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ON SOME CLASSES OF MULTIVALENT STARLIKE FUNCTIONS

BY

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ABSTRACT. Classes of multivalent functions analogous to certain classes of univalent starlike functions are defined and studied. Estimates on coefficients and distortion are made, using a variety of techniques.

1. Let St denote the class of all functions $f(z) = z + \dots$ analytic, univalent and starlike in the unit disc U. Such functions satisfy the condition $\operatorname{Re}(zf'(z)/f(z)) > 0$, $z \in U$.

The problem of defining a corresponding class of multivalent starlike functions has been studied by several authors. Hummel [5] distinguishes six commonly used definitions, a typical one being $f(z)$ belongs to the class $S(p)$ if f has at most p zeros in U and

$$\limsup_{r\to 1} \min_{|z|=r} \operatorname{Re} \frac{zf'(z)}{f(z)} \geqslant 0. \tag{1.1}$$

In this note we will study three classes of multivalent starlike functions which are analogues of certain subclasses of St.

2. Let $S_1(p, \alpha)$, p a positive integer, $0 \leqslant \alpha \leqslant 2p$, denote the class of all functions $f(z) = a_0 + a_1 z + \dots$ analytic in U with precisely p zeros there such that

$$\limsup_{r\to 1} \min_{|z|=r} \operatorname{Re} \frac{zf'(z)}{f(z)} \geqslant \alpha. \tag{2.1}$$

$S_1(p, \alpha)$ is the generalization of the class $S(\alpha)$ of starlike functions of order α introduced by Robertson [11].

THEOREM 2.1. *Let $f(z)$ belong to the class $S_1(p, \alpha)$ and suppose f has zeros at $z_1, z_2, \dots, z_p$. Then $f(z)$ is p-valent in U and there is a function g in the class $S(\alpha/p)$ and a constant A such that*

$$f(z) = A \prod_{j=1}^{p} \Psi(z, z_j)\ [g(z)]^p, \tag{2.2}$$

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where

$$\Psi(z, z_j) = (z - z_j)\,(1 - z_j^* z)/z.$$

(*The asterisk denotes conjugation.*)

PROOF. Since inequality (2.1) implies inequality (1.1), it follows from [5] that $f(z)$ is p-valent and that there is a function $g \in \mathrm{St}$ such that (2.2) holds. We compute

$$\frac{zf'(z)}{f(z)} = \sum_{j=1}^{p} \frac{z\Psi'(z, z_j)}{\Psi(z, z_j)} + p\,\frac{zg'(z)}{g(z)}. \tag{2.3}$$

Since $\operatorname{Re}(z\Psi'/\Psi) = 0$ on $|z| = 1$, it follows from (2.3) and the definition of $S_1(p, \alpha)$ that lim sup min $\operatorname{Re}(zg'(z)/g(z)) \geqslant \alpha/p$ and the result follows from the maximum principle.

For any $f \in S(p)$, let $z_1, \ldots, z_p$ be the zeros of $f(z)$, let $r_i = |z_i|$, $R_M = \max\{r_i\}$, $R_m = \min\{r_i | r_i \neq 0\}$, and let $r = |z|$. We also assume that the constant A of Theorem 2.1 equals 1.

THEOREM 2.2. *Let $S_1(p, \alpha; z_1, \ldots, z_p)$ denote the subclass of $S_1(p, \alpha)$ of functions with zeros at $z_1, \ldots, z_p$. Then the extreme points of the closed convex hull of $S_1(p, \alpha; z_1, \ldots, z_p)$ are precisely the functions of the form*

$$f(z) = \prod_{j=1}^{p} \Psi(z_1 z_j)\, z^p (1 - xz)^{\alpha - 2p}, \qquad |x| = 1.$$

PROOF. It follows from (2.2) and the compactness of $S(\alpha/p)$ that $S_1(p, \alpha; z_1, \ldots, z_p)$ is compact. For $z_1, \ldots, z_p$ fixed the mapping

$$T\colon\ [g(z)]^p \longrightarrow \prod_{j=1}^{p} \Psi(z_1 z_j)[g(z)]^p$$

is a linear homomorphism; thus it suffices to find the extreme points of $\{[g(z)]^p | g \in S(\alpha/p)\}$. Since $p > 0$, the argument in [3] applies and we are done.

