TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 208, 1975

WEIGHTED SHIFTS AND COVARIANCE ALGEBRAS([1])

BY

DONAL P. O'DONOVAN

ABSTRACT. The C^*-algebras generated by bilateral and unilateral shifts are studied in terms of certain covariance algebras. This enables one to obtain an answer to the question of when such shifts are G.C.R., or not, or even when they are N.G.C.R.. In addition these shifts are classified to within algebraic equivalence.

Introduction. This paper is concerned with certain types of bounded linear operators on separable Hilbert spaces. The types are the weighted shifts, both bilateral and unilateral. These operators have been studied quite extensively and have been found to contain examples of many different types of operator behaviour [4], [15], [17]. Among other results, necessary and sufficient conditions are given here for when such shifts are G.C.R. or type I (§3.4), for when the C^*-algebra that they generate contains the compact operators (§2.5, §3.2), and for when two shifts are algebraically equivalent (§2.4, §3.3). In order to answer these questions, it is necessary to obtain a useful description of the C^*-algebras they generate and of their irreducible representations. For this purpose covariance algebras are most appropriate [3], [9], [10], [23], [24].

In the first part of this paper the results on covariance algebras that are needed are presented. Many of these results are known; they appear chiefly in [24]. In the case of the group Z, some of the proofs are conceptually easier and it seemed worthwhile to present them. The principal new result here is Theorem 1.2.1, in which it is shown that a necessary and sufficient condition on a homeomorphism ϕ of a compact space X in order that every ideal in the covariance algebra $C^*(X, \phi)$ contain an element of $C(X)$, is that the periodic points be a "small" set. The theorem is in fact proven for a general $C^*(\mathfrak{A}, Z)$.

The C^*-algebras generated by weighted shifts with closed range are completely characterized in §§2.2 and 3.1 in terms of covariance algebras $C^*(X, \phi)$.

Received by the editors July 3, 1974.

AMS (MOS) subject classifications (1970). Primary 46L05; Secondary 46K10, 47B99, 47C10.

([1]) Much of the following material is contained in a dissertation written under the direction of W. B. Arveson and presented in partial fulfillment of the requirements of the Ph. D. degree at the University of California at Berkeley.

It is shown that the space X has a certain canonical form which for a given shift makes explicit all of its irreducible representations as weighted shifts. Also this canonical form of X classifies shifts to algebraic equivalence (§2.4, §3.3). This last term was introduced by W. B. Arveson [2] to describe two operators T and S for which the map $T \to S$ extends to a *-isomorphism of $C^*(T)$ onto $C^*(S)$. For normal operators this means they have the same spectrum, so for weighted shifts we have an "induced" version of this result.

If the above remarks seem to make little distinction between unilateral and bilateral shifts, this is because as is seen in Parts II and III, the differences are much less than might have been expected. In fact the type of analysis carried out here is almost equally applicable to all classes of centered operators [21].

The terminology and notation used are the standard ones [2], [8]. Thus, for example, $L(H)$ and $C(H)$ denote the bounded linear operators and compact linear operators on a Hilbert space H, $C^*(\{\ \})$ denotes the C^*-algebra generated by $\{\ \}$ and 1, $\mathfrak{A}'$ denotes the commutant of an algebra $\mathfrak{A}$, and H_π denotes the Hilbert space on which a representation π of some C^*-algebra acts. The order of presentation is:

PART I. COVARIANCE ALGEBRAS

1.1. **Representations.** If χ is a *-automorphism of a C^*-algebra $\mathfrak{A}$, the semidirect product or covariance algebra $C^*(\mathfrak{A}, Z)$ is constructed as follows: Let $l^1(\mathfrak{A}, Z)$ be the set of all $\mathfrak{A}$-valued functions F on Z for which the norm

$\|F\|_1 = \Sigma_{n=-\infty}^{\infty}\|F(n)\|$ is finite. $l^1(\mathfrak{A}, Z)$ is a Banach space in this norm and if a multiplication and an involution are defined by

$$(F_1 * F_2)(n) = \sum_k F_1(k)\chi^k(F_2(n-k))$$

and

$$F^*(n) = \chi^n(F(-n)^*),$$

then $l^1(\mathfrak{A}, Z)$ becomes a Banach algebra with approximate identity. Now $C^*(\mathfrak{A}, Z)$ is defined to be the enveloping C^*-algebra [8]. Thus, for $F \in l^1(\mathfrak{A}, Z)$ put

$$\|F\| = \sup_{\pi} \|\pi(F)\|,$$

where π ranges over all irreducible *-representations of $l^1(\mathfrak{A}, Z)$. One can show [9], [24] that $\|F\| = 0 \Rightarrow F = 0$ so $C^*(\mathfrak{A}, Z)$ is defined to be the completion of $l^1(\mathfrak{A}, Z)$ in this norm. More generally, if G is any locally compact group of *-automorphisms of $\mathfrak{A}$, then $C^*(\mathfrak{A}, G)$ can be constructed [9], [23], [24]. To every representation ρ of a covariance algebra $C^*(\mathfrak{A}, Z)$ corresponds a pair (π, U), where π is a representation of $\mathfrak{A}$, and U is a unitary operator on H_π with the property that $U\pi(A)U^{-1} = \pi(\chi(A))$ for all A in $\mathfrak{A}$. In fact if $F \in l^1(\mathfrak{A}, Z)$, then

$$\rho(F) = \sum_{n=-\infty}^{\infty} \pi(F(n))U^n. \tag{1.1}$$

We shall express this relationship by writing $\rho = (\pi, U)$.

Let $\hat{A}$ and $\widehat{A}^\frown$ denote the spectrum, or dual, and the quasi-spectrum of $\mathfrak{A}$ respectively [8]. $\hat{A}$ can be naturally embedded in $\widehat{A}^\frown$, and $\widehat{A}^\frown$ can be endowed with the Mackey Borel structure and the Jacobson topology [8], [13].

Any representation π of $\mathfrak{A}$ has a central decomposition, $\pi = \int_{\widehat{A}}^{\oplus} c(x)\, d\mu(x)$ where μ is a standard Borel measure on $\widehat{A}^\frown$, $c(x)$ is a measurable cross section of the quotient map $\mathrm{Fac}(\mathfrak{A}) \to \widehat{A}^\frown$, and the center of $\pi(\mathfrak{A})$ consists of the diagonalisable operators M. These are the operators M_f (where f is in $\mathfrak{B}(\widehat{A}^\frown)$, the bounded Borel functions on $\widehat{A}^\frown$), defined on F in H_π by $(M_f F)(x) = f(x) \cdot F(x)$ [8].

Any *-automorphism χ of $\mathfrak{A}$ induces an obvious map ϕ of $\widehat{A}^\frown$ into $\widehat{A}^\frown$ which leaves $\hat{A}$ invariant. Further, it is immediate from their definitions that ϕ is an isomorphism for the Borel structure and a homeomorphism for the topology.

For a representation $\rho = (\pi, U)$ of $C^*(\mathfrak{A}, Z)$, let θ denote the inner automorphism of $\pi(\mathfrak{A})$ given by $A \to UAU^{-1}$. Then the center of $\pi(\mathfrak{A})'$, the commutant of $\pi(\mathfrak{A})$, is invariant under θ. As is shown in [13], this leads to the fact that

$$UM_fU^{-1} = M_{f\circ\phi}, \tag{1.2}$$

for each $M_f \in M$. This implies that μ is quasi-invariant with respect to ϕ i.e. $\{\phi^n \circ \mu\}_{n\in Z}$ are pairwise absolutely continuous, where $(\phi \circ \mu)(E) = \mu(\phi(E))$. So if $h = d(\phi \circ \mu)/d\mu$ is the Radon-Nikodym derivative, then defining U_ϕ on $F \in H_\pi$ by

$$(U_\phi F)(x) = \sqrt{h(x)}F(\phi(x)),$$

U_ϕ is unitary and has the property that $U_\phi M_f U_\phi^{-1} = M_{f\circ\phi}$. Thus $UU_\phi^{-1} \in M'$, so $U = B \cdot U_\phi$, where B is a decomposable operator.

If $\rho = (\pi, U)$ is an irreducible representation, then it follows immediately from 1.2 (since $\pi(\mathfrak{A})' = M$), that ϕ must be ergodic with respect to μ, i.e. if $E \subset A$ is measurable and $\phi(E) = E$, then $\mu(E) = 0$ or 1. Since $\phi(\hat{A}) = \hat{A}$, μ_π is based on either $\hat{A}$ or $\widehat{\mathfrak{A}} - \hat{A}$. The former is a necessary and sufficient condition that π be a type I representation [8, Proposition 8.4.8]. If so, then π has a unique decomposition $\pi = \pi_\infty \oplus \pi_1 \oplus \pi_2 \oplus \dots$, where π_i is a representation of multiplicity i, $1 \le i \le \chi_0$. In fact $\pi_i = \int_{B_i}^\oplus c(x)\, d\mu(x)$, where $B_i = \{x\colon\ c(x)$ is quasi-equivalent to some $i \cdot \nu$ with ν irreducible$\}$. Each B_i is clearly ϕ-invariant, so again by ergodicity if π is of type I, then π has uniform multiplicity. In fact, one can go further and conclude that μ is concentrated on some $\hat{A}_{n,i} = \{x\colon\ c(x) = i \cdot \nu$ and $\dim H_\nu = n\}$.

An ergodic quasi-invariant measure μ may have $\mu(0) = 1$ for some orbit 0 of ϕ. In this case the measure is said to be transitive. Otherwise it is called intransitive [20].

Suppose $\rho = (\pi, U)$ is an irreducible representation of $C^*(\mathfrak{A}, Z)$ for which μ_π is transitive, here this means purely atomic, based on the orbit of a_μ in $\widehat{\mathfrak{A}}$ say. If a_μ is not in $\hat{A}$, then one sees readily that ρ is not in fact irreducible. So a_μ is in $\hat{A}$, and then we have seen that $\pi = i \cdot \int_{\hat{A}_n} c(x) \cdot d\mu_\pi(x)$, and π is independent of the particular cross section $c(x)$ chosen [8]. If μ_π is purely atomic, then any cross section is measurable, and one can be chosen so that $U_\phi \pi(A) U_\phi^{-1} = \pi \circ \phi(A)$ for all $A \in \mathfrak{A}$. Then $U = BU_\phi$, where $B \in \pi(\mathfrak{A})'$. Further for such μ_π one sees that if π is not multiplicity free, then ρ is reducible.

