a*-CLOSURES OF LATTICE-ORDERED GROUPS

BY

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ABSTRACT. A convex l-subgroup of an l-group G is *closed* if it contains the join of each of its subsets that has a join in G. An extension of G which preserves the lattice of closed convex l-subgroups of G is called an a*-*extension* of G. In this paper we consider a*-extensions and a*-closures of G.

We continue here the exploration of a*-extensions and a*-closures that was initiated in [4]. Some results in the abelian setting are sharpened, and many are carried over to nonabelian l-groups. The motivation for the study of a*-extensions of l-groups is that for a number of important embedding theorems for l-groups the target turns out to be the a*-closure. (See [4] and also [14].)

The first section of this paper is devoted to a*-extensions, in general, while in the second section the existence of a*-closures, relative to the variety of normal-valued l-groups, is established. The notion of a quasisummand is intro-duced in the third section, and the basic properties of quasisummands are developed. The fourth section treats a*-closures of cardinal sums, cardinal products, and lexicographic sums. The fifth section is devoted to an example which outlines some of the boundaries of the theory. In the sixth section we briefly consider some variations on the concept of an a*-extension. We conclude with a list of questions.

Throughout this paper G and H will denote lattice-ordered groups. Our terminology will be as in [4] and [10] (except where indicated). $C(G)$, $K(G)$, and $P(G)$ will denote, respectively, the lattice of convex l-subgroups of G, the lattice of closed convex l-subgroups of G, and the Boolean algebra of polars in G. If $C \in C(G)$ then $\bar{C} = \{g \in G| \ |g| \text{ is the join in } G \text{ of some subset of } C\}$ is the least closed convex l-subgroup of G containing C [7]. The join operation in $C(G)$ will be denoted by $\vee$. In order to minimize notational confusion no symbol will be used for the join operation in $K(G)$ (or in $P(G)$). Thus, if $K_1, K_2 \in K(G)$, then $K_1 \vee K_2$ is their join in $C(G)$; their join in $K(G)$ is $\overline{K_1 \vee K_2}$.

If G is an l-subgroup of H, and $C \in C(G)$, then we let $\widetilde{C} = \{h \in H| 0 \leqslant |h| \leqslant c \text{ for some } c \in C\}$; $\overline{\widetilde{C}}$ is thus the closure of $\widetilde{C}$ in H.

A *variety* of l-groups is a class of l-groups that is closed under the formation of l-subgroups, cardinal products, and l-homomorphic images. Some important

varieties of l-groups are the variety of abelian l-groups, the variety of representable l-groups, the variety of normal-valued l-groups, and the variety of all l-groups. (Each of these varieties is properly contained in the succeeding ones.)

Let G be an l-subgroup of H. H is an a^*-*extension* of G if $K \longmapsto K \cap G$ is a one-to-one map of $K(H)$ onto $K(G)$; the inverse map is $K \longmapsto \tilde{\tilde{K}}$. H is a^*-*closed*, *relative to a variety* V *of l-groups,* if $H \in V$ and H admits no proper a^*-extension in V. H is an a^*-*closure* for G, *relative to* V, if H is an a^*-extension of G, and H is a^*-closed, relative to V.

The reader is referred to [4] for further background; the notions of P-hull, SP-hull, orthocompletion, and lateral completion (which enter into §3 of this paper) are as formulated in [10].

1. a^*-extension of l-groups.

THEOREM 1.1 (BIGARD [2]). *G is archimedean if and only if $K(G) = P(G)$. Hence [4] G is archimedean if and only if $K(G)$ is a Boolean algebra.*

The next three propositions were established in [4] without employing commutativity.

PROPOSITION 1.2. *Suppose G is an l-subgroup of H. Then*
(i) *if $K \in K(G)$, then $\tilde{\tilde{K}} \cap G = K$, and*
(ii) *if $K \in K(H)$, then $(K \cap G)^{\tilde{}} \subseteq K$ and $(K \cap G)^{\tilde{}} \cap G = K \cap G$.*

PROPOSITION 1.3. *If $K \in K(G)$ and $A \in K(K)$, then $A \in K(G)$. Conversely if $A, K \in K(G)$, and $A \subseteq K$, then $A \in K(K)$. Thus if H is an a^*-extension of G and $K \in K(H)$, then K is an a^*-extension of $K \cap G$.*

PROPOSITION 1.4. *Suppose $K \in K(G)$, K is a normal subgroup of G, $A \in C(G)$, and $A \supseteq K$. If $A/K \in K(G/K)$, then $A \in K(G)$.*

LEMMA 1.5. *If $C \in C(G)$, then $\bar{C}$ is an a^*-extension of C.*

PROOF. Suppose $K \in K(\bar{C})$. Then $K \in K(G)$ by Proposition 1.3. Suppose $c = \bigvee_C c_i$ where $0 \leqslant c_i \in C \cap K$. Then since C is convex, $c = \bigvee_G c_i$ and hence $c \in K$. Thus $c \in K \cap C$, and hence $K \cap C \in K(C)$.

