call it a *[closed] split (C^p, b^*)-embedding.*

A *(C^p, b^*)-partition of unity* for a (C^p, b^*)-manifold M is a topological partition of unity $\{\lambda_\alpha: M \longrightarrow R\}$ for $M(b^*)$ such that each λ_α is a (C^p, b^*)-morphism. (Here we have identified R with R^*.)

We are now ready to state our main results.

THEOREM II. 1. *Let M be a (C^p, b^*)-manifold such that $M(b^*)$ is paracompact. Then M admits a (C^p, b^*)-partition of unity subordinate to any b^*-open cover, $-1 \leq p \leq \infty$.*

The proof of Theorem II. 1 is given in §IV. Note, as an immediate corollary to Theorem II. 1, that the (C^p, b^*)-manifold M admits a C^p-partition of unity subordinate to any b^*-open cover. In Proposition III. 1(d) we show that for any Banach space B, $B^*(b^*)$ is paracompact. Thus we obtain:

THEOREM II. 2 (COROLLARY OF II. 1). *For any Banach space B, B^* admits smooth partitions of unity subordinate to b^*-open covers.*

Thus, for example, $l_1 = c_0^*$ admits smooth partitions of unity subordinate to b^*-open covers. (Here l_1 is the space of all real sequences $\{x_i\}$ with $\|\{x_i\}\| = \Sigma_i|x_i| < \infty$, and c_0 is the space of all real sequences $\{c_i\}$ converging to zero with

$\|\{c_i\}\| = \sup_i\{|c_i|\}$. The isomorphism $l_1 \xrightarrow{\Phi} c_0^*$ is given by $\Phi(\{x_i\})(\{c_i\}) = \Sigma c_i x_i$.) This is in contrast to the fact shown by Bonic and Frampton in [1] that there is no nontrivial differentiable map $f: l_1 \longrightarrow R$ with bounded support. (That there is no such continuously differentiable map f was shown by Kurzweil in [8].)

THEOREM II. 3. *Let M be a (C^p, b^*)-manifold modelled on B^* such that $M(b^*)$ is regular and Lindelöf. If there is a closed linear split embedding $B \oplus B \longrightarrow B$, then there is a closed split (C^p, b^*)-embedding $M \xrightarrow{f} B^*$, $-1 \leqslant p \leqslant \infty$.*

Theorem II. 3 is a corollary of the more general Theorem V. 5 stated and proved in §V.

If B is separable we may weaken the hypothesis "$M(b^*)$ is regular and Lindelöf" occurring in Theorem II. 3. Assuming B is separable, each bounded set in $B^*(b^*)$ is metrizable [3, Theorem 1, p. 426]. Thus, if $\mathcal{O}$ is open in $B^*(b^*)$, $\mathcal{O} \cap B_n^*(b^*)$ is an F_σ set, $n = 1, 2, \cdots$. Since $B_n^*(b^*)$ is compact (Alaoglu's theorem), $\mathcal{O} \cap B_n^*(b^*)$ is thus the countable union of compact sets. It follows that $\mathcal{O}$ is the countable union of compact sets and therefore Lindelöf. But, as shown in Proposition III. 4, if every open subset of the model is Lindelöf, then each component of the paracompact manifold M is Lindelöf. We have proven the following theorem.

THEOREM II. 4 (COROLLARY OF II. 3). *Assume B is separable. Let M be a (C^p, b^*)-manifold modelled on B^* such that $M(b^*)$ is paracompact with at most countably many components. If there is a closed linear split embedding $B \oplus B \longrightarrow B$, then there is a closed spit (C^p, b^*)-embedding $M \xrightarrow{f} B^*$, $-1 \leqslant p \leqslant \infty$.*

REMARK. If B is a Banach space, let b denote the bounded weak topology on B. Define (C^p, b)-manifolds in analogy with (C^p, b^*)-manifolds. If B is reflexive, then the natural isomorphism $B \longrightarrow B^{**}$ identifies the b topology on B with the b^* (i.e. bounded B^*) topology on B^{**}. Hence our theorems yield as special cases analogous theorems about (C^p, b)-manifolds modelled on reflexive Banach spaces.

III. Some preliminaries.

First we recall the following theorem due to Banach and Dieudonné [2]; for the theorem in the form we state it see [3, p. 427].

THEOREM (BANACH-DIEUDONNÉ). *A fundamental system of neighborhoods of the origin for the b^*-topology of B^* consists of the sets $\{x^* \in B^* \mid |x^*(a)| < 1, a \in A\}$ where A is a sequence of elements of B converging to zero in the norm topology.*

Since translation is a homeomorphism on $B^*(b^*)$ it follows that if $x_0 \in B$, a basis for $B^*(b^*)$ at x_0^* is the collection of sets $\{x^* \in B^* |\ |(x^* - x_0^*)(a)| < 1, a \in A\}$ where A is a sequence in B converging to zero in the norm topology.

