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NECESSARY CONDITIONS FOR ISOMORPHISM OF LIE ALGEBRAS OF BLOCK

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ABSTRACT. Two algebras of Block, $\mathfrak{L}(G, \delta, f)$ and $\mathfrak{L}(G', \delta', f')$, are isomorphic only if $m(G) = m(G')$. This is not sufficient for isomorphism.

Let $\mathfrak{L}$ be a simple finite-dimensional Lie algebra over Φ, an algebraically closed field of prime characteristic p. Simplicity allows the identification $x \leftrightarrow \operatorname{ad} x$ for each $x \in \mathfrak{L}$. (That $\mathfrak{L}$ be centerless is sufficient for the identification.) Then if $\mathfrak{D}(\mathfrak{L})$ denotes the derivation algebra of $\mathfrak{L}$ we have $\mathfrak{L}$ $(= \operatorname{ad} \mathfrak{L}) \subset \mathfrak{D}(\mathfrak{L})$. For each $x \in \mathfrak{L}$, $(\operatorname{ad} x)^p$ is a derivation of $\mathfrak{L}$ and, if $(\operatorname{ad} \mathfrak{L})^{p^k}$ is the vector space spanned by $\{(\operatorname{ad} x)^{p^k} | x \in \mathfrak{L}\}$, then $\mathfrak{R}(\mathfrak{L}) = \operatorname{ad} \mathfrak{L} + (\operatorname{ad} \mathfrak{L})^p + (\operatorname{ad} \mathfrak{L})^{p^2} + \cdots$ is a subalgebra of $\mathfrak{D}(\mathfrak{L})$ which is restricted. We will call $\mathfrak{R}(\mathfrak{L})$ the restricted algebra of $\mathfrak{L}$. If $\mathfrak{L}$ is restricted, then $\operatorname{ad} \mathfrak{L} = \mathfrak{R}(\mathfrak{L})$, or under the identification, $\mathfrak{L} = \mathfrak{R}(\mathfrak{L})$. Thus, for any arbitrary centerless algebra $\mathfrak{L}$, $\mathfrak{L} \subseteq \mathfrak{R}(\mathfrak{L}) \subseteq \mathfrak{D}(\mathfrak{L})$. Clearly, any two isomorphic simple algebras $\mathfrak{L}$ and $\mathfrak{L}'$ over Φ must have $\mathfrak{R}(\mathfrak{L}) \cong \mathfrak{R}(\mathfrak{L}')$ and $\mathfrak{D}(\mathfrak{L}) \cong \mathfrak{D}(\mathfrak{L}')$. We will use this relationship to determine isomorphism conditions upon the algebras of Block.

Let G be an elementary abelian p-group written as a direct summand of elementary abelian p-groups, $G = G_0 \oplus G_1 \oplus \cdots \oplus G_m$. Let Φ be an algebraically closed field of characteristic $p > 3$. For each $i = 0, 1, \ldots, m$ define $f\colon G \times G \to \Phi$ such that $f|_{G_i} = f_i\colon G_i \times G_i \to \Phi$ is a skew-symmetric, nondegenerate biadditive form. Then $f = f_0 + f_1 + \cdots + f_m$. For each $i = 1, \ldots, m$, assume that there exist additive functions $g_i, h_i\colon G_i \to \Phi$ such that $f_i(\alpha, \beta) = g_i(\alpha)h_i(\beta) - g_i(\beta)h_i(\alpha)$. Pick $\delta_i \in G_i$ for which $g_i(\delta_i) = 0$, and set $\delta = \delta_1 + \cdots + \delta_m$. Define $\mathfrak{L}(G, \delta, f)$ to be the Lie algebra over Φ with basis $\{u_\alpha | \alpha \in G, \alpha \neq 0, -\delta\}$ where multiplication is given by

$$u_\alpha u_\beta = \sum_{i=0}^{m} f_i(\alpha_i, \beta_i) u_{\alpha+\beta-\delta_i}.$$

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Here α_i and β_i denote the ith components of α and β, respectively, in G and δ_0 is assumed to be zero. $\mathfrak{L}(G, \delta, f)$ is then a simple algebra over Φ called an algebra of Block.

