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A CHARACTERIZATION OF MANIFOLDS

BY

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ABSTRACT. The purpose of this paper is (1) to give a proof of one general theorem characterizing certain manifolds and (2) to illustrate a technique which should be useful in proving various theorems analogous to the one proved here.

THEOREM. *Suppose that* $f: X \Rightarrow [0, 1]$, *where* X *is a compactum, and that* f *has the properties*:

(1) *for* $0 \leqslant x < 1/2$, $f^{-1}(x) = S^n \cong M_0$,

(2) $f^{-1}(1/2) \cong S^n$ *with a tame (or flat) k-sphere* S^k *shrunk to a point*,

(3) *for* $1/2 < x \leqslant 1$, $f^{-1}(x) \cong$ *a compact connected n-manifold* $M_1 \cong S^{n-(k+1)} \times S^{k+1}$ *(a spherical modification of* M_0 *of type k), and*

(4) *there is a continuum C in X such that (letting* $C_x = f^{-1}(x) \cap C$*)* (a) $0 \leqslant x < 1/2$, $C_x \cong S^k$, (b) $C_{1/2} = \{p\}$ *a point*, (c) *for* $1/2 < x \leqslant 1$, $C_x \cong S^{n-(k+1)}$, *and* (d) *each of* $f|(X - C), f|f^{-1}[0, 1/2)$, *and* $f|f^{-1}(1/2, 1]$ *is completely regular.*

Then X is homeomorphic to a differentiable $(n + 1)$*-manifold M whose boundary is the disjoint union of* $\overline{M}_0$ *and* $\overline{M}_1$ *where* $M_i = \overline{M}_i$, $i = 0, 1$.

1. **Introduction.** There are a number of interesting theorems in differential topology that characterize spheres, but their proofs use "smoothness" of both the manifold and the mapping. For example, the theorem of Reeb [15] and Milnor [10], later generalized by Milnor [11] and Rosen [16], is such a characterization.

THEOREM 1 (REEB-MILNOR-ROSEN). *Suppose that M is smooth* (C^∞) *compact manifold and that f is a smooth real-valued function on M with exactly two critical points (degenerate or not). Then M is homeomorphic to a sphere.*

This theorem has a topological version which we gave in [8]. It is as follows.

THEOREM 2 (McAULEY). *Suppose that M is a continuum (compact connected metric space) and that* $f: M \Rightarrow I = [0, 1]$ *is a (continuous) mapping.*

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Furthermore, $f^{-1}(0) = a$ *(point),* $f^{-1}(1) = b$ *(point),* $f|(M - \{a, b\})$ *is completely regular, and* $f^{-1}(x)$ *is homeomorphic to an n-sphere* S^n *for each* $x \in (0, 1)$. *Then M is homeomorphic to* S^{n+1}.

The condition that $f^{-1}(x)$ be an n-sphere S^n is quite natural in view of the following theorem from differential topology.

THEOREM 3. *If* $f: M \longrightarrow N$ *is a smooth mapping between smooth manifolds of dimensions m and n, respectively, where* $m \geqslant n$ *and if* $y \in N$ *is a regular value, then the set* $f^{-1}(y) \subseteq M$ *is a smooth manifold of dimension* $m - n$.

One wonders just what are the topological properties of differential mappings? Also, under what reasonable (topological) conditions is a mapping differential? In the case of nonconstant analytic mappings from the complex plane to the complex plane, the properties of *openness* and *lightness* actually characterize them. Whyburn [21] and Stoilow [18] have shown that if $f: M^2 \Rightarrow N^2$ is a light open mapping between 2-manifolds, then f is topologically equivalent to an analytic mapping. Several researchers, Church in particular, have made considerable progress in obtaining topological properties of differentiable mappings. For references and results, see [2].

At the topology conference held at the University of Oklahoma, March, 1972, I gave a talk containing outlines of proofs of theorems for which Theorems 2 and 4 (below) are special cases. The manuscript for that talk has appeared in the PROCEEDINGS, TOPOLOGY CONFERENCE, University of Oklahoma, 1972.