COROLLARY 2.3. *Let $f(z) = a_0 + \ldots \in S_1(p, \alpha; z_1, \ldots, z_p)$. Then*
(i) $|f(z)| \leqslant \Pi_{j=1}^{p}(r + r_j)\,(1 + r_j r)\,(1 - r)^{\alpha - 2p}$, $|z| < 1$,
(ii) $|f(z)| \geqslant \Pi_{j=1}^{p}(r - r_j)\,(1 - r_j r)\,(1 + r)^{\alpha - 2p}$, $|z| > R_M$,
(iii) $|f(z)| \geqslant \Pi_{j=1}^{p}(r_j - r)\,(1 - r_j r)\,(1 + r)^{\alpha - 2p}$, $|z| < R_m$,
(iv) $|a_n| \leqslant A_n$, *where A_n is the coefficient of z^n in*

$$F(z) = \prod_{j=1}^{p} (z + r_j)(1 + r_j z)(1 - z)^{\alpha - 2p},$$

(v) $|f^{(k)}(z)| \leqslant F^{(k)}(r)$, $k = 1, 2, \ldots.$

We note that it is possible to obtain sharp upper and lower bounds for $\mathrm{Re}(zf'(z)/f(z))$ using the estimates in [6], but we do not state them here.

The problem of determining $\max|a_n|$ when $|a_1|, \ldots, |a_p|$ are fixed was first studied for the class $S(p)$ by Goodman [4]. We are only able to obtain a partial result.

LEMMA 2.4. *Let* $f(z) = (z - z_0)(1 - \bar{z}_0 z) \cdot z^{-1}(z + \sum_{n=2}^{\infty} b_n z^n)^p$. *Then if* $f(z) = \sum_{n=p-1}^{\infty} a_n z^n$,

$$a_{p+1} = \left[pb_3 - \frac{p(p+1)}{2} b_2^2 + \frac{\bar{z}_0}{z_0} \right] a_{p-1} + pb_2 a_p. \tag{2.4}$$

PROOF. Let $\sum_1^{\infty} c_n z^n = (\sum_1^{\infty} b_n z^n)^p$, $b_1 = 1$. Then

$$f(z) = -z_0 c_p z^{p-1} + [-z_0 c_{p+1} + (1 + |z_0|^2) c_p] z^p$$
$$+ [-z_0 c_{p+2} + (1 + |z_0|^2) c_{p+1} - \bar{z}_0 c_p] z^{p+1} + \cdots.$$

Comparing coefficients, we have

$$a_{p-1} = -z_0 c_p, \quad a_p = -z_0 c_{p+1} + (1 + |z_0|^2) c_p,$$
$$a_{p+1} = -z_0 c_{p+2} + (1 + |z_0|^2) c_{p+1} - \bar{z}_0 c_p.$$

An easy calculation yields

$$a_{p+1} = \frac{c_{p+1}}{c_p} a_p + \left[\frac{c_{p+2}}{c_p} - \left(\frac{c_{p+1}}{c_p} \right)^2 + \frac{\bar{z}_0}{z_0} \right] a_{p-1}. \tag{2.5}$$

Since $c_p = 1$, $c_{p+1} = pb_2$, and $c_{p+2} = p(p-1) b_2^2/2 + pb_3$, substitution of (2.5) yields (2.4).

THEOREM 2.5. *Let* $f(z) = a_{p-1} z^{p-1} + a_p z^p + \ldots \in S_1(p, \alpha)$ *with* $p \geqslant 2$, $(2p^2 - 3p - (5p^2 - 4p)^{1/2})/2(p-1) \geqslant \alpha \geqslant 0$, a_p *real. Then*

$$|a_{p+1}| \leqslant [2(p-\alpha)^2 - (p-\alpha) - 1]\, |a_{p-1}| + (p-\alpha)\, |a_p|.$$

PROOF. It follows from Theorem 2.1 that $f(z)$ has a single real zero z_0 not at the origin and that $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ has real coefficients. From (2.4) we have

$$|a_{p+1}| \leqslant \left(1 + p \left| b_3 - \frac{(p+1)}{2} b_2^2 \right| \right) |a_{p-1}| + p|b_2|\, |a_p|.$$

For $\alpha/p \leqslant 1 - 1/p$, it follows from a result of Keogh and Merkes [7] that $|b_3 - (p + 1)b_2^2/2| \leqslant (1 - \alpha/p)(2p - 2\alpha - 1)$ and thus

$$1 + p\left(b_3 - \frac{p+1}{2} b_2^2\right) \geqslant 1 - (p - \alpha)(2p - 2\alpha - 1). \tag{2.6}$$

It remains to find an upper bound for the left-hand side of (2.6). It suffices to prove

$$(p - \alpha)(2p - 2\alpha - 1) - 1 + p\left(\frac{p+1}{2}\right)b_2^2 \geqslant 1 + pb_3,$$

which is certainly true if

$$(p - \alpha)(2p - 2\alpha - 1) \geqslant 2 + pb_3. \tag{2.7}$$

Since $|b_3| \leqslant (1 - \alpha/p)(3 - 2\alpha/p)$ [3], [11], (2.7) is certainly true if $(p - \alpha)(2p - 2\alpha - 1) \geqslant 2 + (p - \alpha)(3 - 2\alpha/p)$, or

$$\alpha \leqslant (2p^2 - 3p - (5p^2 - 4p)^{1/2})/2(p - 1).$$

This proves the theorem, since $|b_2| \leqslant (1 - \alpha/p)$ [3], [11].