The derivations of the algebras of Block have been completely determined in [1]. As they will be utilized later, a brief description follows.

Since G is an elementary abelian p-group it is an n-dimensional vector space over Φ_p (the prime subfield of $\Phi = GF(p)$), each of the G_i's being a subspace of dimension, say, n_i. Pick a basis $\{\sigma_{01}, \sigma_{02}, \cdots, \sigma_{0n_0}\}$ for G_0 and $\{\sigma_{i1}, \cdots, \sigma_{in_i-1}, \delta_i\}$ for G_i, $i = 1, \cdots, m$, such that $f(\sigma_{i1}, \delta_i) = f_i(\sigma_{i1}, \delta_i) \neq 0$. Such is possible since f is nondegenerate. For each $\alpha \in G$, write $\alpha = \Sigma_{i,j} s_{ij}(\alpha)\sigma_{ij} + \Sigma_i s_i(\alpha)\delta_i$. The coefficients $s_{ij}(\alpha)$, $s_i(\alpha)$ of the σ_{ij}'s and δ_i's are unique since the σ_{ij}'s and δ_i's form a basis of G. The derivations of $\mathfrak{L}(G, \delta, f)$ are linear combinations (over Φ) of the elements in the following sets:

(i) $R = \{\text{ad } u_\alpha | \alpha \in G, \alpha \neq 0\}$ (ad $u_{-\delta}$ is included although not an element of $\mathfrak{L}(G, \delta, f)$).

(ii) $S = \{D(\sigma_{k1}, -\delta_k), D(\delta_k, -\delta_k) | k = 1, \cdots, m\}$ where $u_\alpha D(\gamma_k, -\delta_k) = f(\alpha, \gamma_k)u_{\alpha-\delta_k}$ for γ_k in G_k.

(iii) $T = \{D(\sigma_{0k}, 0), D(\sigma_{ij}, 0) | k = 1, \cdots, n_0; i = 1, \cdots, m; j = 2, \cdots, n_i - 1\}$ where $G_0 \neq \{0\}$ and $T = \{D(\delta, 0), D(\sigma_{ij}, 0) | i = 1, \cdots, m; j = 2, \cdots, n_i - 1\}$ when $G_0 = \{0\}$; where $u_\alpha D(\sigma_{ij}, 0) = s_{ij}(\alpha)u_\alpha$ and

$$u_\alpha D(\delta, 0) = \left(-1 + \sum_i s_i(\alpha)\right)u_\alpha$$

($D(\sigma_{i1}, 0)$ is a linear combination of ad u_{δ_i} and the remaining $D(\sigma_{ij}, 0)$'s.)

The set S is, of course, empty when $m = 0$. The dimension of $\mathfrak{L}(G,\delta,f)$ is $np^n - 1$ for $m = 0$ and $np^n - 2$ for $m > 0$, and it follows that the dimension of its derivation algebra, $\mathfrak{D}(\mathfrak{L}(G, \delta, f))$, is

(i) $np^n + n - 1$ when $G = G_0$ or when $G_0 \neq 0$ and $m > 0$.

(ii) $np^n + n$ when $G_0 = 0$.

From the dimensions of the derivation algebras and their derived algebras, Block concludes in [1, Theorem 14, Corollary 1] that necessary conditions for two algebras $\mathfrak{L}(G, \delta, f)$ and $\mathfrak{L}(G', \delta', f')$ to be isomorphic are that either $G_0 = 0$, $G'_0 = 0$, and $m(G) = m(G')$; or $G_0 \neq 0 \neq G'_0$ and $\min\{2, m(G)\} = \min\{2, m(G')\}$. By considering the restricted algebra of $\mathfrak{L}(G, \delta, f)$ we will show that it is necessary that $m(G) = m(G')$ for isomorphism and that, indeed, this is not sufficient.