To summarize, if $\rho = (\pi, U)$ is any irreducible representation of $C^*(\mathfrak{A}, Z)$ for which μ_π is transitive, then μ_π is based on $\hat{A}$, π is a multiplicity free representation, and $U = M_h \cdot U_\phi$, for some $h \in B(\hat{A})$, with $|h| \equiv 1$. If the point a_μ is not periodic, then the family of representations $\rho = (\pi, M_h U_\phi)$ are all possible and all unitarily equivalent. If a_μ is periodic of period k, then again the representations $(\pi, M_h U)$ are all possible and two such are unitarily equivalent if and only if

$$h_1(a_\mu) \cdot h_1(\phi(a_\mu)) \cdots h_1(\phi^{k-1}(a_\mu)) = h_2(a_\mu) \cdot h_2(\phi(a_\mu)) \cdots h_2(\phi^{k-1}(a_\mu)).$$

In general if μ_π is not transitive, then it appears that the relationship

$U_\phi M_f U_{\phi^{-1}} = U_{f\circ\phi}$ cannot always be lifted to give $U_\phi \pi U_\phi^{-1} = \pi \circ \chi$. However for the case of $\mathfrak{A}$ commutative this is of course automatic, so one has in this case that for every quasi-invariant ergodic measure μ on $\hat{A}$, $C^*(\mathfrak{A}, Z)$ has an irreducible representation on $L^2(\hat{A}, \mu)$.

While not much can be said about the intransitive irreducible representations, it is at least clear that one of them cannot be unitarily equivalent to a transitive representation. For if V: $(\pi_1, U_1) \to (\pi_2, U_2)$, then V must implement a unitary equivalence between π_1 and π_2 and thence between the centers of $\pi_1(\mathfrak{A})'$ and $\pi_2(\mathfrak{A})'$. This is clearly impossible if μ_1 is atomic and μ_2 is not.

1.2. **Ideals.** We now want to consider ideals in $C^*(\mathfrak{A}, Z)$. Since, as was remarked earlier, $l^1(\mathfrak{A}, Z) \subset C^*(\mathfrak{A}, Z)$, there is a natural injection i of $\mathfrak{A}$ into $C^*(\mathfrak{A}, Z)$, with $i(A)(n) = A\delta_{0,n}$, for $A \in \mathfrak{A}$. The specific question to be answered is: If I is an arbitrary nonempty selfadjoint ideal in $C^*(\mathfrak{A}, Z)$, under what conditions on $\mathfrak{A}$ and the action of Z on it can it be concluded that $I \cap \mathfrak{A} \neq \{0\}$?

If the induced action ϕ on $\hat{A}$ is free, i.e. no periodic points, then it follows from [24] that the above is true. We shall show the following. Let $H_i = \{x \in \hat{A}: \phi^i(x) = x\}$, $i = 1, 2, \ldots$.

Theorem 1.2.1. *For all nonempty ideals I in $C^*(\mathfrak{A}, Z)$, $I \cap \mathfrak{A} \neq \{0\}$ if and only if* interior $H_i = \emptyset$ *all i.*

Recall that the topology is that of Jacobson. An immediate consequence is the useful

Corollary. *If some nonperiodic point has a dense orbit in $\hat{A}$, then the property is true.*

Before embarking upon a proof of the theorem, we present a sequence of lemmas, at least some of which will be used elsewhere.

Definition. If ϕ is an ergodic quasi-invariant transformation on the finite measure space (X, μ), and $\mathfrak{F}$ is any second countable topology subordinate to the Borel structure, by $\mathrm{supp}_{\mathfrak{F}}\,\mu$ is meant the minimal closed ϕ invariant set whose complement has measure zero.

Alternatively $\mathrm{supp}_{\mathfrak{F}}\,\mu$ is the maximal set on which μ is diffuse, i.e. $\mu(A) > 0$ if A is nonempty and open. With this terminology, it is due to Halmos [14, p. 26] that

Lemma 1.2.1. *For almost every point in X, its orbit under ϕ is dense in* $\mathrm{supp}_{\mathfrak{F}}\,\mu$.

The following is also well known.

Lemma 1.2.2. *If μ is a standard measure on X, and P denotes the set of points periodic under ϕ, a quasi-invariant, ergodic Borel transformation, then $\mu(P) = 0$ or else μ consists of a finite number of atoms.*

Proof. Since μ is standard and ϕ is quasi-invariant, there exists N with $\mu(N) = 0$, and $X|N$ a standard Borel space invariant under ϕ. But then there is a second countable topology subordinate to the Borel structure on $X|N$ which is Hausdorff. Then the orbit of any periodic point is closed, and the conclusion follows for the previous lemma.

If $F \in l^1(\mathfrak{A}, Z)$, define $E_n(F) = F(n)$, $n \in Z$.

Lemma 1.2.3. *E_n is continuous in the C^*-norm on $l^1(\mathfrak{A}, Z)$ and $\|E_n\| \leq 1$.*

Proof. We recall that the C^*-norm on $l^1(\mathfrak{A}, Z)$ is $\|F\| = \sup_\rho \|\rho(F)\|$, where ρ is an irreducible representation of $l^1(\mathfrak{A}, Z)$. If $\nu \in \hat{A}$ is not periodic under ϕ, consider the transitive irreducible representation $\rho_\nu = (\pi, U_\phi)$ defined in §1. Then $H_\pi = \bigoplus_{-\infty}^{\infty} H_i$, with $H_i = H_\nu$ all i, and

$$\|\nu(F(n))\| = \sup_{\substack{\xi \in H_0; \eta \in H_n \\ \|\xi\| = \|\eta\| = 1}} |(\rho_\nu(F)\xi, \eta)|$$

so $\|\nu(F(n))\| \leq \|\rho_\nu(F)\|$.

If ν in $\hat{A}$ is periodic, of period k say, let ρ_ν^g denote the finite dimensional representation $\rho_\nu^g = (\pi, M_g U_\phi)$. If $n \equiv l \pmod{k}$ and if $\xi_i \in H_i$, $H_i = H_\nu$ all i, then

$$(\rho_\nu^g(F)\xi_0, \xi_l) = \left(\sum_{j=-\infty}^{\infty} \pi(F(j))(MgU_\phi)^j \xi_0, \xi_l\right)$$

$$= \sum_{-\infty}^{\infty} (\nu(F(l + jk))\eta_j, \xi_l) \cdot g(\nu) \cdot g(\phi(\nu)) \cdots g(\phi^l(\nu)) \cdot \lambda^j$$

where $\lambda = g(\nu) \cdots g(\phi^{k-1}(\nu))$ and $\eta_j = U_\phi^{l+jk} \xi_0$. But

$$\sup_{|\lambda|=1} \left| \sum_{j=-\infty}^{\infty} (\nu(F(l + jk))\eta_j, \xi_l) \cdot \lambda^j \right| \geq |(\nu(F(l + jk))\eta_j, \xi_l)| \quad \text{for all } j.$$

Hence given ξ, η, there exists g for which

$$|(\rho_\nu^g(F)\xi, \eta)| \geq |(\nu(F(n))U_\phi^n \xi, \eta)|.$$

Taking sups, it has been shown that for all $\nu \in \hat{A}$, there exists ρ_ν with $\|\rho_\nu(F)\| \geq \|\nu(F(n))\|$. Hence, since $\|E(n)\| = \sup_{\nu \in A} \|\nu(F(n))\|$, the result.

Corollary ([9], [24]). *If* $F \in l^1(\mathfrak{A}, Z)$, *then* $\|F\| = 0$ *implies* $F = 0$.

Now E_n is extended by continuity to $C^*(\mathfrak{A}, Z)$. If ρ is any irreducible representation of $C^*(\mathfrak{A}, Z)$ one can attempt to define E_n^ρ: $\rho(C^*(\mathfrak{A}, Z)) \rightarrow \rho(\mathfrak{A})$ by $E_n^\rho(\rho(F)) = \rho(E_n(F))$. For this we need

Lemma 1.2.4. *If* $\rho = (\pi, U)$ *is an irreducible representation for which* μ_π *does not consist of a finite number of atoms, then* E_n^ρ *is both well defined and continuous, for all* $n \in Z$.

Proof. We sketch the argument which is a standard one. With the notation of the last section, it follows from Lemma 1.2.2 that $\pi = \int_X^\oplus c(x)\, d\mu_\pi$, where ϕ is a freely acting Borel isomorphism of the standard Borel space X. Using the fact that the Borel structure is both countably generated and countably separated, given any integer N and $x \in X$, one can find a Borel neighborhood W_x, with $\mu(W_x) > 0$ and $\{\phi^i(W_x)\}_{i=-N}^N$ disjoint. Thence one easily obtains $\{V_i\}_{i \in Z}$ a disjoint, measurable covering of X subordinate to $\{W_x\}_{x \in X}$. Given $\epsilon > 0$, find $H \in l^1(\mathfrak{A}, Z)$ and $N \in Z$, with $H(n) = 0$ if $|n| > N$ and $\|F - H\| < \epsilon$. Then if $\xi \in H_\pi$, and χ_S denotes the characteristic function of the set S,

$$\begin{aligned} \|\pi(E_0(H))\xi\|^2 &= \int_X \|c(y)(E_0(H)) \cdot \xi(y)\|^2\, dy \\ &= \sum_i \int_{V_i} \|c(y)(E_0(H)) \cdot \xi(y)\|^2\, dy \\ &\leq \sum_i \|\rho(H)\|^2 \|M_{\chi_{V_i}} \xi\|^2 \qquad (1.3) \\ &= \|\rho(H)\|^2 \|\xi\|^2 \end{aligned}$$

where (1.3) follows from

$$\begin{aligned} \|\rho(H)M_{\chi_{W_x}} \xi\|^2 &= \int_X \left\| \left(\sum_{-N}^{N} \pi(E_n(H)) \cdot M_{\chi_{\phi^{-n}(W_x)}} U^n \xi \right)(y) \right\|^2 d\mu(y) \\ &\geq \int_{W_x} \left\| \sum_{-N}^{N} c(y)(E_n(H)) M_{\chi_{\phi^{-n}(W_x)}}(y) \cdot (U^n \xi)(y) \right\|^2 d\mu(y). \end{aligned}$$

Thus $\|\pi(E_0(H))\| \leq \|\rho(H)\|$. But E_0 is continuous from the previous lemma, and ρ and π are continuous, and all are of norm ≤ 1, so $\|\pi(E_0(F))\| \leq \|\rho(F)\| + 2\epsilon$. Hence the lemma for $n = 0$. The general case $n = k$ is obtained by considering $\rho(H) \cdot U^{-k}$.