PROPOSITION III. 1. (a) *$B^*(b^*)$ is a locally convex topological vector space.*

(b) *Any open set in $B^*(b^*)$ is unbounded if B is infinite dimensional.*

(c) *If $U = \{x^* \in B^* |\ |x^*(a)| < 1, a \in A\}$ where A is any subset of B, and if $\bar{U}$ is the closure of U in $B^*(b^*)$, then $\bar{U} = \{x^* \in B^* |\ |x^*(a)| \leqslant 1, a \in A\}$.*

(d) *$B^*(b^*)$ is regular, Lindelöf, and paracompact.*

(e) *$B^*(b^*)$ is not first countable if B is infinite dimensional.*

PROOF OF III. 1. (a) is an immediate corollary of the Banach-Dieudonné theorem. For (b), let $V = \{x^* \in B^* |\ |x^*(a)| < 1, a \in A\}$, where A is a sequence in B converging to the origin in the metric topology, be a basic open neighborhood of the origin in $B^*(b^*)$, and let $r > 0$. Set $A' = \{a \in A |\ \|a\| \geqslant 1/r\}$. Then A' is finite, and so, if B is infinite dimensional, the closed linear span of A is not all of B. From the Hahn-Banach theorem it follows [11, Theorem 5.19] that there is a linear functional $y^* \in B^*$ such that $y^*(A') = 0$ and $\|y^*\| = r$. Then $y^* \in V$ and we have shown that V contains elements of arbitrary norm.

For (c), note that the linear functional determined by each $a \in A$ ($x^* \longrightarrow x^*(a), x^* \in B^*$) is continuous on $B^*(b^*)$. Thus the set $\{x^* \in B |\ |x^*(a)| \leqslant 1\}$ is closed in $B^*(b^*)$. This set is exactly the closure of U as each $x^* \in B^*$ is the limit in the b^* topology of a sequence $\{t_n x^*\}$, $\{t_n\}$ being any sequence in (0, 1) converging to 1.

To see that $B^*(b^*)$ is regular, let $U = \{x^* \in B^* |\ |x^*(a)| < 1, a \in A\}$ be a basic open neighborhood of zero. Setting $\frac{1}{2}U = \{x^* \in B^* |\ |x^*(a)| < \frac{1}{2}, a \in A\}$, from (c) we have $0 \in \frac{1}{2}U \subset \overline{\frac{1}{2}U} \subset U$, and regularity follows. $B^*(b^*)$ is Lindelöf since it is the countable union of compact subspaces (Alaoglu theorem). By Morita's theorem [9, Theorem 10] any regular Lindelöf space is paracompact. This completes the proof of (d).

We prove (e), by contradiction. Suppose $B^*(b^*)$ were first countable. For every n, let $S_n^* = \{x^* \in B^* |\ \|x^*\| = n\}$, and let $\bar{S}_n^*$ be the closure of S_n^* in $B^*(b^*)$. From (b) it follows (we are assuming B is infinite dimensional) that $0 \in \bar{S}_1^*$. Hence by our supposition there is a sequence $\{x_i^* | i = 1, 2, \cdots\} \subset S_1^*$ such that $x_i^* \xrightarrow{b^*} 0$ (i.e. $\{x_i^*\}$ converges to 0 in the b^* topology). Similarly, for each i there is a sequence $\{x_{i,k}^* | k = 1, 2, \cdots\} \subset S_{i+1}^*$ converging to x_i^* in the b^* topology. Let $A = \{x_{i,k}^* | i, k = 1, 2, \cdots\}$. Then the b^* closure of A contains 0. By our supposition of first countability there is a sequence $\{a_j | j = 1, 2, \cdots\} \subset A$ such that $a_j \xrightarrow{b^*} 0$. But then clearly also $a_j \xrightarrow{w^*} 0$ so $\{a_j\}$ is bounded.

Hence, for some integer $n \geq 2$, $\{a_j\} \cap S_n^*$ is an infinite sequence convergent to 0. This is a contradiction since $\{a_j\} \cap S_n^*$ must converge to x_n^*.

For convenience we include the following proposition which is probably well known.

PROPOSITION III. 2. *Let* $\lambda: B^* \longrightarrow E^*$ *be a linear map. The following assertions are equivalent and imply that* $\lambda: B^*(\tau^*) \longrightarrow E^*(\tau^*)$ *is continuous*:

(a) $\lambda: B^*(w^*) \longrightarrow E^*(w^*)$ *is continuous.*

(b) $\lambda: B^*(b^*) \longrightarrow E^*(b^*)$ *is continuous.*

(c) λ *is the adjoint of a continuous* (*w.r.t the metric topologies*) *linear map* $\gamma: E \longrightarrow B$.

PROOF OF III. 2. (a) *implies* (b): By [3, Corollary 3, p. 424] the compact sets in $B^*(w^*)$ are the closed sets in $B^*(w^*)$ that are bounded in $B^*(\tau^*)$. Thus, if λ is w^*-continuous (i.e. continuous w.r.t. the w^*-topologies) λ carries bounded sets to bounded sets. It follows at once that λ is b^*-continuous.