THEOREM 4 (McAULEY). *Suppose M is a continuum and that* $f: M \Rightarrow [0, 1]$ *is a mapping such that* (1) $f^{-1}(0) = a$ *(point),* (2) $f^{-1}(1) = b$ *(point),* (3) $f^{-1}(1/4) = f^{-1}(3/4) =$ *a figure eight (two circles with exactly one common point),* (4) *for* $0 < x < 1/4$ *or* $3/4 < x < 1$, $f^{-1}(x) \cong$ *a circle,* (5) *for* $1/4 < x < 3/4$, $f^{-1}(x) \cong$ *a pair of disjoint circles, and* (6) *for* $0 < x < 1$, *there is a "triangulation" of* $f^{-1}(x)$ *which contains exactly four* 1*-simplexes (simple arcs) and f is completely regular with respect to the collection of all* 1*-simplexes. Then* $M \cong$ *torus or Klein bottle.*

The purpose of this paper is to give a proof of one general theorem characterizing certain manifolds. Perhaps a more important objective is our *illustration* of the methods of proof which should be useful in proving various theorems of this kind.

THEOREM 5. *Suppose that X is a compact metric space and that* $f: X \Rightarrow [0, 1]$ *has the following properties*:

(1) *for* $0 \leqslant x < ½, f^{-1}(x) = S^n \cong M_0$,

(2) $f^{-1}(½)$ *is homeomorphic to* S^n *with a tame or flat k-sphere* S^k *shrunk to a point,*

(3) *for* $½ < x \leqslant 1, f^{-1}(x) \cong$ *a compact connected n-manifold* $M_1 \cong S^{n-(k+1)} \times S^{k+1}$ *which is a spherical modification of* M_0 *of type k (a regular neighborhood of* $S^k \subset M_0$*, i.e.,* $S^k \times I^{n-k}$*, is replaced by* $S^{n-(k+1)} \times I^{k+1}$*), and*

(4) *there is a continuum C in X such that (letting* $C_x = f^{-1}(x) \cap C$*)* (a) *for* $0 \leqslant x < ½$, $C_x \cong S^k$, (b) $C_{½} = p$*, a point–the topological critical point of f,* (c) *for* $½ < x \leqslant 1$, $C_x \cong S^{n-(k+1)}$, *and* (d) *each of* $f|(X - C)$, $f|f^{-1}[0, ½)$, *and* $f|f^{-1}(½, 1]$ *is completely regular.*

Then X is homeomorphic to a differentiable $(n + 1)$*-manifold M whose boundary is the disjoint union of* $\overline{M}_0$ *and* $\overline{M}_1$ *where* $M_i \cong \overline{M}_i$, $i = 0, 1$.

PROOF. Let $\overline{M}_0$ and $\overline{M}_1$ be differentiable manifolds homeomorphic to M_0 and M_1, respectively. There is a differentiable manifold M whose boundary is the disjoint union of $\overline{M}_0$ and $\overline{M}_1$ and a differentiable function g on M equal to 0 on $\overline{M}_0$, equal to 1 on $\overline{M}_1$, and otherwise having values between 0 and 1 and having *exactly* one nondegenerate critical point q (with critical value ½, say) with type number $k + 1$ [24]. Now, for $0 \leqslant x < ½$, $g^{-1}(x) \cong S^n \cong M_0 \cong \overline{M}_0$, $g^{-1}(½) \cong S^n$ with a k-sphere shrunk to a point, and for $½ < x \leqslant 1$, $g^{-1}(x) \cong M_1 \cong \overline{M}_1$. Furthermore, there is a "smooth" closed and connected set Z such that (1) for $0 \leqslant x < ½$, $Z_x = Z \cap g^{-1}(x) \cong S^k$, (2) $g^{-1}(½) \cap Z = q$, the critical point of g, (3) for $½ < x \leqslant 1$, $Z_x = Z \cap g^{-1}(x) \cong S^{n-(k+1)}$, and (4) Z is "canonical" in the sense of Wallace [24, p. 88]. Consider the trajectories to the level sets of g. The trajectories starting at points of $Z_0 \cong S^k$ all end at q. As we move through the levels of g from $\overline{M}_0$ to $\overline{M}_1$, the $Z_x \cong S^k$ shrink to q along the orthogonal trajectories. As we continue above the critical level, $g^{-1}(½)$, $Z_x \cong S^{n-(k+1)}$ grows along the orthogonal trajectories from q to $Z_1 \subset \overline{M}_1$. Thus, in this sense, Z is "canonical".