The converse to this last lemma is also true, as if μ_π does consist of a finite number of atoms it is easy to define $F \in l^1(\mathfrak{A}, Z)$, for which $\rho(F) = 0$ but $\rho(E_0(F)) \neq 0$.

The following might be considered as a sort of converse to Lemma 1.2.3. It is one of the keys to understanding the structure of the covariance algebra $C^*(\mathfrak{A}, Z)$.

Lemma 1.2.5. *For $F \in C^*(\mathfrak{A}, Z)$, if $E_n(F) = 0$ for all n, then $F = 0$.*

Proof. What must be shown is that if $\rho(E_n(F)) = 0$ $\forall n, \forall \rho$, then $\rho(F) = 0$ $\forall \rho$. If $\rho = (\pi, U)$ is any irreducible representation, then clearly $\rho_\lambda = (\pi, \lambda \cdot U)$ is also, for any $\lambda \in C$, with $|\lambda| = 1$. Choose ξ, η in H_ρ. For any $H \in l^1(\mathfrak{A}, Z)$, we have

$$(\rho(H)\xi, \eta) = \sum_{i=-\infty}^{\infty} (\pi(E_i(H)) \cdot U^i \xi, \eta).$$

So we can define h: $S^1 \to C$, by

$$h(\lambda) = (\rho_\lambda(H)\xi, \eta) = \sum_{i=-\infty}^{\infty} (\pi(E_i(H)) \cdot U^i \xi, \eta) \cdot \lambda^i = \sum_{i=-\infty}^{\infty} a_i \lambda^i .$$

Since $\Sigma |a_i| < \infty$, $h(\lambda) \in L^2(S^1)$. If $H_k \to F$ in $C^*(\mathfrak{A}, Z)$, then from the definition of the norm in $C^*(\mathfrak{A}, Z)$ we have that

$$h_k(\lambda) \to f(\lambda) = (\rho_\lambda(F)\xi, \eta) \quad \text{uniformly in } \lambda.$$

Hence $h_k(\lambda) \to f(\lambda)$ in $L^2(S^1)$. But if $h_k(\lambda) = \Sigma_{i=-\infty}^{\infty} a_{ik}\lambda^i$, and $f(\lambda) = \Sigma f_i \lambda^i$ then $a_{ik} \to f_i$. However $a_{ik} = (\pi(E_i(H_k))\xi, \eta)$, which by Lemma 1.2.3 converges to zero, each i. Hence $f_i = 0$ for all i, so $f(\lambda) = (\rho_\lambda(F)\xi, \eta) \equiv 0$. In particular $(\rho(F)\xi, \eta) = 0$. Since ξ, η and ρ are all arbitrary, we conclude that $F = 0$.

It is an immediate consequence of the last two lemmas that

Corollary. *If $\rho = (\pi, U)$ is an irreducible representation of $C^*(\mathfrak{A}, Z)$, then ρ is faithful if and only if μ_π is not periodic and π is faithful.*

Remark. From here, when we speak of the support of μ_π, we shall mean with respect to the Jacobson topology. That we may do so and that supp $\mu_\pi = \{\pi_0$: ker $\lambda_0 \supset$ ker $\pi\}$ is shown in [13]. Thus π is faithful if and only if supp $\mu_\pi = A$.

We turn now to the proof of the theorem. Let $H_i = \{a \in \hat{A}: \phi^i(a) = a\}$. Let $P = \bigcup_{i=1}^{\infty} H_i$. Let I be a selfadjoint ideal in $C^*(\mathfrak{A}, Z)$. We want to show that if interior $H_i = \emptyset$, all i, then $I \cap \mathfrak{A} \neq \{0\}$.

Any ideal I is uniquely determined as the kernel of a certain family of irreducible representations $\{\rho_\gamma\}_{\gamma \in S}$ of $C^*(\mathfrak{A}, Z)$. Let $\{\mu_\gamma\}_{\gamma \in S}$ be the corresponding measures on A. If $\operatorname{supp} I = \overline{\bigcup_{\gamma \in S} \operatorname{supp} \mu_\gamma} \neq \widehat{A}$, then from the definition of the topology, for some $T \neq 0$ in $\mathfrak{A}$, $\pi(T) \equiv 0$ for all π in $\operatorname{supp} I$. Thus $T \in I$.

So assume $\operatorname{supp} I = \widehat{A}$. Let $F \neq 0$ be in I. For all nonperiodic ρ_γ, $\rho_\gamma(F) = 0$ implies $\rho_\gamma(E_n(F)) = 0$ for all n by Lemma 1.2.4. Let $\mathfrak{N} = \{\gamma: \rho_\gamma$: is not periodic$\}$, and $Y = \overline{\bigcup_{\gamma \in \mathfrak{N}} \operatorname{supp} \mu_\gamma}$. From the above it follows that $\pi(E_n(F)) = 0$ for all π in Y and all n in Z. Thus if $F \neq 0$, it follows from Lemma 1.2.5 that for some π_0 in Y^c and integer n_0, $\pi_0(E_{n_0}(F)) \neq 0$. There are now two cases to be considered.

Suppose first that π_0 is periodic. Using the fact that I is an ideal it can be assumed that $\|\pi_0(E_0(F))\| = 2$ say. Since $\operatorname{supp} I = A$, it must be that $Y^c \subset \overline{P}$, and the hypothesis that the interior of H_i is empty for all i implies that either

(i) π_0 is the limit of a net $\{\pi_d\}_{d \in D}$ of nonperiodic points,

or that

(ii) it is the limit of a sequence of periodic points whose periods tend to infinity.

This makes use of the fact that ϕ is a homeomorphism in the Jacobson topology. Since Y^c is open and contained in $\overline{P}$, if (i) holds then $\{\pi_d\}_{d \in D}$ is ultimately in $\overline{P}$. But each π_d is not periodic, a net $\{\nu_l\} \subset P$ can be chosen with $\nu_l \to \pi_0$ and period $\nu_l \to \infty$. This is (ii).

Since $Y^c \subset \overline{\bigcup_{\gamma \in M \setminus \mathfrak{N}} \operatorname{supp} \mu_\gamma}$, if (ii) holds then the net $\{\pi_d\}$ can in fact be chosen so that the representations $\rho_d = (\pi_d, M_{g_d} U_\phi)$ are each in $\{\rho_\gamma\}_{\gamma \in M \setminus \mathfrak{N}}$ for some g_d.

On the other hand, if π_0 is not periodic then since $\pi_0 \in \overline{P}$ implies that for some collection $\{\pi_\alpha\} \subset P$, $\ker \pi \supset \bigcap \ker \pi_\alpha$, it follows that $\pi_0(E_0(F)) \neq 0$ implies $\pi_{\alpha_0}(E_0(F)) \neq 0$, some α_0, and we are in the first case above.

Choose $A \in l^1(\mathfrak{A}, Z)$ with $\|A - F\| < 1/4$, and choose $N_0 \geq 0$ with $\|A\|_1 \leq \Sigma_{-N}^N \|E_n(A)\| + 1/4$. If $\rho = (\pi, M_f U_\phi)$ is a transitive irreducible representation of $C^*(\mathfrak{A}, Z)$, based on the orbit of π_0, where π_0 has period k, and if $\lambda = f(\pi_0) \cdot f(\phi(\pi_0)) \cdots f(\phi^{k-1}(\pi_0))$, then for any $\xi, \eta \in H_{\pi_0}$ with $\|\xi\| = \|\eta\| = 1$,

$$\left|\left(\sum_{i=-\infty}^{\infty} \pi_0(E_{0+ik}(A)) \cdot \lambda^i, \xi, \eta\right)\right| = |(\rho(A)\xi, \eta)| \leq \|\rho(A)\|.$$

Thus

$$|(\pi_0(E_0(A))\xi, \eta)| \leq \|\rho(A)\| + \left|\sum_{i \neq 0} (\pi_0(E_{ik}(A))\xi, \eta)\right|.$$

So

$$\|\pi_0(E_0(A))\| \le \|\rho(A)\| + \sum_{i \neq 0} \|\pi_0(E_{ik}(A))\|$$
$$\le \|\rho(A)\| + 1/4 \quad \text{if } k > N_0.$$

Thus ultimately

$$\|\pi_d(E_0(A))\| \le \|\rho(A)\| + 1/4 \le 2$$

since

$$\|\rho_d(A)\| = \|\rho_d(A) - \rho_d(F)\| \le \|A - F\| = 1/4.$$

But then by upper continuity [8, Proposition 3.3.2], it follows that $\|\pi_0(E_0(A))\| \le 1/2$, thence $\|\pi_0(E_0(F))\| \le 3/4$, a contradiction. This is the desired result since it has been shown that if interior $H_i = \emptyset$ all i, then $\operatorname{supp} I = \widehat{A}$ implies that $I = \{0\}$.

For the converse, suppose interior $H_{j_0} \neq \emptyset$, then some π_0 has an open neighborhood $W_{\pi_0} \subset H_{j_0}$. This means that for some $A \in \mathfrak{A}$, $\pi(A) = 0$ for all π in the complement of W_{π_0}, but $\pi_0(A) \neq 0$. For each a in $\hat{A}$, let $\rho_a = (\pi_a, U_\phi)$ denote the transitive irreducible representation based on its orbit. Since $\pi_a(A) = 0$ all a implies $A = 0$, it suffices to find $F \neq 0$ in $l^1(\mathfrak{A}, Z)$, with $\rho_a(F) = 0$ all a in $\hat{A}$.

