

I-RINGS

BY

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ABSTRACT. A ring R , possibly with no identity, is called an I_0 -ring if each one-sided ideal not contained in the Jacobson radical $J(R)$ contains a non-zero idempotent. If, in addition, idempotents can be lifted modulo $J(R)$, R is called an I -ring. A survey of when these properties are inherited by related rings is given. Maximal idempotents are examined and conditions when I_0 -rings have an identity are given. It is shown that, in an I_0 -ring R , primitive idempotents are local and primitive idempotents in $R/J(R)$ can always be lifted. This yields some characterizations of I_0 -rings R such that $R/J(R)$ is primitive with nonzero socle. A ring R (possibly with no identity) is called semiperfect if $R/J(R)$ is semisimple artinian and idempotents can be lifted modulo $J(R)$. These rings are characterized in several new ways: among them as I_0 -rings with no infinite orthogonal family of idempotents, and as I_0 -rings R with $R/J(R)$ semisimple artinian. Several other properties are derived. The connection between I_0 -rings and the notion of a regular module is explored. The rings R which have a regular module M such that $J(R) = \text{ann}(M)$ are studied. In particular they are I_0 -rings. In addition, it is shown that, over an I_0 -ring, the endomorphism ring of a regular module is an I_0 -ring with zero radical.

1. Definitions and examples. Throughout this paper all rings are assumed to be associative but do not necessarily have an identity. When a ring has an identity, modules are assumed to be unital. Unless otherwise stated, all modules are left modules and homomorphisms are written on the right of their arguments. The *Jacobson radical* of a ring R will be denoted by $J(R)$.

LEMMA 1.1. *If R is a ring the following conditions are equivalent:*

- (1) *Every left ideal $L \not\subseteq J(R)$ contains a nonzero idempotent.*
- (2) *Every right ideal $T \not\subseteq J(R)$ contains a nonzero idempotent.*
- (3) *If $a \notin J(R)$ then $xax = x$ for some $x \neq 0$.*

PROOF. Given (1), let $a \notin J(R)$. If $e = ra \in Ra$ is a nonzero idempotent,

Received by the editors November 20, 1973.

AMS (MOS) subject classifications (1970). Primary 16A32; Secondary 16A20, 16A30, 16A48.

Key words and phrases. I -ring, idempotents, lifting idempotents, semiperfect ring, von Neumann regular ring, regular module, endomorphism ring.

⁽¹⁾ This research was supported by N.R.C. (Canada) grant number A8075.

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(3) follows with $x = rar$. The converse is clear and the proof that $(1) \Leftrightarrow (3)$ is analogous. \square

DEFINITION 1.2. A ring R is called an I_0 -ring if it satisfies the conditions of Lemma 1.1. An I_0 -ring in which idempotents can be lifted modulo $J(R)$ is called an I -ring.

An I -ring with a nil Jacobson radical will be called a *Zorn ring*. This terminology is used by Kaplansky [5, p. 19], and Bourbaki [3, p. 75]. However the reader is cautioned that Jacobson [4, p. 210] and Levitzky [8, p. 385] refer to our Zorn rings as I -rings. Many of the results below will be true for Zorn rings, but we shall not mention this fact in most instances. Our primary concern here is with I -rings. A deep study of Zorn rings can be found in Levitzky [8].

The class of I -rings is quite large. It obviously contains all division rings and, more generally, contains all local rings where, in this paper, a ring R will be called *local* if it has an identity and $R/J(R)$ is a division ring. In a different direction, each primitive ring with a minimal left ideal is an I -ring. In fact every semiprime ring with essential socle is an I -ring. On the other hand any I_0 -ring with identity and no divisors of zero is a local ring, so the integers are an example of a noetherian semiprime ring which is not an I -ring.

An element a in a ring R is called (von Neumann) *regular* if $aba = a$ for some $b \in R$. The ring R is called *regular* if each of its elements is regular and R is called π -*regular* if some power of each element is regular. It is easily verified that every π -regular ring is an I -ring (in fact a Zorn ring). In particular, every algebraic algebra is an I -ring [4, p. 210].

LEMMA 1.3. Let R be a ring, let L be a left (right) ideal of R and let $x \in L$. If there is an idempotent f such that $f - x \in J(R)$, then there exists an idempotent $e \in L$ such that $e - x \in J(R)$.