(b) *implies* (c): We know [3, Theorem 6, p. 428 and Theorem 9, p. 421] that the conjugate of $B^*(b^*)$ is B, and the metric topology on B considered as the conjugate of $B^*(b^*)$ is obviously the norm topology τ. If $\lambda: B^*(b^*) \longrightarrow E^*(b^*)$ is continuous, then, as in (a) implies (b), λ is bounded and hence τ^*-continuous. Thus, the adjoint map $\lambda^*: E \longrightarrow B$ is τ-continuous [3, Lemma 2, p. 478]. Clearly λ is the adjoint of λ^*.

(c) *implies* (a): Easy, standard, and left to the reader.

If $B = E \oplus F$ we identify $\{x^* \in B^* | x^*(F) = 0\}$ with E^* and $\{x^* \in B^* | x^*(E) = 0\}$ with F^*. Using this identification we have:

PROPOSITION III. 3. *If* $B = E \oplus F$, *then* $B^* = E^* \oplus F^*$, *and the natural linear map* $E^* \oplus F^* \longrightarrow E^* \times F^*$ *given by* $x^* + y^* \longrightarrow (x^*, y^*)$ *is a homeomorphism w.r.t. the* w^* *and* b^* (*as well as* τ^*) *topologies.*

PROOF OF III. 3. Let $i: E \longrightarrow B$, $j: F \longrightarrow B$ be the inclusions and $p: B \longrightarrow E$, $q: B \longrightarrow F$ the (continuous) projections given by the direct sum $B = E \oplus F$. Then

$$p \circ i = I_E, \quad q \circ j = I_F, \quad \text{and} \quad i \circ p + j \circ q = I_B.$$

By III. 2 the adjoint maps i^*, j^*, p^*, q^* are all continuous w.r.t. the τ^*, w^*, and b^* topologies. Clearly, also,

$$i^* \circ p^* = I_{E^*}, \quad j^* \circ q^* = I_{F^*}, \quad p^* \circ i^* + q^* \circ j^* = I_{B^*}.$$

The proposition follows.

Finally, we establish the following proposition, which was used in §II to establish Theorem II. 4. The technique used in the proof, as noted in [6, Lemma 1.1], is essentially due to Bill Cutler.

PROPOSITION III. 4. *Let M be a connected (Hausdorff) paracompact manifold modelled on the topological vector space F. If each open subset of F is Lindelöf, then M is Lindelöf. If also F is separable, then so also is M.*

PROOF OF III. 4. Let $\{U_\alpha | \alpha \in \mathfrak{A}\}$ be a locally-finite open cover of X by Lindelöf sets. Then for each α, U_α is regular and Lindelöf and hence paracompact (Morita, [9, Theorem 10]). Let $\{\mathcal{O}_\beta | \beta \in \mathfrak{B}\}$ be any given open cover of X. For each $\gamma \in \mathfrak{A}$ consider the cover $\{\mathcal{O}_\beta \cap U_\gamma | \beta \in \mathfrak{B}\}$ of U_γ. Since U_γ is paracompact and Lindelöf, there is a countable, locally-finite open refinement $\{V_i^\gamma | i = 1, 2, \cdots\}$ of $\{\mathcal{O}_\beta \cap U_\gamma | \beta \in \mathfrak{B}\}$ with each V_i^γ meeting only finitely many U_α's. Define a chain on X to be a finite collection of open sets $\{W_i | i = 1, \cdots, n\}$ such that $W_i \cap W_{i+1} \neq \emptyset$, $i = 1, \cdots, n-1$. W_i is called the *ith link* of the chain, and the chain joins x to y if $x \in W_1$ and $y \in W_n$. Given $x \in X$, there are at most countably many chains starting at x (i.e., $x \in W_1$) and made up of links from $\{V_i^\alpha | \alpha \in \mathfrak{A}, i = 1, 2, \cdots\}$. This is because there are only countably many choices for the ith link. The set of points that can be joined to x by such chains is open and closed in connected X, and is therefore all of X. Thus there is such a chain having any given V_i^α as a link. This in turn implies that $\{V_i^\alpha | \alpha \in \mathfrak{A}, i = 1, 2, \cdots\}$ must be countable. Choosing an $\mathcal{O}_\beta$ containing each V_i^α gives the required countable subcover of $\{\mathcal{O}_\beta | \beta \in \mathfrak{B}\}$.

The final assertion of III. 4 follows since the countable union of separable spaces is separable.

IV. **Proof of Theorem II. 1.** From the Banach-Dieudonné theorem (see §III) sets of the form $\{x^* \in B^* | \, |(x^* - x_0^*)(x_i)| < \alpha_i, i = 1, 2, 3, \cdots\}$, where $x_i \in B$, $\|x_i\| = 1$ and $\alpha_i \longrightarrow \infty$, form a basis at $x_0^* \in B^*(b^*)$.