If a has period k and $F \in l^1(\mathfrak{A}, Z)$, then certainly $\rho_a(F) = 0$ if $\sum_{n=-\infty}^{\infty} \phi^l(a)(F(i + nk)) = 0$, $0 \le i, l \le k - 1$. Define

$$F(n) = 0 \qquad \text{if } |n| > j_0^2 + 1,$$
$$F(n) = v_n A \quad \text{if } |n| < j_0^2 + 1,\ v_n \in R.$$

Then since every point in W_{π_0} has period at most j_0, it follows that any nontrivial solution for the v_n's to a system of at most $j_0(j_0 + 1)/2$ equations gives an F with the desired properties.

We may notice that Lemma 1.2.4 implies that every irreducible representation of a covariance algebra $C^*(\mathfrak{A}, Z)$ is in fact a covariance algebra; $C^*(\pi(\mathfrak{A}), Z_k)$ if π is periodic and $C^*(\pi(\mathfrak{A}), Z)$ otherwise. We also have

Theorem 1.2.2. *Let $\rho = (\pi, U)$ be any irreducible representation of $C^*(\mathfrak{A}, Z)$. If I is any nonzero selfadjoint ideal in $\rho(C^*(\mathfrak{A}, Z))$, then $I \cap \pi(\mathfrak{A}) \neq \{0\}$.*

Proof. If π is periodic, then $\rho(C^*(\mathfrak{A}, Z)) \approx \pi(\mathfrak{A}) \otimes M_k$, where $M_k = k \times k$ matrices, and the result is clear. If π is not periodic, then $\mu_\pi(P) = 0$, by Lemma 1.2.2, and μ_π is diffuse on $\operatorname{supp} \mu_\pi$, so interior $H_i' = \emptyset$ each i, where $H_i' = H_i \cap \operatorname{supp} \mu_\pi$. Thus the theorem applies.

1.3. Type I covariance algebras. For a separable C^*-algebra $\mathfrak{A}$, the following are equivalent [2], [8]:

(i) Every irreducible representation of $\mathfrak{A}$ contains the compact operators.

(ii) Any two irreducible representations of $\mathfrak{A}$ with the same kernel are unitarily equivalent.

(iii) Every factor representation of $\mathfrak{A}$ is of type I, i.e. $\pi(\mathfrak{A})''$ is a type I von Neumann algebra.

If any of these hold, $\mathfrak{A}$ is said to be G.C.R. or type I. Takesaki [23] and Zeller-Meier [24] have given conditions under which a general covariance algebra is G.C.R. Utilizing the idea of "induced representations", it is shown that $C^*(\mathfrak{A}, Z)$ is G.C.R. if and only if $\mathfrak{A}$ is G.C.R. and the action of ϕ on $\hat{A}$ is smooth or regular in the sense of Mackey [20]. Glimm [12] and Effros [11] give lists of conditions under which this is true. Since the group here is Z this result can be stated more simply than in general.

Definition. A point $a \in \hat{A}$ is said to be discrete in its orbit (under ϕ) if $\phi^{n_i}(a) \to a$ as $n_i \to +\infty$ or $-\infty$ implies a is periodic (with respect to ϕ).

Theorem 1.3.1. *If* $\mathfrak{A}$ *is G.C.R., then the following are equivalent:*

(i) $C^*(\mathfrak{A}, Z)$ *is G.C.R.*

(ii) *Every* a *in* $\hat{A}$ *is discrete in its orbit.*

(iii) *No two orbits have the same closure.*

(iv) *Every quasi-invariant ergodic measure on* $\hat{A}$ *is transitive.*

Proof. (i) $\Rightarrow$ (ii) If $C^*(\mathfrak{A}, Z)$ is G.C.R., if $a \in \hat{A}$ is not periodic, and if ρ_a is the transitive irreducible representation based on the orbit of a, then $\rho_a(C^*(\mathfrak{A}, Z)) \supset C(H_{\rho_a})$, so by Theorem 1.2.2, for some $A \in \mathfrak{A}$, $\rho_a(A)$ is a rank one projection. In general $\pi_1(A) \oplus \pi_2(A)$ can never be rank one unless $\pi_1(A)$ or $\pi_2(A)$ is zero. We conclude that $\rho_a(A)$ can be rank one only if $\phi^i(a)(A) = 0$ for all $i \neq i_0$. But if $\phi^{n_i}(a) \to a$ this is impossible. Hence (ii).

(ii) $\Rightarrow$ (iii) If (ii) holds and y_1 and y_2 contradict (iii), then $\phi^{n_i}(y_1) \to y_1$ some $\{n_i\}$ and $\phi^{m_i}(y_2) \to y_1$ some $\{m_i\}$. Then that $\phi^{l_i}(y_1) \to y_1$ some $\{l_i\}$, l_i distinct, is immediate.

(iii) $\Rightarrow$ (i) If (iii) holds then no two transitive representations have the same kernel unless they are unitarily equivalent. So if (i) does not hold then there exists an intransitive measure μ. But then it follows from Lemma 1.2.1 that there must be at least two distinct orbits which have supp μ as their closure.

(i) $\Leftrightarrow$ (iv) In the course of the last part, it was shown that (i) $\Rightarrow$ (iv). But from the description given in §1 of the factor representations of $C^*(\mathfrak{A}, Z)$ it follows that if every quasi-invariant ergodic measure is transitive, then every factor representation is the direct sum of irreducibles and hence of type I. Thus (iv) $\Rightarrow$ (i).

1.4. A representation theorem for certain operators with closed range. If $A \in L(H)$, as usual H separable, then among the many ways of expressing the fact that the range of A is closed are the following [7]:

(i) $\exists C > 0$ such that $\|A^*Ax\| \geq C\|x\|$, $\forall x \in (\ker A)^{\perp}$.

(ii) The origin is an isolated point of the spectrum of A^*A.

This last shows in particular that if A has closed range, then every representation of A does. If $A = UD$ is the polar decomposition [2], with U a partial isometry and D a positive operator, then by using (ii) to define a contour integral it is shown in [6] and [7] respectively that U^*U and U belong to $C^*(A)$. We shall also need the following. The proof is entirely straightforward so we omit it.

Lemma 1.4.1. *If π is a representation of a C^*-algebra $\mathfrak{A} \subset L(H)$, and if $A \in \mathfrak{A}$ has polar decomposition $A = UD$ and closed range, then $\pi(A)$ has polar decomposition $\pi(A) = \pi(U) \cdot \pi(D)$.*

Suppose $A = UD$ has closed range. Let ϕ and ϕ^{-1} denote the continuous linear maps of $C^*(A)$ into itself, defined for B in $C^*(A)$ by $\phi(B) = UBU^*$ and $\phi^{-1}(B) = U^*BU$.

Let $\mathfrak{D} = C^*(\{\phi^n(D), U^{(-n)}U^{(n)}\}_{n \in Z}) \subset C^*(A)$, where the notation is $U^{(n)} = U^n$ if $n \geq 0$ and $U^{(n)} = (U^*)^{-n}$ if $n \leq 0$. An alternative description of $\mathfrak{D}$ is that it is the minimal C^*-algebra containing D and 1 and invariant under both ϕ and ϕ^{-1}. Let I_U denote the closed selfadjoint ideal in $C^*(A)$ generated by $U^*U - UU^*$, and let q denote the canonical quotient map from $C^*(A)$ into $C^*(A)/I_U$. Then

Theorem 1.4.1. *If* (C) *holds, namely if for every finite collection $\{D_n\}_{n \in J} \subset \mathfrak{D}$, $\Sigma_{n \in J} D_n U^{(n)} \in I_U$ implies that each $D_n \in I_U$, then $q(C^*(A))$ is *-isomorphic to a covariance algebra $C^*(q(\mathfrak{D}), Z)$.*

Proof. Since $q(C^*(A))$ is a C^*-algebra, we can find a *-isomorphism π of $q(C^*(A))$ into some $L(H_\pi)$, and it may be assumed that π is nondegenerate [8]. For brevity, let B' denote the image of any $B \in C^*(A)$ under the representation $\pi \circ q$. Then by Lemma 1.4.1, we have that $A' = U'D'$ is the polar decomposition. Since $q(U)$ is normal, U' is also, so $N(A') = N(U') = N(U'^*) = N(A'^*)$, and the assumption of nondegeneracy implies that U' is unitary. If $\mathfrak{D}' = \pi \circ q(\mathfrak{D})$, then $\mathfrak{D}'$ is the minimal C^*-algebra containing 1 and D', and invariant under $\phi': B' \to U'B'U'^*$ and $\phi^{-1}: B' \to U'^*B'U'$. Thus $q(C^*(A))$ is *-isomorphic to $C^*(A')$, $q(\mathfrak{D})$ to $\mathfrak{D}'$, and ϕ' and ϕ'^{-1} are inverse *-automorphisms of $\mathfrak{D}'$ implemented by the unitary operator U'. Additionally it follows (C′) that $\Sigma_{n \in J} D'_n U'^n = 0$ implies $D'_n = 0$ each n in J. The primes will now be omitted.

Let B denote the *-subalgebra of $C^*(A)$ consisting of all elements of the form $\Sigma_{n \in J} D_n U^n$, for some finite subset J of Z, and each D_n belonging to $\mathfrak{D}$. Since $A = UD = (UDU^{-1}) \cdot U \in B$, B is clearly dense in $C^*(A)$. Let the action of Z on $\mathfrak{D}$ be given by the automorphism ϕ, and define χ: $B \to l^1(\mathfrak{D}, Z)$ by $\chi(\Sigma_{n \in J} D_n U^n)(k) = \delta_{k,J} D_k$ where $\delta_{k,J}$ is the Kronecker delta. This map is well defined since (C′) guarantees the uniqueness of the representation of elements of B. It is clearly linear and into $l^1(\mathfrak{D}, Z)$. The verifications that χ is multiplicative, *-preserving, and 1-1 are all routine. Defining χ^{-1} on the range of χ we see that it is continuous in the l^1 norm, so that χ^{-1} can be extended to a representation of $l^1(\mathfrak{D}, Z)$. But since this representation is faithful on $\mathfrak{D}$, the corollary to Lemma 1.2.5 gives the conclusion.