PROOF. If $f^2 = f$ and $f - x \in J(R)$, choose $a \in J(R)$ such that $a + (f - x) = a(f - x)$. Then $faf + f - fxf = faf - faxf$ so $f = f(x - ax)f$. Take $e = f(x - ax)$. \square

PROPOSITION 1.4. Let R be a ring in which idempotents can be lifted modulo $J(R)$. Then R is an I -ring if and only if $R/J(R)$ is an I -ring.

A ring R (possibly with no identity) will be called *semiperfect* if $R/J(R)$ is artinian and idempotents can be lifted modulo $J(R)$.

COROLLARY 1.5. Every semiperfect ring is an I -ring.

PROPOSITION 1.6. If R is an I -ring (I_0 -ring) so is each one-sided ideal of R .

PROOF. Let L be a left ideal of R . Then $J(L) = \{a \in L \mid La \subseteq J(R)\}$ [1,

p. 113]. If M is a left ideal of L and $M \not\subseteq J(L)$ then $LM \not\subseteq J(R)$ so there exists $0 \neq e^2 = e \in LM \subseteq M$. Suppose now that $x \in L$, $x^2 - x \in J(L)$. Then $x^4 - x^2 \in J(R)$ so, by Lemma 1.3, choose $e^2 = e \in L$ such that $e - x^2 \in J(R)$. Then $e - x \in J(L)$. \square

COROLLARY 1.7. *If R is an I -ring (I_0 -ring) so is each subring aRb where $a, b \in R$.*

PROOF. aRb is a left ideal of aR . \square

Note that Proposition 1.6 and its corollary are true for Zorn rings as well. We remark here that the center of an I -ring need not be an I -ring (see Example 1.9).

The $n \times n$ matrix ring over a ring R will be denoted by $M_n(R)$. If $r \in R$ let $E_{ij}(r)$ denote the matrix with r in the (i, j) -position and zeros elsewhere.

PROPOSITION 1.8. *If R is an I_0 -ring so is the ring $M_n(R)$ of all $n \times n$ matrices over R .*

PROOF. Let L be a left ideal of $M_n(R)$ with $L \not\subseteq J[M_n(R)] = M_n[J(R)]$. There exists a matrix $A = (a_{ij}) \in L$ with $a_{pq} \notin J(R)$ for some p, q . Then $L_0 = \{x \in R \mid x = x_{pq} \text{ for some } (x_{ij}) \in L\}$ is a left ideal of R and $L_0 \not\subseteq J(R)$. If $0 \neq e^2 = e \in L_0$, let $X = (x_{ij}) \in L$ with $x_{pq} = e$. One verifies that $E_{qp}(e)X = \sum_j E_{qj}(ex_{pj})$ is a nonzero idempotent in L . \square

A natural question is the following: Is $M_n(R)$ an I -ring whenever R is an I -ring? It is sufficient to answer this in the case $n = 2$. The answer is affirmative if R is semiperfect or if $J(R)$ is locally nilpotent. More generally, one can ask: If a ring R is such that idempotents can be lifted modulo $J(R)$, does $M_2(R)$ have the same property? We remark in this connection that it is an open question whether $M_2(R)$ is a nil ring whenever R is a nil ring (see [7]).

It is obvious that a direct sum of rings is an I -ring (an I_0 -ring) if and only if the same is true of each summand. It is also clear that R/A is an I -ring (I_0 -ring) if R is an I -ring (I_0 -ring) and $A \subseteq J(R)$ is an ideal. However the class of I -rings is not closed under homomorphic images.

EXAMPLE 1.9. Let Δ be a division ring and let $S \subseteq \Delta$ be any subring. Let R be the ring of all countably infinite square matrices of the form

$$\begin{pmatrix} A & & 0 \\ & \delta & \\ 0 & & \delta \cdot \cdot \cdot \end{pmatrix}$$

where $A \in M_n(\Delta)$ for some $n \geq 1$ and $\delta \in S$. Denote this matrix by (A, δ) . If

$(A, \delta) \neq 0$ we can assume $A \neq 0$ so, by Proposition 1.8, choose a matrix B such that BA is a nonzero idempotent. Then $(B, 0)(A, \delta) = (BA, 0)$ is an idempotent in R , and it follows that R is an I_0 -ring with $J(R) = 0$. The map $R \rightarrow S$ given by $(A, \delta) \mapsto \delta$ is a ring epimorphism so it is clear that a homomorphic image of an I -ring need not be an I -ring. If S is in the center of Δ , the center of R is isomorphic to S and so need not be an I_0 -ring. \square

2. Primitive idempotents. In any ring R there is a natural partial ordering of the idempotents defined by $f \leq e$ if $f \in eRe$. A nonzero idempotent which is minimal in this partial ordering is called *primitive*. It is easily verified that e is primitive if and only if $0 \neq f^2 = f \in Re$ implies $Rf = Re$.