LEMMA IV. 1. *Let $U = \{x^* \in B^* | \, |(x^* - x_0^*)(x_i)| < \alpha_i, i = 1, 2, 3, \cdots\}$, where $x_i \in B$, $\|x_i\| = 1$ and $\alpha_i \longrightarrow \infty$, be a basic open neighborhood of x_0^* in $B^*(b^*)$. Let $U(2) = \{x^* \in B^* | \, |(x^* - x_0^*)(x_i)| < 2\alpha_i, i = 1, 2, 3, \cdots\}$. Then there exists a (C^∞, b^*)-morphism $\phi: B^* \longrightarrow [0, 1] \subset R$ such that $\phi(B^* \backslash U(2)) = 0$ and $\phi/\overline{U} = 1$, where $\overline{U}$ is the closure of U in $B^*(b^*)$.*

PROOF OF IV. 1. We may assume $x_0^* = 0$. It follows from III. 1(c) that $\overline{U} = \{x^* \in B^* | \, |x^*(x_i)| \leq \alpha_i, i = 1, 2, 3, \cdots\}$. For each i, let $\widetilde{\phi}_i: R \longrightarrow [0, 1]$ be a C^∞ map such that $\widetilde{\phi}_i/[-\alpha_i, \alpha_i] = 1$ and $\widetilde{\phi}_i/(R \backslash (-2\alpha_i, 2\alpha_i)) = 0$. Define $\phi_i: B^* \longrightarrow [0, 1]$ by $\phi_i = \widetilde{\phi}_i \circ \hat{x}_i$ where $\hat{x}_i: B^*(w^*) \longrightarrow R$ is the continuous linear functional $\hat{x}_i(y^*) = y^*(x_i)$. Then $\phi_i: B^* \longrightarrow R$ is a (C^∞, b^*)-morphism. Define

$\phi: B^* \longrightarrow [0, 1]$ by $\phi = \Pi_{i=1}^{\infty}\phi_i$. Since $\alpha_i \longrightarrow \infty$, given any $r > 0$, ϕ restricted to B_r^* reduces to a finite product. It follows easily that ϕ is a (C^∞, b^*)-morphism satisfying the conditions of the lemma.

PROPOSITION IV. 2. *Let A be a closed subset of $B^*(b^*)$ and U an open subset of $B^*(b^*)$ such that $A \subset U$. Then there is a (C^∞, b^*)-morphism $\lambda: B^* \longrightarrow [0, 1] \subset R$ such that $\lambda/A = 1$ and $\lambda/(B^*\backslash U) = 0$.*

PROOF OF IV. 2. For each integer $n \geqslant 1$, let $A_n = A \cap B_n^*$. Since $A_1(b^*)$ is compact (Alaoglu theorem), there are finitely many basic open sets in $B^*(b^*)$, say $V_{1,1}, \cdots, V_{1,k_1}$, covering A_1 such that $V_{1,i}(2) \subset U$, $i = 1, \cdots, k_1$. For every $x^* \in A_2\backslash B_1^*$ let V_{2,x^*} be a basic open neighborhood of x^* in $B^*(b^*)$ such that $V_{2,x^*}(2) \subset U\backslash B_1^*$. Then $V_{1,1}, \cdots, V_{1,k_1}$, $\{V_{2,x^*} | x^* \in A_2\backslash B_1^*\}$ cover the b^* compact A_2, so we may select a finite subcover, say $V_{1,1}, \cdots, V_{1,k_1}$, $V_{2,1}, \cdots, V_{2,k_2}$. Continuing inductively, we obtain a sequence $V_{1,1}, \cdots, V_{1,k_1}$, $V_{2,1}, \cdots, V_{i,1}, \cdots, V_{i,k_i}, \cdots$ of basic b^* open sets covering A such that each $V_{i,j}(2) \subset U\backslash B_{i-1}^*$, $i > 1$. Fix i. Define $\phi_i: B^* \longrightarrow [0, 1]$ as follows. By Lemma IV. 1 for each $j = 1, \cdots, k_i$ there is a (C^∞, b^*)-morphism $\phi_{i,j}: B^* \longrightarrow [0, 1]$ such that $\phi_{i,j}/V_{i,j} = 0$ and $\phi_{i,j}/B^* - V_{i,j}(2) = 1$. Let $\phi_i = \Pi_{j=1}^{k_i}\phi_{i,j}$. Then let $\phi = \Pi_{i=1}^{\infty}\phi_i$. Then, since $\phi/B_n^* = \Pi_{i=1}^{n}\phi_i$, it follows easily that ϕ is a (C^∞, b^*)-morphism. Clearly, also, $\phi/A = 0$ and $\phi/(B^*\backslash U) = 1$. Defining $\lambda(x^*) = 1 - \phi(x^*)$ gives the required (C^∞, b^*)-morphism.

PROOF OF THEOREM II. 1. Having established Proposition IV. 2, Theorem II. 1 follows in rather standard fashion. We omit the proof.