PART II. BILATERAL WEIGHT SHIFTS

2.1. Uniqueness of the basis. Let H be a separable Hilbert space. Then V in $L(H)$ is a bilateral weighted shift means that for some orthonormal basis $\{e_n\}_{n \in Z}$, $Ve_n = d_n e_n$, where d_n are complex scalars. It is well known that such an operator is reducible if and only if some $d_i = 0$ or $\{d_i\}$ is a periodic sequence, and that two such are unitarily equivalent if $|d_i| = |d'_i|$ each i [15], [18]. Thus we restrict ourselves to the case $d_i > 0$.

Theorem 2.1.1. *The basis $\{e_n\}$ with respect to which V has this shift form is unique if and only if V is irreducible.*

Proof. First if V is the ordinary unweighted bilateral shift, i.e. multiplication by z on $L^2(S^1)$, and if $\phi(z)$ is any inner function then $\{z^n \phi\}_{n \in Z}$ form an alternative basis. If V is periodic of period k one simply has to consider $\{z^n \phi(z^k)\}_{n \in Z}$. The converse can be shown by a direct argument or it can be considered a particular case of a more general situation. For this let (X, μ, ϕ), and (Y, ν, χ) be triples of (standard Borel space, finite measure, quasi-invariant Borel isomorphism). For $g > 0$ and $f > 0$ in $L^\infty(Y, \nu)$ and $L^\infty(X, \mu)$ respectively, let $M_g U_\chi$ and $M_f U_\phi$ denote the obvious weighted translation operators on $H_1 = L^2(Y, \nu)$ and $H_2 = L^2(X, \mu)$, respectively.

Lemma 2.1.1. *If $M_f U_\phi$ and $M_g U_\chi$ are irreducible and are unitarily equivalent via $V \in L(H_1, H_2)$ then there exists a nonsingular Borel isomorphism λ of (X, μ) onto (Y, ν) such that $\lambda \circ \phi = \chi \circ \lambda$ and $V = U_\lambda$.*

The theorem now follows if two distinct bases are considered as $L^2(Z, \mu_1)$ and $L^2(Z, \mu_2)$.

For the proof of the lemma, let $\mathfrak{D}_1 = C^*(\{M_{f\circ\phi^n}\}_{n\in Z})$ and $\mathfrak{D}_2 = C^*(\{M_{g\circ\chi^n}\}_{n\in Z})$. Let $\mathfrak{A}_1 = C^*(M_g U_\phi)$, and $\mathfrak{A}_2 = C^*(M_g U_\chi)$. Since the constant function $h \equiv 1$ must be cyclic for the irreducible $M_f U_\phi$, it follows that the C^*-algebra of functions in $\mathfrak{D}$ must have all of $L^2(X, \mu)$ as its L^2 closure. Since μ is finite and we have a C^*-algebra of functions in L^∞, one can readily conclude that every function h in L^2 can be approximated almost everywhere by a sequence h_n of L^∞ functions bounded in L^∞ norm. Then one obtains that $L_{h_n} \to L$ strongly. Thus the von Neumann algebra generated by $\mathfrak{D}_1$ is in fact all of $L^\infty(X, \mu)$. Similarly for $\mathfrak{D}_2$ and $L^\infty(Y, \nu)$. Since $M_f U_\phi$ and $M_g U_\chi$ are the respective polar decompositions, one has $V\mathfrak{D}_1 V^{-1} = \mathfrak{D}_2$, and thence by the above remarks $VL^\infty(X, \mu)V^{-1} = L^\infty(Y, \nu)$. Since we have standard measure algebras, the results of [16] imply that the isomorphism is implemented by a nonsingular point transformation λ. The remaining conclusions of the lemma follow easily.

We should like to thank L. G. Brown for discussions related to the above.

2.2. **The generated C^*-algebra and the canonical diagonal spectrum.** Let the bilateral weighted shift V have polar decomposition $V = UD$, where D is the diagonal operator $De_n = d_n e_n$, and U is the bilateral shift. Using the terminology of Theorem 1.4.1, $I_U = \{0\}$, and $\phi^n(D) = UDU^{-n}$, so $\mathfrak{D} = C^*(\{\phi^n(D)\}_{n\in Z})$ consists of diagonal operators. Let X denote the maximal ideal space or spectrum of the commutative C^*-algebra $\mathfrak{D}$. Let ϕ also denote the induced homeomorphism of X. Then since condition (C) clearly holds, we have

Theorem 2.2.1. *If V is a bilateral weighted shift with closed range, then $C^*(V)$ is *-isomorphic to the covariance algebra $C^*(X, \phi)$.*

If $\{V_\gamma = UD_\gamma\}_{\gamma\in\mu}$ is a collection of bilateral weighted shifts, we may assume that at least one V_{γ_0} has positive weights, so that $V_{\gamma_0} = UD_{\gamma_0}$ with U the bilateral shift and D diagonal is the polar decomposition. Let $\phi(A) = UAU^{-1}$ for $A \in L(H)$ and $\mathfrak{D} = C^*(\{\phi^n(D_\gamma)\}_{n\in Z, \gamma\in\mu})$. Denoting the spectrum of $\mathfrak{D}$ by X and the induced homeomorphism of X by ϕ also, in a manner entirely analogous to the last theorem, one obtains

Theorem 2.2.2. *If V_{γ_0} has closed range then $C^*(\{V_\gamma\}_{\gamma\in\mu})$ is *-isomorphic to $C^*(X, \phi)$.*

Returning to the case of one shift, a natural question is which pairs (X, ϕ) can arise under the correspondence in Theorem 2.2.1. If $n \in Z$, define w_n in $\mathrm{Hom}(\mathfrak{D}, C) = X$ by $w_n(A) = a_n$ if $A = \mathrm{diag}\{a_i\}_{i\in Z}$. Then $\phi^j(w_n)$

$= w_{n-j}$, and $\{w_n\}_{n\epsilon Z}$ is dense in X, since if $f \in C(X)$, with $f(w_n) = 0$, all n, then the inverse Gelfand transform of f is zero, and hence $f \equiv 0$. So the pair (X, ϕ) has the property that there exists an orbit of ϕ which is dense in X.

Let Y denote the spectrum of D. Define $T: X \to \Pi_{-\infty}^{+\infty} Y$, by

$$T(x) = (\ldots, x(\phi(D)), x(D), x(\phi^{-1}(D)), \ldots).$$

If $T(X)$ is given the topology induced by the product topology on ΠY, then since $\mathfrak{D}$ is generated as a C^*-algebra by $\{\phi^n(D)\}_{n\epsilon Z}$, T is both continuous and 1-1. Consequently since X is compact, it is a homeomorphism of X onto $T(X)$. $T(X)$ will be denoted by X_c and referred to as the *canonical form* of the *diagonal spectrum* X. Note that under T, ϕ becomes the usual right shift on a product space.

Theorem 2.2.3. *If ϕ is a homeomorphism of a compact Hausdorff space X, then there exists a bilateral weighted shift V, with $C^*(V)$ naturally *-isomorphic to $C^*(X, \phi)$ if and only if*

(i) *there exists a point $x_0 \in X$, with dense orbit under ϕ, and*

(ii) *there exists f in $C(X, R)$, such that $\{f \circ \phi^n\}_{n\epsilon Z}$ separates points in X.*

Proof. The necessity of (i) has been shown. For (ii), let $p_0 \in C(X_c, R)$ be the projection on the zero coordinate and pull back to X. Conversely, assume first that x_0 is not periodic under ϕ. If $f \in C(X, R)$ satisfies (ii), let $f'(x) = f(x) + 2\|f\|$. Let μ be a transitive ergodic measure on the orbit of x_0, and let $\rho_{x_0} = (\pi_\mu, U_\phi)$ be the corresponding representation of $C^*(X, \phi)$. By the corollary to Lemma 1.2.5, ρ is a faithful representation and if V is a bilateral weighted shift with weights $d_n = f'(\phi^n(x_0))$, then V is the image under ρ of an obvious $F \in C^*(X, \phi)$ and has positive weights and closed range, so an application of the Stone-Weierstrass theorem gives the desired conclusion. If x_0 is periodic, X consists of a finite number of points, say k, and any shift with nonzero weights of period k will clearly suffice.

From the proof it is immediate that

Corollary 1. *$C^*(X, \phi)$ is naturally *-isomorphic to a C^*-algebra generated by a family of bilateral weighted shifts if and only if ϕ has a dense orbit.*

We also have, using the real and imaginary parts as in the theorem,

Corollary 2. *If ϕ has a dense orbit, then $C^*(X, \phi)$ is *-isomorphic to a*

C^*-algebra generated by two bilateral weighted shifts if there exists f in $C(X, C)$ such that $\{f \circ \phi^n\}_{n \in Z}$ separates points in X.

This enables us to give an example of a C^*-algebra generated by a pair of shifts, which is not generated by a single one.

Example 1. With $S^1 = \{z: z \in C, |z| = 1\}$. Let $X = \Pi_{-\infty}^{\infty} S^1$ and ϕ be the usual shift. This pair clearly satisfies the conditions of Corollary 2, but if for $f \in C(X, R)$, $\{f \circ \phi^n\}_{n \in Z}$ separates points, then letting $T: X \to \Pi_{-\infty}^{\infty} f(X)$ be given by $T(x) = (\dots, f(\phi(x)), f(x), f(\phi^{-n}(x)), \dots)$ then T is a homeomorphism onto its range. Now $\chi = T \circ \phi \circ T^{-1}$ is the usual shift on $T(X) \subset \Pi_{-\infty}^{\infty} f(X)$, and the set of fixed points of ϕ must be homeomorphic to those of Y. But this gives T homeomorphic to $f(X) \subset R$, which is impossible.

Some examples of how the theorem itself applies follow:

Example 2. Let Y be any compact subset of the real line, and $X = \Pi_{-\infty}^{\infty} Y$ with the product topology. Let ϕ be the usual shift. Since, with respect to the natural product measure, ϕ is measure preserving and ergodic, by the lemma of Halmos quoted earlier, Lemma 1.2.1, almost every point has dense orbit, and clearly any coordinate function works.