Clearly, Z is homeomorphic to C (in the hypothesis) and M is homeomorphic to $P = (S^n \times [0, ½]) \cup ((S^{n-(k+1)} \times S^{k+1}) \times [½, 1])$ where $(S^n, ½)$ and $(S^{n-(k+1)} \times S^{k+1}, ½)$ are sewed together in the obvious manner (indicated below). Thus, there is (I) a tame k-sphere S^k in S^n, (II) a tame $n - (k + 1)$ sphere $S^{n-(k+1)}$ in $(S^{n-(k+1)} \times S^{k+1})$, and (III) a continuous mapping m: $P \Rightarrow M$ such that (1) $m|(S^n, x)$, $0 \leqslant x < ½$, is a homeomorphism taking (S^n, x) onto $g^{-1}(x)$, (2) $m|(S^{n-(k+1)}, x)$, $½ < x \leqslant 1$, is a homeomorphism taking $(S^{n-(k+1)}, x)$ onto $g^{-1}(x)$, and (3) each of $m|(S^n, ½)$ and $m|(S^{n-(k+1)} \times S^{k+1}, ½)$ is a homeomorphism off $(S^k, ½)$ and $(S^{n-(k+1)}, ½)$, respectively, which takes $(S^n - S^k)$ and $(S^{n-(k+1)} \times S^{k+1} - S^{n-(k+1)})$ onto $g^{-1}(\frac{1}{2}) - \{q\}$ and takes $(S^k, \frac{1}{2})$ and $(S^{n-(k+1)}, \frac{1}{2})$ onto q. In the following, it is more convenient to work with P than with M.

Let h_1 denote a mapping of $S^k \times [0, \frac{1}{2}]$ into $f^{-1}[0, \frac{1}{2}]$ (actually, onto $f^{-1}[0, \frac{1}{2}] \cap C$) such that h_1 takes (S^k, t) homeomorphically onto $C_t = f^{-1}(t) \cap C$ for $0 \leqslant t < \frac{1}{2}$ and takes $(S^k, \frac{1}{2})$ onto $f^{-1}(\frac{1}{2}) \cap C = p$. Similarly, let h_2: $S^{n-(k+1)} \times [\frac{1}{2}, 1] \longrightarrow f^{-1}[\frac{1}{2}, 1]$ take $(S^{n-(k+1)}, t)$ homeomorphically onto $C_t = f^{-1}(t) \cap C$ for $\frac{1}{2} < t \leqslant 1$ and takes $(S^{n-(k+1)}, \frac{1}{2})$ onto $p = C_{\frac{1}{2}} = f^{-1}(\frac{1}{2}) \cap C$.

For $0 \leqslant t < \frac{1}{2}$, let K_t be the space of all homeomorphisms of S^n onto $f^{-1}(t)$ taking $x \in S^k$ onto $h_1(x, t)$. Similarly, let K_t be the space of all homeomorphisms of $S^{n-(k+1)} \times S^{k+1}$ onto $f^{-1}(t)$ taking $x \in S^{n-(k+1)}$ onto $h_2(x, t)$ for $\frac{1}{2} < t \leqslant 1$.

Let $K^0_{\frac{1}{2}}$ be the space of all mappings w of S^n onto $f^{-1}(\frac{1}{2})$ taking $x \in S^k$ to $h_1(x, \frac{1}{2}) = p$ such that $w|(s^n - S^k)$ is a homeomorphism. Similarly, let $K^1_{\frac{1}{2}}$ be the space of mappings w of $S^{n-(k+1)} \times S^{k+1}$ onto $f^{-1}(\frac{1}{2})$ taking $x \in S^{n-(k+1)}$ to $h_2(x, \frac{1}{2}) = p$ such that $w|\{S^{n-(k+1)} \times S^{k+1} - S^{n-(k+1)}\}$ is a homeomorphism.