We say that $e = e^2$ is a *local idempotent* if eRe is a local ring. It is well known that every primitive idempotent in a semiperfect or regular ring is local. We generalize this as follows:

PROPOSITION 2.1. *The following conditions are equivalent for an idempotent e in an I_0 -ring R :*

- (1) e is primitive.
- (2) If $L \subseteq Re$ is a left ideal and $L \not\subseteq J(R)$ then $L = Re$.
- (3) e is local.

PROOF. (1) \Rightarrow (2) by the above remark (since R is an I_0 -ring) and (3) \Rightarrow (1) is obvious. Assume (2). If $a \in eRe$ and $a \notin J(eRe) = eRe \cap J(R)$, then $Ra = Re$ by (2). Hence a has a left inverse in eRe , proving (3). \square

An immediate consequence of this is that an I_0 -ring has a unique nonzero idempotent if and only if it has the form $L \oplus A$ where L is local and $J(A) = A$.

Two idempotents e and f in a ring R are said to be *equivalent* if there exist $x \in eRf$ and $y \in fRe$ such that $e = xy$ and $f = yx$.

COROLLARY 2.2. *Two primitive idempotents e and f in an I_0 -ring R are equivalent if and only if $eRf \not\subseteq J(R)$.*

PROOF. If $x \in eRf$, $x \notin J(R)$, we have $Rx = Rf$ by Proposition 2.1. Write $f = yx$ with $y \in fRe$. Then $0 \neq (xy)^2 = xy \in eRe$ so $xy = e$. The converse is trivial. \square

We have the following result on the existence of primitive idempotents:

PROPOSITION 2.3. *An I_0 -ring with $J(R) = 0$ has a primitive idempotent if and only if it has a maximal left (right) annihilator.*

PROOF. Let $L = \{a \in R \mid aS = 0\}$ be a maximal left annihilator. If $0 \neq x \in S$ and $0 \neq e^2 = e \in xR$, we have $L = \{a \mid ae = 0\}$ by maximality. We claim

e is primitive. If $0 \neq f^2 = f \in eRe$ we have $L = \{a \mid af = 0\}$ so $e - f \in L$. But then $0 = (e - f)e = e - f$. The converse follows from Proposition 2.1. \square

We have immediately the following which retrieves a result of Koh [6] when R is regular.

COROLLARY 2.4. *An I_0 -ring R with $J(R) = 0$ is primitive with nonzero socle if and only if it is a prime ring with a maximal left (right) annihilator.*

The next result shows that, in an I_0 -ring, primitive idempotents in $R/J(R)$ can always be lifted to R . It will be referred to several times below.

LEMMA 2.5. *Let R be an I_0 -ring and suppose $x \in R$ is such that $x + J(R)$ is a primitive idempotent in $R/J(R)$. There exists an idempotent $e \in R$ such that $e - x \in J(R)$.*

PROOF. Choose $0 \neq f^2 = f \in Rx$. Then, in $\bar{R} = R/J(R)$, $\bar{f} \in \bar{R}\bar{x}$ so $\bar{R}\bar{f} = \bar{R}\bar{x}$ by Proposition 2.1. If we set $e = f + xf - fxf$, then $e^2 = e \neq 0$ and $\bar{e} = \bar{x}$. \square

PROPOSITION 2.6. *The following are equivalent for a ring R :*

- (1) *R is an I_0 -ring and every nonzero idempotent contains a primitive idempotent.*
- (2) *Each left (right) ideal $L \not\subseteq J(R)$ contains a primitive idempotent.*

PROOF. If (2) holds and $e^2 = e \neq 0$ let $f^2 = f \in Re$ be primitive. Then $(ef)^2 = ef \in eRe$ and ef is primitive by Proposition 2.1 since $Ref = Rf$. Hence (2) \Rightarrow (1); the converse is clear. \square

We say a ring R has primitive idempotents if it satisfies these conditions. Lemmas 2.5 and 1.3 immediately yield

COROLLARY 2.7. *An I_0 -ring R has primitive idempotents if and only if the same is true of $R/J(R)$.*

Hence every semiperfect ring has primitive idempotents. It is well known that a primitive ring with nonzero socle has primitive idempotents (the socle is large as a left ideal). The ring in Example 1.9 is easily shown to be primitive with nonzero socle and so a homomorphic image of a ring with primitive idempotents need not have this property. On the other hand, the methods of §1 show that, if R has primitive idempotents, the same is true of any one-sided ideal of R , any subring of the form aRb , and any matrix ring $M_n(R)$.