Example 3. Let X be the n-torus, and let ϕ be an ergodic rotation of this topological group. Again from Lemma 1.2.1, this time with respect to Haar measure, almost every point has dense orbit. In fact, every point must have. Define $f \in C(X, R)$ by $f(z_1, z_2, \dots, z_m) = \mathrm{Re}(z_1 + z_2 + \dots + z_m)$. A simple argument using the fact that a Vandermonde matrix is invertible if and only if the elements are distinct [17] shows that $\{f \circ \phi^n\}_{n \in Z}$ separates points.

Returning to the canonical diagonal spectrum, recall that we had $T: X \to X_c$ by (2.2). So for the dense subset $\{w_n\}_{n \in Z}$ in X, that was defined previously, we have $T(w_k) = (\dots, d_{k-1}, d_k, d_{k+1}, \dots)$. (Recall $V = UD$, with $D = \mathrm{diag}\{d_i\}_{i=2}$.) Thus X_c is simply the closed subset of $\Pi_{-\infty}^{\infty}$(spectr. D) generated by D and its translates. And if χ is the usual shift, V is algebraically equivalent to the element F_V in $C^*(X_c, \chi)$ of the form

$$F_V(n) = p_0 \quad \text{if } n = 1, \qquad = 0 \quad \text{if } n \neq 1.$$

Thus at least when V is G.C.R. it is a simple matter using the representation theory described earlier to write down the irreducible representations of V.

Example. Let $V = UD$ with

$$D = \mathrm{diag}\{\dots, 1, 1, \dot{1}, 1, 2, 1, 1, 2, 2, 1, 1, 1, 2, 2, 2, \dots\}.$$

Then X_c is easily determined and one ascertains that V has, to within unitary equivalence, only the following distinct irreducible representations:

(i) the identity representation,

(ii) a representation as a shift with weights $\{\ldots, 1, 1, 1, 2, 2, 2, \ldots\}$,

(iii) as a shift with weights $\{\ldots, 2, 2, 2, 1, 1, 1, \ldots\}$, and

(iv) for each $\lambda \in \mathbf{C}$, a one dimensional representation as λ and 2λ.

2.3. Shifts without closed range. It was necessary that V have closed range in Theorem 2.2.1 in order that the diagonal algebra $\mathfrak{D}$ be a subalgebra of $C^*(V)$ and so that for all $B \in \mathfrak{D}$, we should have $B \cdot U^k$ in $C^*(V)$ for all $k \in Z$. In general if V does not have closed range, one can only say that $C^*(V)$ is *-isomorphic to some subalgebra of $C^*(X, \phi)$. For certain shifts, one can say more. If V is essentially (or almost) normal i.e. $V^*V - VV^* \in C(H)$, then except for $D = \lambda 1$, which we ignore, V is irreducible, so $C^*(V) \supset C(H)$ [2]. If $K(H)$ denotes the compact diagonal operators, then $\mathfrak{D} = C^*(D) + K(H) \subset C^*(V)$, and clearly $\mathfrak{D}^1 \cdot U^k \subset C^*(V)$ for $\mathfrak{D}^1 = \mathfrak{D} - \{\lambda I\}_{\lambda \in C}$, so the proof of Theorem 1.4.1 shows that $C^*(V)$ is *-isomorphic to $C^*(X_0, \phi)$, with X_0 only locally compact here. In particular, every representation ρ consists of a pair (π, L), and $\rho(V) = \pi(D) \cdot L$. For an arbitrary shift this is no longer true. In fact, one can show quite easily

Theorem 2.3.1. *$C^*(V = DU)$ has the property that for every irreducible representation ρ, $\rho(V) = \pi(D) \cdot L$ for some representation π of $\mathfrak{D}$ and unitary operator L if and only if*

(O) $\quad d_{n_i} \longrightarrow 0$ *implies* $d_{n_{i+k}} \longrightarrow 0$, *all* $k \in Z$.

One can go further and show that

Theorem 2.3.2. *$C^*(V)$ is a covariance algebra $C^*(X, \phi)$ if and only if V satisfies condition* (O).

Proof. The previous theorem shows the necessity. Let $\mathfrak{D}^\#$ be the subalgebra of $C(V)$ consisting of all diagonal operators. Let $A_n = (V^n V^{*n})^{1/2}$ and $B_n = (V^{*n}V^n)^{1/2}$, all $n \geq 1$. As always $\mathfrak{D} = C^*(\{\phi^n(D)\}_{n \in Z}) \approx C(X)$. If $w \in X$, then it follows from (O) that $w(\phi^n(D)) \neq 0$, all n. But then using the fact that A_n and B_n are in $\mathfrak{D}^\#$, the "functions" in $\mathfrak{D}^\#$ are seen to separate the points of X. $\mathfrak{D}^\#$ is a *-algebra, so the Stone-Weierstrass theorem gives $\mathfrak{D}^\# = \mathfrak{D}$. If $\mathfrak{D}^\#_n$ denotes $\{B \in \mathfrak{D}: B \cdot U^n \in C^*(V)\}$, then $\mathfrak{D}^\#_n$ contains the selfadjoint algebra $D \cdot \mathfrak{D}$, the elements of which also must separate the points of X. Thus $\mathfrak{D}^\#_n = \mathfrak{D}$, each n, and the conclusion of the theorem follows.

2.4. Algebraic equivalence. We turn now to the question of when two irreducible bilateral weighted shifts V_1 and V_2 are algebraically equivalent, i.e. when does there exist a faithful representation π of $C^*(V_1)$ with $\pi(V_1) = V_2$. The first lemma shows that we may assume that V_1, and hence necessarily V_2, have closed range.

Lemma 2.4.1. *Let $A \in L(H)$ have polar decomposition $A = UD$. If π is any representation of $C^*(A)$ for which $N(\pi(A)) = N(\pi(A^*)) = \{0\}$, then π has an extension to a representation of $C^*(A, U)$ on the same Hilbert space H_π.*

Proof. From [8, Proposition 2.10.2] there exists an extension π' of π to $C^*(A, U)$ such that $H_{\pi'} \supset H_\pi$ and $\pi'(B)|_{H_\pi} = \pi(B)$, for all B in $C^*(A)$. Then $\pi'(A)|_{H_\pi} = \pi'(U) \circ \pi'(D)|_{H_\pi}$ implies, since $\overline{\pi'(D)H_\pi} = H_\pi$ by hypothesis, that $\pi'(U)H_\pi \subset H_\pi$. Similarly considering $\pi'(A^*)$, one obtains $\pi'(U^*)H_\pi \subset H_\pi$. So π' is reduced by H_π and the corresponding subrepresentation is the desired one.

Returning to the irreducible weighted shifts V_1 and V_2, we may now assume closed range. Let X_{ic}, $i = 1, 2$, denote their respective canonical diagonal spectrum. Then

Theorem 2.4.1. *V_1 is algebraically equivalent to V_2 if and only if $X_{1c} = X_{2c}$.*

Proof. By Theorem 2.2.1, each $C^*(V_i)$ is naturally *-isomorphic to $C^*(X_{ic}, \chi)$ where χ is the usual shift on a product space, and under this *-isomorphism, V_i is carried to F_i in $C^*(X_{ic}, \chi)$, where $F_i(1) = p_0$, the zero coordinate projection and $F_i(n) = 0$ if $n \neq 1$. Hence the sufficiency.

If V_1 is algebraically equivalent to V_2, then $C^*(V_2)$ is the image of a faithful irreducible representation ρ of $C^*(X_{ic}, \chi)$ with $\rho(F_1) = V_2$. By §1.1, if $\rho = (\pi, L)$, then μ_π is transitive, and if it is based on the orbit of x_0, then $\rho(F_1)$ is a shift whose weights have as their absolute values the coordinates of x_0. But this sequence and its translates generates X_{2c}, so $X_{2c} \subset X_{1c}$, and symmetry reverses the inequality.

As previously remarked, if $X = \Pi_{-\infty}^{\infty}\{1, 2\}$, and χ is the usual shift, then with respect to the usual product measure, almost every point has dense orbit. In this sense

Corollary. *Almost all bilateral weighted shifts whose weights are 1 or 2 are algebraically equivalent.*

Remark. If the weights of a particular shift form an almost periodic function on Z [19], then the diagonal spectrum X is a topological group. There is a natural homeomorphism of X onto a subgroup of $\Pi_{-\infty}^{\infty} S^1$ and it is possible to formulate a condition for algebraic equivalence in terms of this subgroup.

2.5. **N.G.C.R. shifts.** Recall that a C^*-algebra is said to be N.G.C.R. if it has no C.C.R. ideal [2]. So if V is any irreducible operator, $C^*(V)$ is N.G.C.R. means simply that $C^*(V) \cap C(H) = \{0\}$. If V has closed range then $C^*(V) \approx C^*(X, \phi)$ and known conditions apply [24]. In fact, these conditions are derived in Part I. Actually the same condition applies whether or not the range is closed. Let $V = UD$, where $D = \text{diag}\{d_k\}_{k \in Z}$.

Theorem 2.5.1. *$C^*(V) \cap C(H) = \{0\}$ if and only if there exists $n_i \to \infty$ with $d_{n_i+k} \to d_k$, all $k \in Z$.*

Proof. We know that $C^*(V)$ is *-isomorphic to some subalgebra of $C^*(X, \phi)$, where X is the spectrum of $\mathfrak{D} = C^*(\{\phi^n(D)\}_{n \in Z})$ and $\phi(D) = UDU^*$. Let $\mathfrak{D}^\#$ be the C^*-subalgebra of $C^*(V)$ consisting of all diagonal operators. Let $X^\#$ denote its spectrum. We shall see that $\mathfrak{D}^\#$ is a "large enough" subalgebra of $\mathfrak{D}$, so that we can work with it.

Clearly, if the weights are periodic, $C^*(V) \cap C(H) = \{0\}$, so we may restrict ourselves to the case of V irreducible. Then $C^*(V) \cap C(H) \neq \{0\}$ implies $C^*(V) \cap C(H) \equiv C(H)$ [2]. Thus $\mathfrak{D}^\#$ is a representation π of $C(X^\#)$ and so from the well-known structure of these [2], $\mathfrak{D}^\#$ will contain rank one operators if and only if

(i) π is multiplicity free, and

(ii) μ_π has an isolated atom.