We shall consider the collection L_0 of all K_t, $0 \leqslant t < \frac{1}{2}$ plus $K^0_{\frac{1}{2}}$ and the collection L_1 of all K_t, $\frac{1}{2} < t \leqslant 1$ plus $K^1_{\frac{1}{2}}$. Now, L_i^* will denote the union of the elements of L_i. Next, we define a metric for L_i^*. If $m \in L_i^*$, let $\hat{m}$ denote the graph of m in $P \times X$. Thus, for each pair $m, n \in L_i^*$ where $m \in K_a$ and $n \in K_b$, let $D(m, n) = H(\hat{m}, \hat{n})$ where H denotes the Hausdorff metric on the space of all closed subsets of $P \cap X$. Now, (L_i^*, D) is a topologically complete metric space. For a proof, see an argument in [7, Theorem 1] for an analogous result. We let ρ denote a complete metric for L_i^*.

LEMMA 1. *Each K_t and $K^i_{\frac{1}{2}}$, $i = 0, 1$, is LC^0 (in the homotopy sense). Indeed, each is locally contractible.*

PROOF. For $0 \leqslant t < \frac{1}{2}$, it should be clear that K_t is homeomorphic to the space of all homeomorphisms of S^n onto itself with a tame (or flat) k-sphere S^k fixed. Thus, by [4], it follows that K_t is locally contractible. Similarly, for $\frac{1}{2} < t \leqslant 1$, K_t is locally contractible. Now, $K^i_{\frac{1}{2}}$ is the space of all homeomorphisms of a compact polyhedron T onto itself keeping a point s fixed where T is the result of shrinking S^k (or $S^{n-(k+1)}$) in S^n (or $S^{n-(k+1)} \times S^{k+1}$) to a point. It follows from [22] that $K^i_{\frac{1}{2}}$ is locally contractible.

LEMMA 2. *The collections L_i, $i = 0, 1$, are equi-LC^n.*

PROOF. Each L_i^* is a complete metric space with metric ρ. Note that

$f|f^{-1}[0, \frac{1}{2})$ and $f|f^{-1}(\frac{1}{2}, 1]$ are completely regular in the sense of Dyer and Hamstrom [3]. It follows by an argument analogous to that given in [3] that the collection of all K_t is equi-LC^n for each n. To show that L_0 is equi-LC^n, we need only consider $\epsilon > 0$ and $g \in K^0_{\frac{1}{2}}$.

Since $K^0_{\frac{1}{2}}$ is LC^n, there is a $\delta_1 > 0$ such that each mapping r: $S^k \longrightarrow K^0_{\frac{1}{2}} \cap N_{\delta_1}(g)$, for $0 \leqslant k \leqslant n$, can be extended to a mapping R: $I^{k+1} \longrightarrow K^0_{\frac{1}{2}} \cap N_{\epsilon/2}(g)$. Sincc $f|(X - C)$ is completely regular, there is $\alpha > 0$ such that if $\frac{1}{2} - b < \alpha$, $b \in [0, \frac{1}{2}]$, there is a mapping m: $f^{-1}(b) \Rightarrow f^{-1}(\frac{1}{2})$ such that $m(C_b) = C_{\frac{1}{2}}$, $m|(f^{-1}(b) - C_b)$ is a homeomorphism, and m moves no point as much as $\delta_1/2$.