If R is a ring with identity, a module $M \neq 0$ is called *local* if it is projective and $Rx = M$ for each $x \in M - \text{rad } M$. If $e^2 = e \in R$ then $\text{rad}(Re) = J(R)e$ and it follows easily from Proposition 2.1 that Re is local if and only if e is a local idempotent.

THEOREM 2.8. *The following are equivalent for an I_0 -ring R :*

(1) *R has primitive idempotents and any two primitive idempotents are equivalent.*

(2) *R contains a primitive idempotent and $J(R)$ is a prime ideal.*

(3) *$R/J(R)$ is a primitive ring with nonzero socle.*

If R has an identity these are equivalent to

(4) *R has a local module and $J(R)$ is a prime ideal.*

PROOF. If R has an identity, (2) \Leftrightarrow (4) by the remark above; (3) \Rightarrow (1) by Corollary 2.7 and [4, p. 51]; and (2) \Rightarrow (3) is clear. If (1) holds let $aRb \subseteq J(R)$ where $a, b \in R - J(R)$. Choose primitive idempotents $e \in Ra$ and $f \in bR$. By hypothesis, $e = xy$ where $x \in eRf$ and $y \in fRe$ so $e \in RaRbR \subseteq J(R)$, a contradiction. This means $J(R)$ is a prime ideal so (1) \Rightarrow (2). \square

In fact, the equivalence of (2) and (4) can be proved under the weaker assumption that $J(R)$ is small in R as a left ideal. In this form it generalizes the fact that a prime ring is primitive with nonzero socle if and only if it has an irreducible projective module.

3. Maximal idempotents. An idempotent e in a ring R is said to be the *greatest* idempotent if $e \geq f$ for every idempotent f in R . Such an idempotent is central since $e \geq e + er - ere$ and $e \geq e + re - ere$ for each $r \in R$. Hence:

LEMMA 3.1. *An I_0 -ring with a greatest idempotent has the form $S \oplus A$ where S has an identity and $J(A) = A$.*

An idempotent e is called *maximal* if it is not the greatest idempotent and $e \leq f, f^2 = f$ implies $e = f$ or f is the greatest idempotent.

LEMMA 3.2. *Let R be a ring with no greatest idempotent. The following are equivalent for an idempotent $e \in R$:*

(1) *e is maximal.*

(2) *If $Re \subseteq Rf$ where $f^2 = f$ then $Re = Rf$.*

(3) *If $ef = 0$ where $f^2 = f$ then $f = 0$.*

Moreover the left-right analogs of (2) and (3) are equivalent to (1).

PROOF. (1) \Rightarrow (2). If $Re \subseteq Rf$ let $g = e + f - fe$. Then $g^2 = g \geq e$ and (2) follows.

(2) \Rightarrow (3). If $ef = 0$ let $g = e + f - fe$. Then $g^2 = g$ and $Re \subseteq Rg$ so $g = ge = e$ by (2). Hence $f = f^2 = (fe)f = 0$.

(3) \Rightarrow (1). If $g^2 = g \geq e$ then $e(g - e) = 0$ so $g = e$. \square

THEOREM 3.3. *If an I_0 -ring R has a maximal idempotent then $R/J(R)$ has*

an identity and the following are equivalent:

- (1) $x \in Rx$ for each $x \in R$.
- (2) $J(R)$ is small as a right ideal.
- (3) R has a left identity.

PROOF. If R has a greatest idempotent the result follows from Lemma 3.1. Otherwise, let $e^2 = e$ be maximal. The right ideal $T = \{a \in R \mid ea = 0\}$ is contained in $J(R)$ by Lemma 3.2 so $r - er \in J(R)$ for every $r \in R$. Similarly $r - re \in J(R)$ so $R/J(R)$ has an identity.

Now (3) \Rightarrow (2) is well known and (2) \Rightarrow (3) follows since $R = eR + J(R)$. If (1) holds let $r \in R$ and write $x = r - er$ where $e^2 = e$ is maximal. If $x = tx$, $t \in R$ then $x = (t - te)x$ so $x = 0$ (since $t - te \in J(R)$). Hence (1) \Rightarrow (3); the converse is obvious. \square

In particular a regular ring with a maximal idempotent has an identity. The next result generalizes the well-known fact that a semisimple artinian ring has an identity.