For all $k \in Z$, define w_k' in $X^\#$ by $w_k'(B) = b_n$ for $B = \text{diag}\{b_n\}_{n \in Z}$. Then $\{w_k'\}_{k \in Z}$ is dense in $X^\#$, and it is seen that the measure μ_π is purely atomic with atoms $\{w_k'\}_{k \in Z}$.

Put $A_n = \sqrt{V^n V^{*n}} = \phi(D) \cdot \phi^2(D) \cdots \phi^n(D)$, and $B_n = \sqrt{V^{*n} V^n} = D \cdot \phi^{-1}(D), \ldots, \phi^{-n+1}(D)$. Since these all belong to $\mathfrak{D}^\#$, by the continuing assumption of nonzero weights each w_k' has a *unique* extension to the obvious w_k in X. In particular $w_i' = w_j' \Rightarrow w_i = w_j \Rightarrow w^i(\phi^k(D)) = w^j(\phi^k(D))$, all k, $\Rightarrow D$ is periodic. We assumed otherwise, so $\{w_k'\}_{k \in Z}$ is distinct, i.e. π is multiplicity free. Thus we are reduced to considering whether w_0' is an isolated point of $X^\#$ or not. But again, since $w_k'(A_n) \neq 0$ and $w_k'(B_n) \neq 0$, a simple argument shows that $w_{n_i}' \to w_0'$ if and only if $w_{n_i} \to w_0$. Putting w_{n_i} in canonical form, this is exactly the condition of the theorem.

PART III. UNILATERAL WEIGHTED SHIFTS

3.1. **The generated C^*-algebra.** If H is a separable Hilbert space, an operator W is a unilateral weighted shift means that for some orthonormal basis $\{e_n\}_{n\in Z^+}$, $We_i = d_i e_{i+1}$, with $d_i \in C$. It is well known that such a W is irreducible if and only if each $d_i \neq 0$. We shall always assume this. Further, to unitary equivalence it may be assumed that $d_i > 0$ [15]. Then W has polar decomposition $W = S \cdot D$ where $D = \text{diag}\{d_n\}_{n\in Z^+}$ and S is the unilateral shift.

If W has closed range, i.e. $\{d_i\}_{i\in Z^+}$ is bounded away from zero or alternatively D is invertible, we can apply Theorem 1.4.1. With the notation introduced there, I_S is generated by $S^*S - SS^*$ a projection of rank one. Any ideal in an irreducible C^*-algebra is irreducible, so $I_S = C(H)$ [2]. Now

$$\mathfrak{D} = C^*(\{\phi^n(D), S^{*(n)}S^{(n)}\}_{n\in Z}) = C^*(\{\phi(D)\}_{n\in Z}),$$

since D has closed range, is commutative. Let X denote the spectrum of $\mathfrak{D}$. If q is the quotient map $L(H) \longrightarrow L(H)/C(H)$, then $q(\mathfrak{D})$ is also commutative. Denote its spectrum by $q(X)$, and call it the essential diagonal spectrum of W. Now ϕ induces an automorphism of $q(\mathfrak{D})$, and hence a homeomorphism, also denoted by ϕ, of $q(X)$. Condition (C) is easily verified so it follows that

Theorem 3.1.1. *If W is an irreducible unilateral weighted shift with closed range, then $C^*(W)/C(H)$ is *-isomorphic to the covariance algebra $C^*(q(X), \phi)$.*

It is not difficult to do as in §2.3 and extend this result to certain types of shifts which do not necessarily have closed range. We omit the details.

Since $C^*(W)$ contains the irreducible ideal $C(H)$, it is known [2], [8] that every representation is a direct sum of two subrepresentations ρ_1 and ρ_2 with $\rho_1(C(H)) \neq 0$, and $\rho_2(C(H)) = 0$. Then ρ_1 must be equivalent to a multiple of the identity representation, so with the representations ρ_2 of $C^*(q(X), \phi)$ having been described in Part I, quite a good description of the representations of $C^*(W)$ is possible; in particular, a complete description in case W is G.C.R. Additionally, the conditions given in §1.3 characterize which $C^*(W)/C(H)$ and thence $C^*(W)$ are G.C.R. We must postpone the corresponding characterization for those shifts without closed range until later.

As was done for the bilateral operators, since $\mathfrak{D}$ consists of diagonal operators, define $w_n \in X$, $n = 0, 1, 2$, by $w_n(B) = b_n$ if $B = \text{diag}\{b_n\}_{n\in Z^+}$. By the usual argument $\{w_n\}_{n\in Z^+}$ is dense in X.

Now $\phi\colon D \longrightarrow SDS^*$ and $\phi^{-1}\colon D \longrightarrow S^*DS$ are both continuous, linear, multiplicative maps of $\mathfrak{D}$ into itself, the former (1-1) and the latter onto. If the zero homomorphism is adjoined to X, then ϕ and ϕ^{-1} induce continuous maps of X into X, whose action on $\{w_n\}_{n \in Z^+}$ is given by

$$\phi(w_n) = w_{n-1} \quad \text{if } n > 0,$$
$$ = 0 \quad \text{if } n = 0$$

and

$$\phi^{-n}(w_0) = w_n \quad \text{if } n \geq 0.$$

Let $X_d = \{0, w_0, w_1, \ldots\}$ and $X_l = X - X_d$. If $y \in X_l$, then $y = \lim w_{n_i} = \lim \phi^{-n_i}(w_0)$, so $\phi^{-1} \circ \phi(y) = y$ since $\phi \circ \phi^{-1} = \mathrm{id}_X$, and ϕ is seen to be a homeomorphism of X_l. Of course X_l is simply the essential diagonal spectrum $q(X)$. For, if $x = \lim w_{n_i}$, and $K \in \mathfrak{D}$ is compact, then $x(K) = \lim w_{n_i}(K) = 0$. So $X_l \subset q(X)$. But $q(X) \subset X$ is always true and clearly no point of X_d, except 0, is in $q(X)$. So we have $X_l = q(X)$.

We again consider the canonical form of the diagonal spectrum. Thus if $x \in X$, let $T(X)$ be as in (2.2). Then T is a homeomorphism of X into $\Pi_{-\infty}^{\infty} Y$ where Y is the spectrum of D. In particular $T(w_0) = (\ldots, 0, 0, 0, \underline{d_0}, d_1, d_2, \ldots)$ and $T(w_1) = (\ldots, 0, 0, d_0, d_1, d_2, \ldots)$, etc. and $T(X) = \overline{\{T(w_n)\}}_{n \in Z} = T(X_d) + T(X_l)$, where + denotes disjoint union.

Theorem 3.1.2. *Let ϕ be a homeomorphism of a compact Hausdorff space Y. Then there exists a unilateral weighted shift W for which $C^*(W)/C(H)$ is naturally *-isomorphic to $C^*(Y, \phi)$ if and only if*

(i) $\exists f \in C(Y, R)$ with $\{f \circ \phi^n\}_{n \in Z}$ separating points.

(ii) If T denotes the natural isomorphism

$$T(x) \to (\ldots, f(\phi(x)), f(x), f(\phi^{-n}(x)), \ldots)$$

then $\exists \{d_i\}_{i \geq 0}$, bounded, $\subset R$, such that if $D = (\ldots, 0, 0, 0, d_0, d_1, d_2, \ldots)$ and χ is the backward shift, then $\overline{\{\chi^k(D)\}_{k \geq 0}} = T(Y) + \{\chi^k(D)\}_{k \geq 0}$.

Proof. The remarks preceding the theorem show the necessity. For sufficiency, if we choose $W = S \cdot (D + 2 \sup |d_i| \cdot 1)$, where S = unilateral shift, then the canonical form of X_l described above shows that $q(X) = X_l = T(Y)$, and hence that $C^*(W)/C(H)$ is indeed *-isomorphic to $C^*(Y, \phi)$.

It is unfortunate that as even very simple examples show it is necessary in the above to consider points outside Y. There is a simpler condition that is sufficient.

Corollary. *For sufficiency* (ii) *may be replaced by*

(ii)′ *for some $x \in Y$, either $\{\phi^n(x)\}_{n \in Z^+}$ or $\{\phi^{-n}(x)\}_{n \in Z^+}$ is dense in Y.*

Proof. Just let $d_i = f(\phi^{-i}(x))$, $i \geq 0$, in case $\{\phi^n(x)\}_{n \in Z^+}$ is dense.

Among the many pairs (Y, ϕ) to which the corollary applies are those of Examples 2 and 3 of Part II.

3.2. **N.G.C.R. shifts.** As was previously remarked, an irreducible operator W is N.G.C.R. if and only if $C^*(W) \cap C(H) = \{0\}$.

Theorem 3.2.1. *If $W = S \cdot D$ is an irreducible unilateral weighted shift, then W is N.G.C.R. if and only if there exists $n_i \to +\infty$ such that $d_{n_i+k} \to$ if $k \geq 0$ and $d_{n_i+k} \to 0$ if $k < 0$.*

Proof. After the description of the canonical diagonal spectrum given in §3.1, the theorem is proven in exactly the same manner that Theorem 2.5.1 was established.

The condition is clearly not a vacuous one, so the existence of such shifts is established. The result may also be expressed as follows.

Theorem 3.2.2. *If W is an irreducible unilateral weighted shift, $W = S \cdot D$, and $\mathfrak{D} = C^*(\{\phi^n(D)\}_{n \epsilon Z})$, then $C^*(W)$ is N.G.C.R. if and only if ϕ^{-1} is an isomorphism of $\mathfrak{D}$.*

Proof. kernel $\phi^{-1} = \{\text{diag}\{b, 0, 0, 0, \ldots\}\} \subset C(H)$, so ϕ^{-1} is 1-1 if and only if $\mathfrak{D} \cap C(H) = \{0\}$.