For $u_\alpha, u_\beta \in \mathfrak{L}(G, \delta, f)$ it is easily shown by induction on m that

$$u_\alpha(\mathrm{ad}\, u_\beta)^p = \sum_{i=0}^{m} f(\alpha_i, \beta_i) f(\alpha_i - \delta_i, \beta_i) \cdots f(\alpha_i - (p-1)\delta_i, \beta_i) u_\alpha$$

The following lemma then shows that

$$u_\alpha(\mathrm{ad}\, u_\beta)^p = \sum_{i=0}^{m} \{f(\alpha_i, \beta_i)^p - f(\alpha_i, \beta_i) f(\delta_i, \beta_i)^{p-1}\} u_\alpha$$

Lemma 1. *Let* $a, b \in \Phi$, char $\Phi = p > 0$. *Then*

$$a(a-b)(a-2b) \cdots (a-(p-1)b) = a^p - ab^{p-1}.$$

Proof. The polynomial $x^p - xb^{p-1}$ has roots ib for $i = 0, \ldots, p-1$. Hence, $x^p - xb^{p-1} = \Pi_{i=0}^{p-1}(x - ib)$. Substituting a for x yields the desired result.

It is evident that

$$u_\alpha(\mathrm{ad}\, u_\beta)^{p^2} = \left\{ \sum_{i=0}^{m} f(\alpha_i, \beta_i)^p - f(\alpha_i, \beta_i) f(\delta_i, \beta_i)^{p-1} \right\}^p u_\alpha$$

$$= \sum_{i=0}^{m} \{f(\alpha_i, \beta_i)^{p^2} - f(\alpha_i, \beta_i)^p f(\delta_i, \beta_i)^{p(p-1)}\} u_\alpha,$$

and more generally that

$$u_\alpha(\mathrm{ad}\, u_\beta)^{p^k} = \sum_{i=0}^{m} \{f(\alpha_i, \beta_i)^{p^k} - f(\alpha_i, \beta_i)^{p^{k-1}} f(\delta_i, \beta_i)^{p^{k-1}(p-1)}\} u_\alpha.$$

Suppose that $\alpha = \Sigma_{i=0}^m \alpha_i = \Sigma_{i=1}^m (\Sigma_{j=1}^{n_i - 1} s_{ij}(\alpha)\sigma_{ij} + s_i(\alpha)\delta_i) + \Sigma_{j=1}^{n_0} s_{0j}(\alpha)\sigma_{0j}$. Then

$$\sum_{i=0}^{m} \{f(\alpha_i, \beta_i)^{p^k} - f(\alpha_i, \beta_i)^{p^{k-1}} f(\delta_i, \beta_i)^{p^{k-1}(p-1)}\}$$

$$= \sum_{i=0}^{m} \left(\sum_{j=1}^{q_i} s_{ij}(\alpha) \{f(\sigma_{ij}, \beta_i)^{p^k} - f(\sigma_{ij}, \beta_i)^{p^{k-1}} f(\delta_i, \beta_i)^{p^{k-1}(p-1)}\} \right.$$

$$\left. + s_i(\alpha) \{f(\delta_i, \beta_i)^{p^k} - f(\delta_i, \beta_i)^{p^{k-1}} f(\delta_i, \beta_i)^{p^{k-1}(p-1)}\} \right),$$

or

$$\mathrm{ad}\, u_\beta^{p^k} = \sum_{i=0}^{m} \left(\sum_{j=1}^{q_i} \{f(\sigma_{ij}, \beta_i)^{p^k} \right.$$

(1)

$$\left. - f(\sigma_{ij}, \beta_i)^{p^{k-1}} f(\delta_i, \beta_i)^{p^{k-1}(p-1)}\} D(\sigma_{ij}, 0) \right),$$

where $q_0 = n_0$ and $q_i = n_i - 1$ for $i = 1, \dots, m$. The restricted algebra $\mathfrak{R}(\mathfrak{L}(G, \delta, f))$ is therefore contained within the span of $R \cup T$. In the following discussion we will show that a basis for $\mathfrak{R}(\mathfrak{L}(G, \delta, f))$ is $R \cup T \setminus \{\text{ad } u_{-\delta}\}$ when $G_0 \neq \{0\}$ and $R \cup T \setminus \{\text{ad } u_{-\delta}, D(\delta, 0)\}$ when $G_0 = \{0\}$. It follows that $\dim \mathfrak{R}(\mathfrak{L}(G, \delta, f)) = \dim \mathfrak{L}(G, \delta, f) + n - 2m$.