COROLLARY 3.4. *Let R be an I_0 -ring with $J(R) = 0$. If R has a minimal right annihilator of the form $r(e) = \{a \in R \mid ea = 0\}$, $e^2 = e$ then R has an identity.*

PROOF. It suffices to show R has a greatest idempotent by Lemma 3.1. If not, let $r(e)$ be minimal. If $e \leq f$, $f^2 = f$ then $r(f) \subseteq r(e)$ so either $r(f) = r(e)$ (so $e = f$) or $r(f) = 0$ (so $fR = R$). It follows that R has a maximal idempotent and so has an identity, a contradiction. \square

We remark finally that an I -ring R has a maximal idempotent if and only if $R/J(R)$ has an identity and, in this case, the maximal idempotents are just the pre-images of the identity in $R/J(R)$.

4. Finiteness conditions. In this section we study I_0 -rings which satisfy chain conditions on the set of idempotents and obtain some new characterizations of semiperfect rings.

LEMMA 4.1. *Let R be any ring. The following are equivalent:*

- (1) R has maximum condition on idempotents.
- (2) R has maximum condition on left ideals Re , $e^2 = e$ (on right ideals eR , $e^2 = e$).
- (3) R has no infinite orthogonal family of idempotents.

PROOF. The proofs of (2) \Rightarrow (3) \Rightarrow (1) are omitted. Given (1) let $Re_1 \subseteq Re_2 \subseteq \dots$, $e_i^2 = e_i$. Define f_1, f_2, \dots as follows: $f_1 = e_1$, $f_{k+1} = f_k + e_{k+1} - e_{k+1}f_k$ for $k \geq 1$. Then $f_k \in Re_k$ for each k and consequently $f_k^2 = f_k$

and $f_1 \leq f_2 \leq \dots$. If $f_n = f_{n+1} = \dots$ for some n then $Re_{n+1} = Re_{n+2} = \dots$. Hence (1) \Rightarrow (2). \square

We remark that a ring R with minimum condition in left annihilators of idempotents has maximum condition on idempotents. Moreover, the converse is true if R has no total right annihilators; that is if $Ra = 0$ implies $a = 0$.

It is surprising that the minimum condition on idempotents is, in general, weaker than the maximum condition. (Consider $R = \Delta \oplus \Delta \oplus \dots$ where Δ is a division ring.) The two are equivalent for rings with identity as the next result shows.

LEMMA 4.2. *The following are equivalent in any ring R :*

- (1) R has minimum condition on idempotents.
- (2) R has minimum condition on left ideals Re , $e^2 = e$ (on right ideals eR , $e^2 = e$).
- (3) Any bounded ascending chain of idempotents terminates.

PROOF. Given (1) let $Re_1 \supseteq Re_2 \supseteq \dots$, $e_k^2 = e_k$. Then $e_1 e_2 \dots e_k$ is an idempotent for each k , $Re_1 e_2 \dots e_k = Re_k$ and $e_1 \geq e_1 e_2 \geq \dots$. Hence (1) \Rightarrow (2). We leave (2) \Rightarrow (3) \Rightarrow (1) to the reader. \square

The next result contains some new characterizations of semiperfect rings (defined following Proposition 1.4).

THEOREM 4.3. *The following are equivalent for any ring R :*

- (1) R is semiperfect.
- (2) R is an I_0 -ring with maximum condition on idempotents.
- (3) If $L \subseteq R$ is a left (right) ideal there exist an idempotent $e \in L$ and a left (right) ideal $M \subseteq J(R)$ such that $L = Re + M$.
- (4) R is an I_0 -ring and $R/J(R)$ is semisimple artinian.

PROOF. (1) \Rightarrow (2). If R is semiperfect it is an I_0 -ring by Corollary 1.5. If $e_1 \leq e_2 \leq \dots$ are idempotents in R then $\bar{e}_n = \bar{e}_{n+1} = \dots$ in $R/J(R)$ for some n . Hence $(e_{k+1} - e_k)^2 = (e_{k+1} - e_k) \in J(R)$ for each $k \geq n$ and so $e_n = e_{n+1} = \dots$.