V. Proof of Theorem II. 3.

PROPOSITION V. 1. *Let $f: M \longrightarrow N$ be a local split (C^p, b^*)-embedding, $-1 \leqslant p \leqslant \infty$, which is also a topological embedding $M(b^*) \longrightarrow N(b^*)$. Then f is a split (C^p, b^*)-embedding.*

PROOF OF V. 1. We must show that $f(M)$ is a (C^p, b^*)-submanifold of N and that $f^{-1}: f(M) \longrightarrow M$ is a (C^p, b^*)-morphism. Let $m \in M$. Since f is a local split (C^p, b^*)-embedding there is a (C^p, b^*)-chart (U, ϕ) at m such that $f: U \longrightarrow f(U)$ is a (C^p, b^*)-isomorphism onto a (C^p, b^*)-submanifold of N. By the defining property of (C^p, b^*)-submanifold there is a (C^p, b^*)-chart (V, ψ) about $f(m)$ in N such that $\psi(V) = V_1 \oplus V_2 \subset B^*$, $\psi(V \cap f(U)) = V_1$, where V_i is open in $E_i^*(b^*)$, $i = 1, 2$, and $B = E_1 \oplus E_2$. Since $f(U)$ is open in $f(M)$ regarded as a subspace of $N(b^*)$, $f(U) = f(M) \cap W$ for some b^*-open set W in N. Certainly we can require $V \subset W$. Then $\psi(V \cap f(M)) = \psi(V \cap f(U)) = V_1$. It follows that $f(M)$ is a (C^p, b^*)-submanifold of N.

To see that f^{-1} is a (C^p, b^*)-morphism let $(V', \psi') = (V \cap f(M), \psi_1/V \cap f(M))$ where ψ_1 is the composite of ψ and the projection $V_1 \oplus V_2 \longrightarrow V_1$. Then (V', ψ') is a (C^p, b^*)-chart at $f(m)$ in $f(U)$, and hence (since $f\colon U \longrightarrow f(U)$ is a (C^p, b^*)-isomorphism), $\phi f \psi'^{-1}$ is a (C^p, b^*)-morphism. But (V, ψ') is also a (C^p, b^*)-chart at $f(m)$ in $f(M)$, so this shows $f^{-1}\colon f(M) \longrightarrow M$ is a (C^p, b^*)-morphism at $f(m)$.

DEFINITION V. 2 (CF. [7]). Let ΣB^* be the countable direct limit of B^*. Again we have two topologies on ΣB^*: the τ^* topology obtained by regarding ΣB^* as $\Sigma B^*(\tau^*)$ and the b^* topology obtained by regarding ΣB^* as $\Sigma B^*(b^*)$. For every $n \geqslant 1$, $\Sigma B^* = (B^*)^n \oplus {}_nB^*$ where $(B^*)^n = \Sigma_{i=1}^n B^*$ and ${}_nB^* = \{x = \{x_i^*\} \in \Sigma B^* | x_i^* = 0, i \leqslant n\}$. Let E be a Banach space. A continuous linear injection $\chi\colon \Sigma B^*(b^*) \longrightarrow E^*(b^*)$ will be called a *(τ^*, b^*)-Banach completion of ΣB^** if, for every n,

(a) $\chi/(B^*(b^*))^n$ is a closed b^*-embedding.

(b) there is a b^*-closed linear subspace ${}_nE^*$ of E^* such that $E^* = \chi((B^*)^n) \oplus {}_nE^*$ where $\chi((B^*)^n) \oplus {}_nE^*$ is of the form $F_{1,n}^* \oplus F_{2,n}^*$ with $E = F_{1,n} \oplus F_{2,n}$, $\chi({}_nB^*) \subset {}_nE^*$ and ${}_{(n+1)}E^* \subset {}_nE^*$.

Note that by III. 2, if χ is a (τ^*, b^*)-Banach completion, then $\chi/(B^*)^n$ is also an embedding w.r.t. the metric topologies.

PROPOSITION V. 3. *If there is a closed linear split embedding (metric topologies) $B \oplus B \longrightarrow B$, then there is a (τ^*, b^*)-Banach completion $\Sigma B^* \longrightarrow B^*$.*

PROOF OF V. 3. The lemma of [7] asserts that our hypothesis implies the existence of a continuous linear injection $\rho\colon \Sigma B \longrightarrow B$ (metric topologies) such that for $n \geqslant 1$, ρ/B^n is a closed embedding and, writing $B = B^n \oplus {}_nB$, there is a closed linear subspace of B, ${}_nE$, such that $B = \rho(B^n) \oplus {}_nE$, ${}_{(n+1)}E \subset {}_nE$, and $\rho({}_nB) \subset {}_nE$. For $n \geqslant 1$, let $\rho_n = \rho/B^n$. Define $\chi\colon \Sigma B^* \longrightarrow B^*$ by $\chi/(B^*)^n = (\rho_n^{-1})^*$. (Here we identify $(B^*)^n$ with $(B^n)^*$.) One checks that χ is well defined and gives the required (τ^*, b^*)-Banach completion.