Definition. The column rank over Φ_p of a matrix A with entries from Φ is the dimension of the vector space over Φ_p spanned by the columns of A. Denote this dimension by $\text{col rank}_{\Phi_p}(A)$.

Lemma 2. *If* $G = G_0$, *then* $\text{col rank}_{\Phi_p}(f(\sigma_{0i}, \sigma_{0j})^p) = n$, *the dimension of* G *over* Φ_p.

Proof. Suppose $\text{col rank}_{\Phi_p}(f(\sigma_{0i}, \sigma_{0j})^p) < n$, that is, suppose that there exist elements $a_1, a_2, \dots, a_n \in \Phi_p$, not all zero, such that

$$\sum_{j=1}^{n} a_j f(\sigma_{0i}, \sigma_{0j})^p = 0$$

for $i = 1, \dots, n$. Then from the biadditivity of f we conclude that $f(\sigma_{0i}, \sum_{j=1}^n a_j \sigma_{0j})^p = 0$, or $f(\sigma_{0i}, \sum_{j=1}^n a_j \sigma_{0j}) = 0$ for $i = 1, \dots, n$. This contradicts the nondegeneracy of f, whence the lemma is proved.

Lemma 3. *Suppose* $G = G_1$. *Let* $\{\beta_1, \dots, \beta_k, \delta\}$ *be a basis for* G *where* $f(\beta_1, \delta) \neq 0$, *and let*

$$A = \begin{bmatrix} 0 & f(\beta_1, \beta_2)^p - f(\beta_1, \beta_2)f(\beta_1, \delta)^{p-1} & \cdots & f(\beta_1, \beta_k)^p - f(\beta_1, \beta_k)f(\beta_1, \delta)^{p-1} \\ \vdots & & & \vdots \\ f(\beta_k, \beta_1)^p - f(\beta_k, \beta_1)f(\beta_k, \delta)^{p-1} & \cdots & f(\beta_k, \beta_{k-1}) - f(\beta_k, \beta_{k-1})f(\beta_k, \delta)^{p-1} & 0 \\ f(\delta, \beta_1)^p & \cdot \quad \cdot & \cdot & f(\delta, \beta_k)^p \end{bmatrix}$$

Then $\text{col rank}_{\Phi_p}(A) = k$.

Proof. Suppose $\text{col rank}_{\Phi_p}(A) < k$. Then there exist $a_1, \dots, a_k \in \Phi_p$, not all zero, such that

$$\sum_{j=1}^{k} a_j\{f(\beta_i, \beta_j)^p - f(\beta_i, \beta_j)f(\beta_i, \delta)^{p-1}\} = 0$$

for $i = 1, \dots, k$ and $\sum_{j=1}^k a_j f(\delta, \beta_j)^p = 0$. For each of the first k equalities we have

$$f\left(\beta_i, \sum_{j=1}^{k} a_j\beta_j\right)^p = f\left(\beta_i, \sum_{j=1}^{k} a_j\beta_j\right) f(\beta_i, \delta)^{p-1}$$

or

$$\left(f\left(\beta_i, \sum_{j=1}^{k} a_j\beta_j\right)\Big/ f(\beta_i, \delta)\right)^{p-1} = 1,$$

if $f(\beta_i, \delta) \neq 0$. Thus, for each $i = 1, \ldots, k$, there exists $c_i \in \Phi_p$ such that $f(\beta_i, \sum_{j=1}^k a_j\beta_j) = c_i f(\beta_i, \delta)$ (if $f(\beta_i, \delta) = 0$, then $c_i = 0$). Now define $g: G \to \Phi$ and $h: G \to \Phi$ by $g(\alpha) = f(\alpha, \delta)$ and $h(\alpha) = f(\beta_1, \alpha)[f(\beta_1, \delta)]^{-1}$, whence $f(\alpha, \beta) = g(\alpha)h(\beta) - g(\beta)h(\alpha)$. Now