(2) \Rightarrow (3). If $L \subseteq J(R)$ take $e = 0$ and $M = L$. Otherwise let $0 \neq e^2 = e \in L$ be maximal in L . Then $L = Re + M$ where $M = \{x \in L | xe = 0\}$. If $M \not\subseteq J(R)$ choose $0 \neq f^2 = f \in M$. Then $g = e + f - ef$ is an idempotent in L and $e \leq g$. Hence $e = g$ by the choice of e and so $f = f^2 = f(ef) = 0$, a contradiction. Hence $M \subseteq J(R)$.

(3) \Rightarrow (4). This is obvious.

(4) \Rightarrow (1). Since R is an I_0 -ring, primitive idempotents in $R/J(R)$ can be

lifted by Lemma 2.5. But every idempotent in $R/J(R)$ is a sum of orthogonal primitive idempotents and so can be lifted by standard techniques. \square

If R has an identity it is not hard to show (using Lemma 2.3 of [2]) that condition (3) is equivalent to the condition that every cyclic R -module has a projective cover. The following consequences of Theorem 4.3 give, in the case when R has an identity, some new proofs of well-known facts.

COROLLARY 4.4. *Let R be a semiperfect ring. A subring of R is semiperfect if and only if it is an I_0 -ring. In particular, one-sided ideals of R and subrings of the form aRb , $a, b \in R$, are semiperfect.*

PROOF. Simply observe that the maximum condition on idempotents is inherited by subrings and use Proposition 1.6. \square

COROLLARY 4.5. *A ring R is semiperfect if and only if the ring $M_n(R)$ is semiperfect for all $n \geq 1$ (some $n \geq 1$).*

PROOF. If R is semiperfect then $M_n(R)$ is an I_0 -ring by Proposition 1.8 and $M_n(R)/J[M_n(R)] \cong M_n[R/J(R)]$ is semisimple artinian. Conversely, R can be embedded in $M_n(R)$ as the subring of all matrices with zeros in all positions except $(1, 1)$. Since this is the intersection of a left and a right ideal, Corollary 4.4 completes the proof. \square

COROLLARY 4.6. *Every homomorphic image of a semiperfect ring is semiperfect.*

PROOF. If R is semiperfect and $A \subseteq R$ is an ideal, let $L \supseteq A$ be a left ideal. Choose an idempotent $e \in L$ and a left ideal $M \subseteq J(R)$ such that $L = Re + M$. Then, in $\bar{R} = R/A$, $\bar{L} = \bar{R}\bar{e} + \bar{M}$ and $\bar{M} \subseteq J(\bar{R})$. Hence R/A is semiperfect. \square

We remark that if R is a semiperfect ring and A is an ideal of R then $J(R/A) = (A + J)/A$ where J denotes $J(R)$. Indeed, there is a ring epimorphism $R/A \rightarrow R/(A + J)$ with kernel $(A + J)/A$. But $R/(A + J)$ is a homomorphic image of R/J so $J(R/A) \subseteq (A + J)/A$. The reverse inclusion always holds.

5. Regular idempotents. The notion of regularity has been extended to projective modules by Ware [9] and to a wider class of modules by Zelmanowitz [10]. In this section we shall study the connection between these modules and the I_0 -rings. In particular we show that, over an I_0 -ring, the endomorphism ring of a regular module is an I_0 -ring with zero radical.

Let M be a left R -module and let $M^* = \text{Hom}_R(M, R)$. Zelmanowitz calls M *regular* if, for any $x \in M$, there exists $\alpha \in M^*$ such that $(x\alpha)x = x$. If M is an R -module, $x \in M$ and $\alpha \in M^*$, define a map $[\alpha, x] \in \text{end}(M)$ by $y[\alpha, x] = (y\alpha)x$

for every $y \in M$. We say M is *projective* if it has a *dual basis*, that is if there exist subsets $\{x_\nu | \nu \in I\} \subseteq M$ and $\{\alpha_\nu | \nu \in I\} \subseteq M^*$ (indexed by the same set I) such that, for each $x \in M$, $x\alpha_\nu = 0$ for all but a finite number of $\nu \in I$ and $x = \sum_\nu x[\alpha_\nu, x_\nu]$. The following easily verified facts from [10] will be needed.

LEMMA 5.1. *Let M be a regular module, let $x \in M$, and suppose $\alpha \in M^*$ satisfies $(x\alpha)x = x$. Then:*

- (1) $e = x\alpha$ is an idempotent and $ex = x$.
- (2) $\alpha|_{Rx}: Rx \rightarrow Re$ is an isomorphism so Rx is projective.
- (3) $M = Rx \oplus \ker[\alpha, x]$.