DEFINITION V. 4. Let M be a (C^p, b^*)-manifold, $-1 \leqslant p \leqslant \infty$. A closed (C^p, b^*)-embedding $f\colon M \longrightarrow \Sigma B^*$ is a closed topological embedding $f\colon M(b^*) \longrightarrow \Sigma B^*(b^*)$ such that for each $x \in M$ there is a neighborhood U of x in $M(b^*)$ and an integer n (U, n depending on x) such that $f(U) \subset (B^*)^n$ and $f\colon U \longrightarrow (B^*)^n$ is a split (C^p, b^*)-embedding as defined in §II. Here we identify $(B^*)^n$ with $(B^n)^*$ and regard $(B^*)^n$ as a (C^p, b^*)-manifold modelled on $(B^n)^*$.

THEOREM V. 5. *Let M be a (C^p, b^*)-manifold modelled on B^* such that $M(b^*)$ is regular and Lindelöf. Then there is a closed (C^p, b^*)-embedding $f\colon M \longrightarrow \Sigma B^*$ such that if $\chi\colon \Sigma B^* \longrightarrow E^*$ is a (τ^*, b^*)-Banach completion, then*

$\chi \circ f: M \longrightarrow E^*$ is a closed split (C^p, b^*)-embedding, $-1 \leqslant p \leqslant \infty$.

Combined with Proposition V. 3, Theorem V. 5 immediately implies Theorem II. 3. Note that the proof of Theorem II. 4 given in §II shows that here, too, if B is separable we may replace the hypothesis "$M(b^*)$ is regular and Lindelöf" by "$M(b^*)$ is paracompact with at most countably many components".

PROOF OF V. 5. In this proof closures of sets will always be taken in the b^* topology.

By Morita's theorem [9, Theorem 10], $M(b^*)$ being regular and Lindelöf implies $M(b^*)$ is also paracompact. For each $m \in M$, let (U_m, ϕ_m) be a (C^p, b^*)-chart with $m \in U_m$. Choose G_m open in $B^*(b^*)$ such that

$$\phi_m(m) \in G_m \subset \bar{G}_m \subset \phi_m(U_m).$$

Note that for any set $U \subset \phi_m^{-1}(G_m)$ we have $\phi_m(\bar{U}) = \overline{\phi_m(U)}$. Using this together with the fact that $M^*(b^*)$ is paracompact and Lindelöf, one obtains a countable (C^p, b^*)-atlas $\{(U_i, \phi_i) | i = 1, 2, \cdots\}$ for M such that $\{U_i\}$ is a locally-finite collection of nonempty sets, and each ϕ_i extends to a closed b^*-embedding $\overline{\phi_i}$: $\overline{U_i} \longrightarrow \overline{\phi_i(U_i)}$. Let $\{W_i\}$, $\{O_i\}$, $\{V_i\}$ be precise b^*-open refinements of $\{U_i\}$ such that for each i,

$$\varnothing \neq W_i \subset \bar{W}_i \subset O_i \subset \bar{O}_i \subset V_i \subset \bar{V}_i \subset U_i.$$

By IV. 2 there is for each i a (C^p, b^*)-morphism λ_i': $B^* \longrightarrow [0, 1]$ such that $\lambda_i'/\phi_i(\bar{W}_i) = 1$ and $\lambda_i'/(B^* \backslash \phi_i(O_i)) = 0$. Define λ_i: $M \longrightarrow [0, 1]$ by $\lambda_i | U_i = \lambda_i' \circ \phi_i$, $\lambda_i/(M \backslash \bar{O}_i) = 0$. Then λ_i is a (C^p, b^*)-morphism with $\lambda_i/\bar{W}_i = 1$ and $\lambda_i/(M \backslash O_i) = 0$. Similarly, using the V_i, construct for each i a (C^p, b^*)-morphism μ_i: $M \longrightarrow [0, 1]$ such that $\mu_i/\bar{O}_i = 1$ and $\mu_i/(M \backslash U_i) = 0$. Fix a nonzero element a^* of B^*. Define a (C^p, b^*)-morphism $\tilde{\phi}_i$: $M \longrightarrow B^* \times B^*$ by

$$\tilde{\phi}_i(m) = \begin{cases} (\lambda_i(m)\phi_i(m), \lambda_i(m) \cdot a^*), & m \in U_i, \\ (0, 0), & m \notin \bar{O}_i. \end{cases}$$

Since $\lambda_i = 1$ on W_i, each $\tilde{\phi}_i$ restricted to W_i is a split (C^p, b^*)-embedding. Define f: $M \longrightarrow \Sigma B^*$ by

$$f(m) = \left(\sum_{i=1}^{\infty} i \cdot \mu_i(m) a^*, \tilde{\phi}_1(m), \tilde{\phi}_2(m), \cdots \right).$$

The local-finiteness of the U_i insures that the sum is finite and that $f(m) \in \Sigma B^*$ (i.e. at most finitely many $\tilde{\phi}_i(m)$ are nonzero for each m).