$$h\left(\sum_{j=1}^{k} a_j\beta_j - c_1\delta\right) = f\left(\beta_1, \sum_{j=1}^{k} a_j\beta_j\right)[f(\beta_1, \delta)]^{-1} - f(\beta_1, c_1\delta)[f(\beta_1, \delta)]^{-1}$$

$$= \left\{f\left(\beta_1, \sum_{j=1}^{k} a_j\beta_j\right) - c_1 f(\beta_1, \delta)\right\}[f(\beta_1, \delta)]^{-1} = 0$$

and

$$g\left(\sum_{j=1}^{k} a_j\beta_j - c_1\delta\right) = f\left(\sum_{j=1}^{k} a_j\beta_j - c_1\delta, \delta\right) = \sum_{j=1}^{k} a_j f(\beta_j, \delta).$$

But $\sum a_j f(\delta, \beta_j)^p = 0$, so $g(\sum_{j=1}^k a_j\beta_j - c_1\delta) = 0$. This implies the contradiction $f(\alpha, \sum_{j=1}^k a_j\beta_j - c_1\delta) = 0$ for all α, implying that col $\text{rank}_{\Phi_p}(A) = k$.

Now suppose G is arbitrary and $\{\sigma_{01}, \ldots, \sigma_{0n_0}, \sigma_{11}, \ldots, \sigma_{1,n_1-1}, \delta_1, \ldots, \sigma_{m1}, \ldots, \delta_m\}$ is a basis of G over Φ_p. Equation (1) shows that $\mathfrak{R}(\mathfrak{L}(G, \delta, f)) \subseteq \langle D(\sigma_{ij}, 0), \text{ad } x \mid x \in \mathfrak{L}(G, \delta, f)\rangle$ and for the special case $k = 1$ we have a matrix equation of the form:

$$\begin{bmatrix} \text{ad } u^p_{\sigma_{01}} \\ \vdots \\ \text{ad } u^p_{\sigma_{0n_0}} \\ \vdots \\ \text{ad } u^p_{\sigma_{m1}} \\ \vdots \\ \text{ad } u^p_{\delta_m} \end{bmatrix} = \begin{bmatrix} C_0 & & 0 \\ & \ddots & \\ 0 & & C_m \end{bmatrix} \begin{bmatrix} D(\sigma_{01}, 0) \\ \vdots \\ D(\sigma_{0n_0}, 0) \\ \vdots \\ D(\sigma_{m1}, 0) \\ \vdots \\ D(\sigma_{m,n_m-1}, 0) \end{bmatrix},$$

where $C_0 = (f(\sigma_{0i}, \sigma_{0j})^p)$ and, for $i > 0$, C_i is an $n_i \times (n_i - 1)$ matrix of the form of the matrix in Lemma 3. Denote this matrix by C. To determine the coefficient matrix of the $D(\sigma_{ij}, 0)$'s for higher powers of p, one merely raises the elements in C to the appropriate pth power.

Lemma 4. *Let $A = (a_{ij})$ be an $r \times s$ matrix over a field Φ of characteristic $p > 0$ and let $A_{p^t} = (a_{ij}^{p^t})$ for $t \geq 0$. If* col $\text{rank}_{\Phi_p} A = s$, *then* $\text{rank}_\Phi (A\, A_p \cdots A_{p^t})^T = s$ *for sufficiently large t (^{T} denotes transpose).*