3. Let $S_2(p, \alpha)$ denote the subclass of $S(p)$ consisting of all functions $f(z)$ for which

$$\limsup \min \operatorname{Re}\left[\alpha \frac{1 + zf''(z)}{f'(z)} + (1 - \alpha)\frac{zf'(z)}{f(z)}\right] \geqslant 0.$$

The class $S_2(p, \alpha)$ is the analog of the class $C(\alpha)$ of α-convex functions defined by Mocanu [10]. A short calculation gives

THEOREM 3.1. *Let $f(z) \in S_2(p, \alpha)$. Then $f(z) = \prod_{j=1}^{p} \Psi(z, z_j)\, [g(z)]^p$, where $g(z) \in C(\alpha/p)$.*

We cannot state an analogue of Theorem 2.2 since as yet the extreme points of $C(\beta)$ are not known for all β. However, we can obtain the results of Corollary 2.3.

THEOREM 3.2. *Let $f(z) = a_0 + \ldots \in S_2(p, \alpha)$, $\alpha > 0$. Then:*

(i) $$|f(z)| \leqslant \prod_{j=1}^{p} (r + r_j)(1 + r_j r)\, r^{-p}\left[\frac{p}{\alpha}\int_0^r (1 - t)^{-p/\alpha} t^{p/\alpha - 1}\, dt\right]^{\alpha}, \quad |z| < 1,$$

(ii) $$|f(z)| \geqslant \prod_{j=1}^{p} (r - r_j)(1 - r_j r)\, r^{-p}\left[\frac{p}{\alpha}\int_0^r (1 + t)^{-p/\alpha} t^{p/\alpha - 1}\, dt\right]^{\alpha}, \quad |z| > R_M,$$

(iii) $$|f(z)| \geqslant \prod_{j=1}^{p} (r_j - r)(1 - r_j r)\, r^{-p}\left[\frac{p}{\alpha}\int_0^r (1 + t)^{-p/\alpha} t^{p/\alpha - 1}\, dt\right]^{\alpha}, \quad |z| < R_m,$$

(iv) $|a_n| \leqslant A_n$; *where*

$$F(z) = \prod_{j=1}^{p} \Psi(z, -r_j)\left[\frac{p}{\alpha}\int_0^z (1-t)^{-p/\alpha} t^{p/\alpha - 1}\, dt\right]^{\alpha} = \sum_{n=0}^{\infty} A_n z^n,$$

(v) $|f^{(k)}(z)| \leqslant F^{(k)}(r)$, $k = 1, 2, \ldots$.

PROOF. Inequalities (i), (ii) and (iii) follow from Theorem 3.1 and the distortion theorem for α-convex functions (see Miller [9]). Statements (iv) and (v) follow from the recent result of P. Kulshrestha [8].

We mention without proof that a result similar to Theorem 2.5 holds for $S_2(p, \alpha)$ using the technique of Theorem 2.5 and a result of J. Syzmal [12].

4. Let $S_3(p, \alpha)$, $0 \leqslant \alpha \leqslant 1$, denote the subclass of $S(p)$ consisting of all functions f for which

$$\limsup \max |\arg(zf'(z)/f(z))| \leqslant \alpha\pi/2.$$

This extends the class $S^*(\alpha)$ of strongly starlike functions of order α defined by D. Brannan and W. Kirwan [2]. Note that a single valued branch of $\arg(zf'(z)/f(z))$ can be defined in some annulus $\rho < |z| < 1$.

THEOREM 4.1. *Let $f(z) \in S_3(p, \alpha)$. Then there is a function $g(z) \in S^*(\alpha)$ such that $f(z) = \Pi_{j=1}^{p} \Psi(z, z_j)\, g(z)^p$.*

PROOF. This follows from the equation (2.3) since $\mathrm{Re}(z\Psi'(z, z_j)/\Psi(z, z_j)) = 0$ on $|z| = 1$.

COROLLARY 4.2. *If $f \in S_3(p, \alpha)$, then f is bounded in U.*

PROOF. This follows from [2, Theorem 2.1] and the previous theorem.