Now let $K_1, K_2 \in K(\bar{C})$ and suppose $K_1 \cap C = K_2 \cap C$. Let $0 < x \in K_1$. Then $x = \bigvee_G c_i$ where $0 \leqslant c_i \in C$. Thus $c_i \in K_1 \cap C = K_2 \cap C$. Since $x \in \bar{C}$ we have $x = \bigvee_{\bar{C}} c_i$, and hence $x \in K_2$. Thus $K_1 \subseteq K_2$, and similarly $K_2 \subseteq K_1$.

It follows now from Proposition 1.2 that $\bar{C}$ is an a^*-extension of C.

LEMMA 1.6. *If $C \in C(G)$ and $g > C$ for some $g \in G$, then $g > \bar{C}$.*

PROOF. If $0 \leqslant x \in \bar{C}$, then $x = \bigvee_G x_i$ where $x_i \in C$. Since g exceeds each x_i, we have $g > x$.

G is a *large l*-subgroup of H if $C \cap G \neq 0$ for all $0 \neq C \in C(H)$.

THEOREM 1.7. *Let G be an l-subgroup of H. If $K \cap G \neq 0$ for all $0 \neq K \in K(H)$, then G is large in H. (The converse is trivially true.)*

PROOF. Let $0 \neq C \in C(H)$. If C is not archimedean, then C has a bounded convex *l*-subgroup $D \neq 0$, and by Lemma 1.6 we can assume $D \in K(H)$. Thus $0 \neq D \cap G \subseteq C \cap G$.

Suppose now that C is archimedean. By Lemma 1.5 $\bar{C}$ is an a^*-extension of C, and hence by Theorem 1.1 $\bar{C}$ is archimedean. Let $0 \neq P \in P(\bar{C})$. Then $P \in K(\bar{C})$ and hence by Proposition 1.3 $P \in K(H)$. Thus $P \cap G \neq 0$ by hypothesis. Thus $P \cap (G \cap \bar{C}) \neq 0$, and hence by [9, Theorem 3.7] $G \cap \bar{C}$ is large in $\bar{C}$. $C \in C(\bar{C})$ and hence $C \cap (G \cap \bar{C}) \neq 0$. But clearly $C \cap G = C \cap (G \cap \bar{C})$. Thus $C \cap G \neq 0$ and the proposition is proved.

COROLLARY. *If H is an a^*-extension of G, then G is a large l-subgroup of H.*

THEOREM 1.8. *Suppose G is a large l-subgroup of H. If $\{a_i | i \in I\}$ is a subset of G and $g = \bigvee_G a_i$, then $g = \bigvee_H a_i$.*

PROOF. Suppose the conclusion is false. Then there exists $h \in H$ with $g > h \geqslant a_i$ for all $i \in I$. Since G is large in H, there exist $x \in G$ and an integer $n > 0$ such that $n(g - h) \geqslant x > 0$.

Now $-x + (n - 1)(g - h) + g \geqslant h \geqslant a_{i_1}$ for all $i_1 \in I$, and hence

$$(n - 1)(g - h) \geqslant x + a_{i_1} - g \quad \text{for all } i_1 \in I.$$

Thus $-(x + a_{i_1} - g) + (n - 2)(g - h) + g > h \geqslant a_{i_2}$ for all $i_1, i_2 \in I$, and hence $(n - 2)(g - h) \geqslant x + a_{i_1} - g + a_{i_2} - g$ for all $i_1, i_2 \in I$. Continuing in this way, we eventually arrive at

$$0 \geqslant x + a_{i_1} - g + a_{i_2} - g + \cdots + a_{i_n} - g \quad \text{for all } i_1, i_2, \ldots, i_n \in I.$$

We note that each of the terms in this last inequality is an element of G.

Now $-(x + a_{i_1} - g + \cdots + a_{i_{n-1}} - g) + g \geqslant a_{i_n}$ for all $i_1, \ldots, i_n \in I$, and hence $-(x + a_{i_1} - g + \cdots + a_{i_{n-1}} - g) + g \geqslant g$ for all $i_1, \ldots, i_{n-1} \in I$. Thus $0 \geqslant x + a_{i_1} - g + \cdots + a_{i_{n-1}} - g$, and hence

$$-(x + a_{i_1} - g + \cdots + a_{i_{n-2}} - g) + g \geqslant a_{i_{n-1}}, \quad \text{for all } i_1, \ldots, i_{n-1} \in I,$$

and thus $-(x + a_{i_1} - g + \cdots + a_{i_{n-2}} - g) + g \geqslant g$ for all $i_1, \ldots, i_{n-2} \in I$. Thus we have

$$0 \geqslant x + a_{i_1} - g + \cdots + a_{i_{n-2}} - g \quad \text{for all } i_1, \ldots, i_{n-2} \in I.$$

This process can be repeated until eventually we arrive at $0 \geqslant x + a_{i_1} - g$ for all

$i_1 \in I$, from which with one more repetition we get $0 \geqslant x$, a contradiction.