This shows that, if M is regular and $x \in M$, then Rx is a projective summand of M . Zelmanowitz proves the converse [10, Theorem 2.2], and also shows that $Rx_1 + \cdots + Rx_n$ is a projective summand for any $x_i \in M$.

LEMMA 5.2. *If $e^2 = e \in R$, the following are equivalent:*

- (1) Re is a regular R -module.
- (2) For each $x \in Re$ there exists $y \in R$ such that $xyx = x$.
- (3) For each $x \in Re$ there exists $f^2 = f \in Re$ with $Rx = Rf$.

The easy proof is left to the reader. An idempotent e in a ring R will be called *left regular* if Re is a regular R -module. Clearly, every idempotent in a regular ring is left regular. Also, eRe is a regular ring if e is a left regular idempotent (the converse is false as an example below will show).

The following is a strengthening of a remark in [10, p. 346].

LEMMA 5.3. *Let R be a ring and let M be a regular R -module. If $a \notin \text{ann } M$ then aR contains a nonzero left regular idempotent. In particular $J(R) \subseteq \text{ann } M$.*

PROOF. If $ax \neq 0$, $x \in M$, choose $\alpha \in M^*$ such that $[(ax)\alpha]ax = ax$. Then $(ax)\alpha \in aR$ is a nonzero left regular idempotent by Lemma 5.1 and the fact that submodules of a regular module are regular. \square

Prompted by this, we say that a ring R has *left regular idempotents* if each right ideal $T \not\subseteq J(R)$ contains a nonzero left regular idempotent.

THEOREM 5.4. *The following are equivalent for a ring R :*

- (1) $J(R)$ is an intersection of annihilators of regular left modules.
- (2) $J(R) = \text{ann } M$ for some regular left module M .
- (3) R has left regular idempotents.

Moreover, when this is the case, $J(R)$ is the intersection of the annihilators of the cyclic regular left ideals Re , $e^2 = e \in R$.

PROOF. (1) \Rightarrow (2) since a direct sum of regular modules is again regular [10, Theorem 2.8]. (2) \Rightarrow (3) by Lemma 5.3. Hence assume (3). If $a \notin J(R)$ choose $0 \neq e^2 = e \in aR$ such that Re is regular. Then $a \notin \text{ann}(Re)$ so $J(R) \supseteq \bigcap \{\text{ann}(Re) | e^2 = e \text{ left regular}\}$. This must be equality by Lemma 5.3 so the proof is complete. \square

COROLLARY 5.5. *If M is a regular left R -module and $A = \text{ann } M$, then R/A has left regular idempotents and $J(R/A) = 0$.*

PROPOSITION 5.6. *Let R be a ring, let $A \subseteq J(R)$ be an ideal, let $T \subseteq R$ be a right ideal and let $e^2 = e \in R$. If R has left regular idempotents so do the rings R/A , T and eRe .*

PROOF. Let M be a regular R module with $\text{ann } M = J(R)$. Then M is a regular R/A -module and $\text{ann}_{R/A}(M) = J(R/A)$.

Turning to eRe , eM is an eRe -module and $\text{ann}_{eRe}(eM) = J(R) \cap eRe = J(eRe)$. We show that eM is regular. If $x \in eM$ let $\alpha: M \rightarrow R$ satisfy $(x\alpha)x = x$. Define $\bar{\alpha}: eM \rightarrow eRe$ by $y\bar{\alpha} = (y\alpha)e$. Then $\bar{\alpha} \in (eM)^*$ and $(x\bar{\alpha})x = (x\alpha)ex = x$.

Finally, TM is a T -module and $\text{ann}_T(TM) = \{t \in T | tTM = 0\} = \{t \in T | tT \subseteq J(R)\} = J(T)$. Now let $x \in TM$ and suppose $\alpha: M \rightarrow R$ satisfies $(x\alpha)x = x$. Since $(TM)\alpha \subseteq T(M\alpha) \subseteq T$, the map $\alpha|_{TM}: TM \rightarrow T$ is a T -homomorphism and $(x\alpha|_{TM})x = x$. This shows TM is a regular T -module. \square

PROPOSITION 5.7. *If a ring R has left regular idempotents the same is true of the matrix ring $M_n(R)$.*