Choose δ, $0 < \delta < \min(\delta_1/2, \frac{1}{2})$ such that if $K_b \cap N_\delta(g) \neq \emptyset$, then $\frac{1}{2} - b < \alpha$. Now, let ϕ: $S^k \longrightarrow K_b \cap N_\delta(g)$. We wish to show that ϕ can be extended to ϕ: $I^{k+1} \longrightarrow K_b \cap N_\epsilon(g)$. Let $c = \frac{1}{2}$. We can define a 1-1 mapping H_{bc}: $K_b \longrightarrow K_c$ as follows: For $e \in K_b$, let $H_{bc}(e) = me \in K_c$. Clearly, $H_{bc}|(K_b \cap N_\delta(g))$ maps $K_b \cap N_\delta(g)$ into $K_c \cap N_{\delta_1}(g)$. In fact, H_{bc} maps K_b onto K_c. Furthermore, $r = [H_{bc}|\phi(S^k)]\phi$ maps S^k into $K_c \cap N_{\delta_1}(g)$ and can be extended to a mapping R: $I^{k+1} \longrightarrow K_c \cap N_{\epsilon/2}(g)$ such that for each $p \in I^{k+1}$, $R(p) \in H_{bc}(K_b) \subset K_c$ since $H_{bc}(K_b)$ is LC^n. Now, define H_{cb}: $H_{bc}(K_b) \longrightarrow K_b$ as $H_{cb}(me) = e$. Clearly, H_{cb} is the inverse of H_{bc} and H_{bc} is a homeomorphism. Now, $\Phi = [H_{cb}|H_{bc}(K_b) \cap N_{\epsilon/2}(g)]R$ maps I^{k+1} into $K_b \cap N(g)$ and agrees with ϕ on S^k the boundary of I^{k+1}. Thus, L_0 is equi-LC^n. Similarly, it follows that L_1 is equi-LC^n.

LEMMA 3. *The collections L_i are lower semicontinuous* (*lsc*) *in the sense that if* $\{x_i\} \longrightarrow x$ *in* $[0, \frac{1}{2}]$ *or* $[\frac{1}{2}, 1]$, *then* K_x *is in the closure of* $\bigcup K_{x_i}$.

A proof follows easily from the fact that each of $f|f^{-1}[0, \frac{1}{2})$, $f|f^{-1}(\frac{1}{2}, 1]$, and $f|(X - C)$ is completely regular.

Next, let F: $L^*_0 \Rightarrow [0, \frac{1}{2}]$ be the *function* defined by $F(k) = x$ iff $k \in K_x$. Thus, the collection of point inverses under F is the collection L_0 which is 1sc and equi-LC^n. Also, L^*_0 is a complete metric space. Given $x \in [0, \frac{1}{2}]$, let $\phi(x) \in K_x$. By Michael's section theorem [9], there is an open set U of $[0, \frac{1}{2}]$ with $x \in U$ and a continuous extension of ϕ to U (denote it by Φ) with the property that $\Phi(u) \in K_u$ for each $u \in U$. Clearly, $[0, \frac{1}{2}]$ is covered by a finite number of closed intervals $[a_i, b_i]$ where $a_0 = 0 < b_0 = a_1 < b_1 = a_2 < b_2 \ldots < b_t = \frac{1}{2}$ with mappings m_i: $S^n \times [a_i, b_i] \Rightarrow f^{-1}[a_i, b_i]$ where m_i is a homeomorphism for $i = 1, 2, \ldots, t - 1$ and m_t is a homeomorphism off $(S^k, \frac{1}{2})$ and takes $(S^k, \frac{1}{2})$ to p. Next, we sew the pieces together in the obvious way. Identify $h_i(x, a_i)$ with $h_{i+1}(x, a_i)$ for $i = 0, 1, \ldots, t - 1$. We obtain a mapping

H_0: $S^n \times [0, ½] \Rightarrow f^{-1}[0, ½]$ which is a homeomorphism except on $(S^k, ½)$. In a similar way, we obtain a mapping H_1: $(S^{n-(k+1)} \times S^{k+1}) \times [½, 1] \Rightarrow f^{-1}[½, 1]$ which is a homeomorphism except on $(S^{n-(k+1)}, ½)$ which maps to p. We sew these together to obtain a mapping H: $P \Rightarrow f^{-1}[0, 1] = X$ (recalling that $P = (S^n \times [0, ½]) \cup ((S^{n-(k+1)} \times S^{k-1}) \times [½, 1]))$ such that $h = Hm^{-1}$: $M \Rightarrow X$ is a homeomorphism (again, recalling that m: $P \Rightarrow M$ has certain properties). Consequently, X is homeomorphic to the differentiable $(n + 1)$-manifold M. Theorem 5 is proved.

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