Proof. Since $\text{rank}_\Phi (A \cdots A_{p^i})^T \leq \text{rank}_\Phi (A \cdots A_{p^{i+1}})^T \leq s$ for all i there exists some t such that $\text{rank}_\Phi (A \cdots A_{p^t})^T = \text{rank}_\Phi (A \cdots A_{p^{t+1}})^T$. If $\text{rank}_\Phi (A \cdots A_{p^t})^T < s$, then there exist $b_1, \ldots, b_s \in \Phi$, not all zero, such that $\Sigma_{j=1}^s b_j a_{ij}^{p^v} = 0$ for all i, $1 \leq i \leq r$, and all v, $0 \leq v \leq t$. Note that this, and the choice of t, implies $\Sigma_{i=1}^s b_j a_{ij}^{p^{t+1}} = 0$. Assume that the b's have been chosen so that the number of nonzero b_j is minimal. In addition, assume $b_1 = 1$. Then

$$0 = \left(\sum_{j=1}^s b_j a_{ij}^{p^v} \right)^p = \sum_{j=1}^s b_j^p a_{ij}^{p^{v+1}} .$$

On the other hand, since $\Sigma_{j=1}^s b_j a_{ij}^{p^{v+1}} = 0$ for $0 \leq v \leq t$ we have

$$\sum_{j=1}^s (b_j^p - b_j) a_{ij}^{p^{v+1}} = 0.$$

Extracting pth roots and using the minimality of the b's (recall $b_1 = 1$) gives $b_j^p - b_j = 0$ for all j, that is, $b_j \in \Phi_p$ for all j. This contradicts the assumption that col $\text{rank}_{\Phi_p} A = s$.

Returning to C, recall that the nondegeneracy of f guarantees that col $\text{rank}_{\Phi_p} C_0 = n_0$ and col $\text{rank}_{\Phi_p} C_i = n_i - 1$ for $i > 0$. Lemma 4 then allows us to conclude that

$$\langle D(\sigma_{ij}, 0), \text{ad}\, x \mid x \in \mathfrak{L}(G, \delta, f) \rangle \subseteq \mathfrak{R}(\mathfrak{L}(G, \delta, f)).$$

Inclusion in the other direction was illustrated earlier, completing the proof of the main theorem.

Theorem. *Let $\mathfrak{L}(G, \delta, f)$ be a simple Lie algebra of Block. Then* $\dim \mathfrak{R}(\mathfrak{L}(G, \delta, f)) = \dim \mathfrak{L}(G, \delta, f) + n - 2m$.

Corollary. *Two algebras of Block of the same dimension, $\mathfrak{L}(G, \delta, f)$ and $\mathfrak{L}(G', \delta', f')$, are isomorphic only if $m(G) = m(G')$.*

From the preceding discussion it is evident that for $\mathfrak{L}(G, \delta, f)$ and $\mathfrak{L}(G', \delta', f')$ of the same dimension isomorphism is not guaranteed by the equality $m(G) = m(G')$. This follows from the fact that $\mathfrak{R}(\mathfrak{L}(G, \delta, f))$ need not be isomorphic to $\mathfrak{R}(\mathfrak{L}(G', \delta', f'))$. For example, let $m(G) = m(G') = 0$ and $n = 4$. Suppose $\{\beta_1, \beta_2, \beta_3, \beta_4\}$ and $\{\beta'_1, \beta'_2, \beta'_3, \beta'_4\}$ are bases for G and G', respectively, where the matrices $(f(\beta_i, \beta_j))$ and $(f(\beta'_i, \beta'_j))$ are

$$\begin{bmatrix} 0 & 1 & 0 & x \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -x & 0 & -1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & x & 0 & 1 \\ -x & 0 & 1 & 0 \\ 0 & -1 & 0 & -1/x \\ -1 & 0 & 1/x & 0 \end{bmatrix},$$

respectively, $x \notin \Phi_p$. In the first case, $\mathfrak{R}(\mathfrak{L}(G, \delta, f)) = \operatorname{ad} \mathfrak{L} + (\operatorname{ad} \mathfrak{L})^p$ while this is not true in the second.

BIBLIOGRAPHY

1. Richard Block, *New simple Lie algebras of prime characteristic*, Trans. Amer. Math. Soc. 89 (1958), 421–449. MR 20 #6446.

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