PROOF. Let $T \not\subseteq J[M_n(R)]$ be a right ideal. As in the proof of Proposition 1.8, there exists an idempotent X in T of the form $X = \sum_{i=1}^n E_{ip}(r_i e)$. It is easy to show that each element in $M_n(R)X$ has the form $Y = \sum_{i=1}^n E_{ip}(t_i)$ where each $t_i \in Re$. By Lemma 5.2, we must show each such Y is regular in $M_n(R)$. By Lemma 5.2, let $t_1 s_1 t_1 = t_1$. Then $Y E_{p1}(s_1) Y - Y = \sum_{i=2}^n E_{ip}(t_i s_1 t_1 - t_i)$ and it suffices to show this is regular in $M_n(R)$. In other words, we may assume Y has the form $Y = \sum_{i=2}^n E_{ip}(t_i)$, $t_i \in Re$. This procedure can be continued to complete the proof. \square

EXAMPLES AND REMARKS. 1. Every regular ring has left (and right) regular idempotents.

2. Every ring R with $J(R) = 0$ and which has primitive idempotents has left (and right) regular idempotents.

3. If a prime ring R has a nonzero regular left module then it has left regular idempotents and $J(R) = 0$.

4. Any local ring R with $J(R) \neq 0$ is an example of an I -ring which does not have regular left (or right) idempotents.

5. The ring R in Example 1.9 is primitive with nonzero socle and so has left (and right) regular idempotents. Hence, if R has left regular idempotents, the same need not be true of the center of R or of a homomorphic image of R .

6. Let Δ be a division ring and let $R = \begin{bmatrix} \Delta & \Delta \\ 0 & 0 \end{bmatrix} \subseteq M_2(\Delta)$. If $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ then Re is a regular module and $\text{ann}(Re) = J(R)$. On the other hand $fR = R$ for any $0 \neq f^2 = f \in R$ so R has no regular right modules. In particular, Re can be regular while eR is not even when eRe is a division ring.

We conclude this section with some remarks about the endomorphism ring $E(M)$ of a regular module M . We shall write $A(M) = \{\alpha \in E(M) \mid \ker \alpha \text{ is large in } M\}$. It is well known that $A(M)$ is an ideal of $E(M)$.

THEOREM 5.8. *If M is a regular R -module, each one-sided ideal of $E(M)$ which is not contained in $A(M)$ contains a nonzero idempotent.*

PROOF. If $\alpha \in E(M) - A(M)$, choose $0 \neq x \in M$ such that $Rx \cap \ker \alpha = 0$. Then $\alpha|_{Rx}: Rx \rightarrow Rx$ is an isomorphism. We have $M = Rx\alpha \oplus K$ by Lemma 5.1 so define $\beta: M \rightarrow M$ by setting $K\beta = 0$ and $\beta|_{Rx\alpha} = (\alpha|_{Rx})^{-1}$. Then $\beta\alpha\beta = \beta \neq 0$ and the result follows. \square

COROLLARY 5.9. *Let M be a regular module such that every monomorphism in $E(M)$ has a right inverse. Then $E(M)$ is an I_0 -ring and $J[E(M)] = A(M)$.*

PROOF. Since $\ker \alpha \cap \ker(1 - \alpha) = 0$ for every $\alpha \in E(M)$, our hypothesis implies $A(M) \subseteq J[E(M)]$. This must be equality since $J[E(M)]$ contains no nonzero idempotent. \square

COROLLARY 5.10. *Let M be a regular R -module with the property that $M\gamma \subseteq J(R)$ for every $\gamma \in M^*$ with large kernel. Then $E(M)$ is an I_0 -ring with $J[E(M)] = 0$.*

PROOF. Suppose $\alpha \in A(M)$ and let $y \in M\alpha$. There exists $\gamma \in M^*$ such that $(y\gamma)y = y$. But $\ker \alpha \subseteq \ker(\alpha\gamma)$ and consequently $M\alpha\gamma \subseteq J(R)$. Thus $y\gamma$ is an idempotent in $J(R)$ and so $y\gamma = 0$. It follows that $M\alpha = 0$ and hence that $A(M) = 0$. \square

One situation where the condition in Corollary 5.10 is met is when R is an I_0 -ring.

COROLLARY 5.11. *If R is an I_0 -ring and M is a regular module then $E(M)$ is an I_0 -ring with $J[E(M)] = 0$.*

PROOF. Suppose $\gamma \in M^*$ has large kernel and $M\gamma \not\subseteq J(R)$. Let $e = y\gamma$ be a nonzero idempotent and put $x = ey$. Then $x\gamma = e \neq 0$ and $x = ex$ and it follows that $Rx \cap \ker \gamma = 0$. This is a contradiction since $Rx \neq 0$ and so the result follows from Corollary 5.10. \square

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