Let $m_0 \in M$. Find an integer j such that $m_0 \in W_j$ and then a b^* neighborhood V of m_0 such that $V \subset W_j$ and V intersects only $U_1, \cdots, U_n$ some n.

Then for every $m \in V$,

$$f(m) = \left(\sum_{i=1}^{\infty} i\mu_i(m)a^*, \widetilde{\phi}_1(m), \cdots, \widetilde{\phi}_n(m) \right) \in (B^*)^{2n+1}.$$

For convenience, define f_i: $V \longrightarrow B^*$, $i = 1, \cdots, 2n + 1$, such that

$$f(m) = \left(\sum_{i=1}^{n} i\mu_i(m)a^*, \widetilde{\phi}_1(m), \cdots, \widetilde{\phi}_n(m) \right) = (f_1(m), \cdots, f_{2n+1}(m)).$$

Note that $f_{2j}(m) = \phi_j(m)$. Let $G = \Sigma_{i=1}^{2n+1} X_i$ where $X_i = B^*$, $i \neq 2j$, and $X_{2j} = \phi_j(V)$. Define ψ: $G \longrightarrow G$ by $\psi(y_1, \cdots, y_{2n+1}) = (z_1, \cdots, z_{2n+1})$ where $z_i = y_i - f_i\phi_j^{-1}(y_{2j})$, $i \neq 2j$ and $z_{2j} = y_{2j}$. Then ψ is a (C^p, b^*)-isomorphism onto G, and (G, ψ) is a (C^p, b^*)-chart at $f(m_0)$ in $(B^*)^{2n+1}$. Clearly $\psi(f(V) \cap G) = 0 \times \cdots \times 0 \times \phi_j(V) \times 0 \times \cdots \times 0$, which shows $f(V)$ is a (C^p, b^*)-submanifold of $(B^*)^{2n+1}$. The map $f(V) \longrightarrow V$ given by $(y_1, \cdots, y_{2n+1}) \longrightarrow \phi_j^{-1}(y_{2j})$ is f^{-1} and is clearly a (C^p, b^*)-morphism. Thus we have shown that each $m_0 \in M$ has a neighborhood V in $M(b^*)$ such that $f(V) \subset (B^*)^{2n+1}$, some n, and such that f: $V \longrightarrow (B^*)^{2n+1}$ is a split (C^p, b^*)-embedding.

To finish the proof that f: $M \longrightarrow \Sigma B^*$ is a closed (C^p, b^*)-embedding it is only left to show that f: $M(b^*) \longrightarrow \Sigma B^*(b^*)$ is a topological closed embedding. Clearly f is injective. Let A be closed in $M(b^*)$. Let $\{f(m_\alpha) | \alpha \in \mathfrak{A}\}$ be a net in $f(A)$ converging to y in $\Sigma B^*(b^*)$. (We work with nets since the b^* topology is generally not first countable, see III. 1(e).) Writing $y = (y_1, y_2, \cdots)$ we then must have $\{\Sigma_{i=1}^{\infty} i\mu_i(m_\alpha)a^* | \alpha \in \mathfrak{A}\}$ converging to y_1 in $B^*(b^*)$. Hence, for some $\rho \in \mathfrak{A}$ and some integer n, $\Sigma_{i=1}^{\infty} i\mu_i(m_\alpha) < n$ for $\alpha > \rho$. Since $\bigcup_{i=1}^{\infty} \lambda_i^{-1}\{1\} \supset \bigcup_{i=1}^{\infty} W_i \supset M$ and $\mu_i \geqslant \lambda_i$, it now follows that $\{m_\alpha | \alpha > \rho\} \subset \lambda_1^{-1}\{1\} \cup \cdots \cup \lambda_n^{-1}\{1\}$. It follows that for some $k \leqslant n$ and some cofinal $\mathfrak{D} \subset \mathfrak{A}$, $\{m_\alpha | \alpha \in \mathfrak{D}\} \subset \lambda_k^{-1}\{1\}$. Since $\{f(m_\alpha) | \alpha \in \mathfrak{D}\}$ converges to y, we then have $\{\phi_k(m_\alpha) | \alpha \in \mathfrak{D}\}$ converging to y_{2k}. But on $\lambda_k^{-1}\{1\}$, ϕ_k is a closed b^*-embedding. It follows that $y_{2k} = \phi_k(m)$, some $m \in \lambda_k^{-1}\{1\} \cap A$, and $\{m_\alpha | \alpha \in \mathfrak{D}\}$ converges to m in $M(b^*)$. The continuity of f then assures us that $\{f(m_\alpha) | \alpha \in \mathfrak{D}\}$, and hence $\{f(m_\alpha) | \alpha \in \mathfrak{A}\}$, converges to $f(m)$ in $B^*(b^*)$ so that $y = f(m)$. Thus $f(A)$ is closed in $\Sigma B^*(b^*)$, and we have shown that f: $M(b^*) \longrightarrow \Sigma B^*(b^*)$ is a closed embedding. This completes the proof that f: $M \longrightarrow \Sigma B^*$ is a closed (C^p, b^*)-embedding.