3.3. **Algebraic equivalence.** We want to consider when two irreducible unilateral weighted shifts are algebraically equivalent. Firstly

Theorem 3.3.1. *If W_1 is any irreducible unilateral weighted shift with closed range, then W_1 is algebraically equivalent to another shift W_2 if and only if $W_1^* W_1 = W_2^* W_2$.*

Proof. Notice that W_2 is not assumed to be irreducible. It is well known that W_1 is unitarily eqiuvalent to W_2 if and only if $W_1^* W_1 = W_2^* W_2$. Hence the sufficiency. For the necessity we must first show that W_2 is in fact necessarily irreducible. For this, suppose W_2 has polar decomposition $W_2 = S' \cdot D_2$. Since W_1 has closed range, the unilateral shift S belongs to $C^*(W)$, and it is a consequence of Lemma 1.4.1 that if π implements the algebraic equivalence, then $\pi(S) = S'$. Hence S' is an isometry. But W_2 is a unilateral weighted shift, so it must be that $S' = S$, i.e. W_2 is irreducible.

Now since $C^*(W_1) \supset C(H)$, and π is an irreducible representation, π must be unitarily implemented [2]. So the conclusion.

If W_1 is not of closed range, it is possible that W_2 be reducible.

Example 4. Let $W_1 = S \cdot D_1$ where

$$D_1 = \mathrm{diag}\{1, 1, 1/2, 1, 1, 1/3, 1, 1, 1/4, \ldots\}$$

then W_1 has a three dimensional representation as the operator with matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Thus W_1 has a faithful representation as the reducible shift with weights $\{1, 1, 0, 1, 1, \frac{1}{2}, \ldots\}$.

It is true for any irreducible operator that is not N.G.C.R., that every faithful irreducible representation is unitarily implemented. But we have seen that shifts with nonclosed range may be N.G.C.R. However, as has been true throughout most of this part, the unilateral case is not markedly different from the bilateral.

Theorem 3.3.2. *If W_1 and W_2 are irreducible unilateral weighted shifts, then W_1 and W_2 are algebraically equivalent if and only if $X_{1c} = X_{2c}$ (where X_{ic} denotes the canonical diagonal spectrum defined previously in* §3.1).

Proof. Let ρ be the representation implementing the algebraic equivalence. Extend ρ to a representation ρ' of $C^*(W, S)$ on some $H_{\pi'} \supset H_\pi$ in the usual way [8]. If ρ is not unitarily implemented, then certainly both W_1 and W_2 must be N.G.C.R. Let $\widetilde{W}_1$ denote W_1 with all the weights increased by one. Then ρ' is an irreducible representation of $C^*(\widetilde{W}_1) \approx C^*(\widetilde{X}_{1c}, \chi)$, for which $\rho'(W_1)$ is reducible and $\rho'(W_1)|_{H_\pi} = W_2$. This cannot occur unless ρ' is a transitive representation based on the orbit of a point in $\widetilde{X}_{1c}$ of the form $\{\ldots, a_{-2}, a_{-1}, 1, 1 + a_1, 1 + a_2, \ldots\}$ where $\{a_n\}_{n \epsilon Z^-}$ are irrelevant and W_2 has weights $\{a_n\}_{n\epsilon Z^+}$. But W_2 is N.G.C.R. so by Theorem 3.2.1 some sequence of translates converges to $\{\ldots, 1, 1, 1, 1 + a_1, 1 + a_2, 1 + a_3, \ldots\}$. This says that $X_{1c} \supset X_{2c}$, so by symmetry we are done.

3.4. **G.C.R. shifts.** For weighted shifts, either unilateral or bilateral, with nonzero weights and closed range, a characterization of those which are G.C.R. follows from §1.3. In [4], it is shown that any shift whose weights consist only of 0's and 1's is G.C.R. More generally, suppose $V = U \cdot D_1$ ($W = S \cdot D_2$) is a bilateral (unilateral) weighted shift. Let Y_1 (Y_2) be the subset of the diagonal spectrum X_1 (X_2) given by $Y_i = \{w \in X_i\colon$ if $w(\phi^k(D)) = 0$ then $w(\phi^{k+n}(D)) = 0$, all $n \geq 0$ *or* all $n \leq 0\}$.

Theorem 3.4.1. *V (resp. W) is G.C.R. if and only if every point of Y_1 (Y_2) is discrete in its orbit.*

Proof. Since every irreducible representation of $C^*(W)$ is either unitarily equivalent to the identity representation or else is a representation of $C^*(W)/C(H)$ (these are not mutually exclusive) the argument for the unilateral case reduces to the following for the bilateral case.

To every point of X_1 there corresponds the transitive irreducible representation of the covariance algebra $C^*(V, U)$ described in §1.1. If the point is in Y_1, then by restricting to $C^*(V)$ and possibly taking a subrepresentation, one obtains an irreducible representation ρ of V as a bilateral weighted shift or a unilateral weighted shift or the adjoint of the last. In any of these cases, if the point of U is not discrete in its orbit, then by Theorem 2.5.1 or Theorem 3.2.1 it follows that $\rho(C^*(V)) \cap C(H) = \{0\}$. Thus V is not G.C.R.

Conversely, suppose ρ is an irreducible representation of $C^*(V)$ for which $\rho(C^*(V)) \cap C(H_\rho) = \{0\}$. Since V is a centered operator, it follows that $\rho(V)$ is one [21]. If $\rho(V)$ and $\rho(V^*)$ have zero null space, then by Lemma 2.4.1, ρ extends to a representation ρ' of $C^*(V, U)$, which is a covariance algebra, also on H_ρ. If $\rho' = (\pi, L)$ has μ_π transitive, then $\rho(V)$ is an N.G.C.R. irreducible bilateral weighted shift, and so by Thoerem 2.5.1, there exists $y \in Y_1$, not discrete in its orbit. If μ_π is intransitive, then by Theorem 1.3.1 there exist points in X_1, in fact a set of nonzero measure of them, which are not discrete in their orbit. Then $N(\rho(V^*)) = \{0\}$ implies that some point in Y_1 has this property, i.e. $N((M_f L_F U_\phi)^*) = \{0\} \Rightarrow \mu_\pi(\{y: f(y) = 0\}) = 0$. Finally, if $\rho(V)$ is an irreducible unilateral shift or the adjoint of one, then an argument almost identical to that at the end of Theorem 3.3.2 gives the conclusion. By the decomposition of centered operators given in [21], the above are the only possibilities for $\rho(V)$.

REFERENCES

1. W.B.Arveson, *Analyticity in operator algebras*, Amer. J. Math. 89 (1967), 578–642. MR 36 #6946.

2. ———, *Representations of C^*-algebras*, Lecture Notes in Math., Springer-Verlag, New York (to appear).

3. W.B. Arveson and K.B. Josephson, *Operator algebras and measure preserving automorphisms*. II, J. Functional Analysis 4 (1969), 100–134. MR 40 #3322.

4. J.W. Bunce and J.A. Deddens, *C^*-algebras generated by weighted shifts*, Indiana Univ. Math. J. 23 (1973), 257–271.

5. L.A. Coburn, *The C^*-algebra generated by an isometry*, Bull. Amer. Math. Soc. 73 (1967), 722–726. MR 35 #4760.

6. L.A. Coburn and A. Lebow, *Algebraic theory of Fredholm operators*, J. Math. Mech. 15 (1966), 577–584. MR 33 #569.

7. H.O. Cordes, *On a class of C^*-algebras*, Math. Ann. 170 (1967), 283–313. MR 35 #749.

8. J. Dixmier, *Les C^*-algèbres et leurs représentations*, Cahiers scientifique, fasc. 29, Gauthier-Villars, Paris, 1964. MR 30 #1404.

9. S. Doplicher, D. Kastler and D.W. Robinson, *Covariance algebras in field theory and statistical mechanics*, Comm. Math. Phys. **3** (1966), 1–28, MR **34** #4930.

10. E.G. Effros and F. Hahn, *Locally compact transformation groups and C*-algebras*, Mem. Amer. Math. Soc. No. 75 (1967). MR **37** #2895.

11. E.G. Effros, *Transformation groups and C*-algebras*, Ann. of Math. (2) **81** (1965), 38–55. MR **30** #5175.

12. J.G. Glimm, *Locally compact transformation groups*, Trans. Amer. Math. Soc. **101** (1961), 124–138. MR **25** #146.

13. A. Guichardet, *Utilisation des sous-groupes distingués ouverts dans l'étude des représentations unitaires des groupes localement compacts*, Compositio Math. **17** (1965), 1–35. MR **32** #5787.

14. P.R. Halmos, *Lectures on ergodic theory*, Publ. Math. Soc. Japan, no. 3, Math. Soc. Japan, Tokyo, 1956; reprint, Chelsea, New York, 1960. MR **20** #3958; **22** #2677.

15. ———, *A Hilbert space problem book*, Van Nostrand, Princeton, N.J., 1967. MR **34** #8178.

16. P.R. Halmos and J. von Neumann, *Operator methods in classical mechanics*. II, Ann. of Math. (2) **43** (1942), 332–350. MR **4**, 14.

17. K. Hoffman and R. Kunze, *Linear algebra*, Prentice-Hall Math. Ser., Prentice-Hall, Englewood Cliffs, N.J., 1961. MR **23** #A3146.

18. R.L. Kelley, *Weighted shifts on Hilbert space*, Ph. D. Dissertation, University of Michigan, Ann Arbor, Mich., 1966.

19. L.H. Loomis, *An introduction to abstract harmonic analysis*, Van Nostrand, Princeton, N.J., 1953. MR **14**, 883.

20. G.W. Mackey, *Unitary representations of group extensions*. I, Acta Math. **99** (1958), 265–311. MR **20** #4789.

21. B.S. Morrel and P.S. Muhly, *Centered operators* (preprint).

22. S. K. Parrot, *Weighted translation operators*, Ph.D. Dissertation, University of Michigan, Ann Arbor, Mich., 1965.

23. M. Takesaki, *Covariant representations of C*-algebras and their locally compact transformation groups*, Acta Math. **119** (1967), 273–303.

24. G. Zeller-Meier, *Porduits croisés d'une C*-algèbre par un groupe d'automorphismes*, J. Math. Pures Appl. (9) **47** (1968), 101–239. MR **39** #3329.

DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK, STONY BROOK, NEW YORK 11794

Current address: Department of Mathematics, Dalhousie University, Halifax, Nova Scotia, Canada.