LEMMA 4.3. *Let $g(z) = z + b_2 z^2 + \ldots \in S^*(\alpha)$. Then if either $\lambda \geqslant 3/4$ or $3/4 - 1/4\alpha \geqslant \lambda$, $|b_3 - \lambda_2^2| \leqslant |3\alpha^2 - 4\lambda\alpha^2|$, and this result is sharp.*

PROOF. Using the notation of [1, Theorem 2.1], we have

$$b_3 - \lambda b_2^2 = \frac{\alpha}{2}\left[p_2 + \frac{3\alpha - 1 - 4\lambda\alpha}{2} p_1^2\right],$$

where $P(z) = 1 + p_1 z + p_2 z^2 + \ldots$ has $\mathrm{Re}\, P(z) > 0$ in U. We have

$$b_3 - \lambda b_2^2 = \frac{\alpha}{2} \cdot 2\int_0^{2\pi} e^{-2i\theta}\, d\mu(\theta) + \frac{\alpha}{2}\left(\frac{3\alpha - 1 - 4\lambda\alpha}{2}\right)\left(2\int_0^{2\pi} e^{-i\theta}\, d\mu(\theta)\right)^2,$$

where $\mu(\theta)$ is an increasing function on $[0, 2\pi]$ with $\mu(2\pi) - \mu(0) = 1$. Hence

$$\frac{2}{\alpha}\operatorname{Re}(b_3 - \lambda b_2^2) = 2\int_0^{2\pi}\cos\theta\, d\mu(\theta)$$

$$+ (6\alpha - 2 - 8\lambda\alpha)\left[\left(\int_0^{2\pi}\cos\theta\, d\mu(\theta)\right)^2 - \left(\int_0^{2\pi}\sin\theta\, d\mu(\theta)\right)^2\right].$$

Suppose first that $6\alpha - 2 - 8\lambda\alpha \geqslant 0$. Then

$$\frac{2}{\alpha}\operatorname{Re}(b_3 - \lambda b_2^2) \leqslant 2\int_0^{2\pi}\cos 2\theta\, d\mu(\theta) + (6\alpha - 2 - 8\lambda\alpha)\left(\int_0^{2\pi}\cos\theta\, d\mu(\theta)\right)^2$$

$$\leqslant 4\int_0^{2\pi}\cos^2\theta\, d\mu(\theta) - 2 + (6\alpha - 2 - 8\lambda\alpha)\int_0^{2\pi}\cos^2\theta\, d\mu(\theta)$$

$$\leqslant 6\alpha - 8\lambda\alpha,$$

where we have used Jensen's inequality and the estimate $\int_0^{2\pi}\cos^2\theta\, d\mu(\theta) \leqslant 1$.

The case $6\alpha - 2 - 8\lambda\alpha < 0$ is treated in a similar manner.

To show that the inequality is sharp, we consider the function $g(z)$ defined by

$$zg'(z)/g(z) = ((1 + z)/(1 - z))^\alpha \tag{4.1}$$

for which $a_2 = 2\alpha$, $b_3 = 3\alpha^2$.

THEOREM 4.4. *Let* $f(z) = a_{p-1}z^{p-1} + a_p z^p + \cdots \in S_3(p, \alpha)$ *and suppose each* a_n *is real. If* $p \geqslant 3$ *and* $\alpha \geqslant \min(1/3, (p^2 - 2p)^{-1/2})$,

$$|a_{p+1}| \leqslant (2p^2\alpha^2 - p\alpha^2 - 1)\,|a_{p-1}| + 2\alpha p|a_p|,$$

and this result is sharp.

PROOF. By Lemma 2.4, it is sufficient to show that

$$\left|1 + pb_3 - p\left(\frac{p+1}{2}\right)b_2^2\right| \leqslant 2p^2\alpha^2 - p\alpha^2 - 1,$$

since $|b_2| \leqslant 2\alpha$ [1]. By Lemma 2.3, with $\lambda = (p + 1)/2$,

$$1 + pb_3 - p((p + 1)/2)\, b_2^2 \geqslant 1 + p\alpha^2 - 2p^2\alpha^2$$

and hence it suffices to show

$$1 + pb_3 \leqslant 2p^2\alpha^2 - p\alpha^2 - 1. \tag{4.2}$$

Since $|b_3| \leqslant 3\alpha^2$ if $\alpha \geqslant 1/3$, (4.2) follows if $1 + 3p\alpha^2 \leqslant 2p^2\alpha^2 - p\alpha^2 - 1$, which is certainly true if $p \geqslant 3$.

The sharpness of the result follows from the fact that the function $g(z)$ defined by (4.1) simultaneously maximizes $|b_2|$ and $|b_3 - (p + 1)b_2^2/2|$.

Notes. 1. If $p \geqslant 4$, the result holds for all $\alpha \geqslant 1/3$.

2. For $0 < \alpha \leqslant 1/3$, a similar (but not sharp) result holds for $p > [\alpha^2 + \alpha + (17\alpha^4 + 2\alpha^3 + \alpha^2)]\ (4\alpha^2)^{-1}$.

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