Now suppose χ: $\Sigma B^* \longrightarrow E^*$ is a (τ^*, b^*)-Banach completion. Let $m \in M$. Since f: $M \longrightarrow \Sigma B^*$ is a closed (C^p, b^*)-embedding there is a b^*-neighborhood V of m in M and an integer n such that $f(V) \subset (B^*)^n$ and f: $V \longrightarrow (B^*)^n$ is a split (C^p, b^*)-embedding. But $\chi/(B^*)^n$ is a linear split (C^p, b^*)-embedding, so $(\chi \circ f)/V$ is a split (C^p, b^*)-embedding into E^*. Thus, $\chi \circ f$: $M \longrightarrow E^*$ is a local split (C^p, b^*)-embedding.

By V. 1, $\chi \circ f$ will be a closed split (C^p, b^*)-embedding if $\chi \circ f\colon M(b^*) \longrightarrow E^*(b^*)$ is a closed topological embedding. In order to show this let A be closed in $M(b^*)$, and let $\{m_\alpha | \alpha \in \mathfrak{A}\}$ be a net in A such that $\{\chi \circ f(m_\alpha)\}$ converges to $y \in E^*$. Let $f_1(m_\alpha) = \Sigma_{i=1}^{\infty} i\mu_i(m_\alpha)a^*$ and let $\pi\colon E^* \longrightarrow \chi((B^*)^1)$ be the natural projection, where $E^* = \chi((B^*)^1) \oplus {}_1E^*$. Since $\chi({}_1B^*) \subset {}_1E^*$ we obtain

$$((\chi/(B^*)^1)^{-1} \circ \pi \circ \chi \circ f)(m_\alpha) = f_1(m_\alpha).$$

Hence

$$f_1(m_\alpha) \longrightarrow ((\chi/(B^*)^1)^{-1} \circ \pi)(y) = \gamma a^* \quad \text{some } \gamma \geqslant 0.$$

Thus, there is a $\rho \in \mathfrak{A}$ and an integer n such that $\mu_i(m_\alpha) < 1$, and hence $\lambda_i(m_\alpha) = 0$, for $\alpha > \rho$ and $i > n$. Therefore, $f(m_\alpha) \in f(A) \cap (B^*)^{2n+1}$ if $\alpha > \rho$. Since $\chi/(B^*)^{2n+1}$ is a closed embedding it now follows that $y \in (\chi \circ f)(A)$. We have shown that $\chi \circ f$ is a closed split (C^p, b^*)-embedding. This completes the proof of Theorem V. 5 (and hence also Theorem II. 3).

VI. **Remark.** Given a differentiable map $g\colon B^* \longrightarrow E^*$ and $x^* \in B^*$, let $Dg(x^*)\colon B^* \longrightarrow E^*$ be the derivative of g at x^*. Note that the (C^∞, b^*)-morphism $\phi\colon B^* \longrightarrow R$ constructed in the proof of Lemma IV. 1 satisfies the property that $D\phi(x^*)\colon B^* \longrightarrow R$ is also a (C^∞, b^*)-morphism for each $x^* \in B^*$. This is because for each $r > 0$ there is an integer n such that $\phi/B_r^* = \Pi_{i=1}^n \widetilde{\phi}_i \circ \hat{x}_i$ ($\widetilde{\phi}_i$ and $\hat{x}_i$ as defined in the proof of IV. 1) and each

$$D(\widetilde{\phi}_i \circ \hat{x}_i)(x^*) = D\widetilde{\phi}_i(x^*(x_i)) \circ \hat{x}_i\colon B^*(b^*) \longrightarrow R$$

is continuous. Similar reasoning establishes that the λ constructed in Proposition IV. 2 also satisfies the property that $D\lambda(x^*)\colon B^* \longrightarrow R$ is a (C^∞, b^*)-morphism for each $x^* \in B^*$.

Now define a *S*(*trong*)-(C^p, b^*)-manifold M by requiring that M be a (C^p, b^*)-manifold such that the transition maps $\phi_j\phi_i^{-1}$ also satisfy the requirement that $D(\phi_j\phi_i^{-1})(x^*)\colon B_i^*(b^*) \longrightarrow B_j^*(b^*)$ be continuous for each $x^* \in B_i^*$. Similarly define S-(C^p, b^*)-morphisms, etc. We have just indicated that the λ constructed in Proposition IV. 2 is an S-(C^∞, b^*)-morphism. Using this, the proofs given in this paper yield analogous theorems about S-(C^p, b^*)-manifolds.

In reference to the remark at the end of §II, observe that there is no distinction between (C^p, b)-manifolds and S-(C^p, b)-manifolds. This is because any linear map between Banach spaces which is continuous with respect to the metric topology is automatically continuous with respect to the weak, and hence also bounded weak, topology (cf. Proposition III. 2).

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