COROLLARY. *If G is a large l-subgroup of H, and $K \in K(H)$, then $K \cap G$* $\in K(G)$.

PROOF. Suppose $g \in G$ and $g = \bigvee_G g_\alpha$ where $g_\alpha \in K \cap G$. Then by Theorem 1.8 $g = \bigvee_H g_\alpha$, and hence $g \in K$. Thus $g \in K \cap G$, and hence $K \cap G$ $\in K(G)$.

THEOREM 1.9. *Let G be an l-subgroup of H. For $K \in K(H)$ let $K\tau = K \cap$* *G, and for $M \in K(G)$ let $M\delta = \tilde{M}$. The following are equivalent*:
    (i) *H is an a^*-extension of G.*
    (ii) *τ is one-to-one.*
    (iii) *δ maps $K(G)$ onto $K(H)$.*

PROOF. (ii) *implies* (i). By the corollaries to Theorems 1.7 and 1.8 we have $K \cap G \in K(G)$ for all $K \in K(H)$. Now by Proposition 1.2 we have (ii) implies (i).
    (i) *implies* (iii). Immediate.
    (iii) *implies* (ii). Suppose $K_1, K_2 \in K(H)$ and $K_1 \cap G = K_2 \cap G$. By (iii) there exist $M_1, M_2 \in K(G)$ such that $K_1 = \tilde{M}_1$ and $K_2 = \tilde{M}_2$. Thus by Proposition 1.2 we get $M_1 = M_2$, and hence $K_1 = K_2$.

2. **Existence of a^*-closures, relative to the variety of normal-valued l-groups.** The first four lemmas in this section are valid for all l-groups, and the fifth holds for all groups.

LEMMA 2.1. *Let F be an l-subgroup of G, and G an l-subgroup of H. Then H is an a^*-extension of F if and only if H is an a^*-extension of G and G is an a^*-extension of F.*

PROOF. Immediate from Theorem 1.9 and Proposition 1.2 (or see [4]).

LEMMA 2.2. *Let $\{H_\alpha\}$ be a chain of l-groups each of which is an l-subgroup of the members of the chain which contain it and each of which is an a^*-extension of G. Then $H = \bigcup H_\alpha$ is an a^*-extension of G.*

PROOF. In view of the results in the previous section, the proof of [4, Lemma 3.3] goes over to the present setting.

LEMMA 2.3. *Let $A \in K(G)$ and $B \in C(G)$. If A is a normal subgroup of B, then A is a normal subgroup of $\bar{B}$.*

PROOF. If $0 < g \in \bar{B}$, then $g = \bigvee_G b_i$ where $b_i \in B$. Thus since A is closed, we have (see [6]) $A + \bigvee b_i = \bigvee(A + b_i) = \bigvee(b_i + A) = \bigvee b_i + A$.

LEMMA 2.4. *Suppose H is an a*-extension of G, $A \in K(H)$, A is a normal subgroup of H, and $A \subseteq G$. Then H/A is an a*-extension of G/A.*

PROOF. Suppose K_1/A, $K_2/A \in K(H/A)$ and $K_1/A \cap G/A = K_2/A \cap G/A$. Then $K_1 \cap G = K_2 \cap G$ (by the isomorphism theorem for groups). Moreover, $K_1, K_2 \in K(H)$ by Proposition 1.4. Thus $K_1 = K_2$, and hence $K_1/A = K_2/A$. By Theorem 1.9 H/A is an a*-extension of G/A.

The following crucial lemma is due to Steve McCleary.

LEMMA 2.5. *Let $\{(K^\alpha, K_\alpha)|\alpha \in A\}$ be a set of pairs of subgroups of a group G such that $K_\alpha \subset K^\alpha$ and for each $0 \neq g \in G$ there exists $\alpha \in A$ with $g \in K^\alpha \backslash K_\alpha$. Then there exists a one-to-one function from G into the (set-theoretic) cartesian product $\Pi_{\alpha \in A} K^\alpha/K_\alpha$, where K^α/K_α is the set of left cosets of K_α in K^α.*

PROOF. For each $\alpha \in A$ and each coset X of K^α in G pick $x \in G$ such that $X = K^\alpha + x$. Then the map 0_X which takes $K_\alpha + y$ to $K_\alpha + y - x$ is a one-to-one map of the set of those cosets of K_α in G that are contained in X onto K^α/K_α. Now define $\psi: G \longrightarrow \Pi K^\alpha/K_\alpha$ by $g\psi = (-(K_\alpha + g)0_{K^\alpha + g}-)$. Suppose $g, h \in G$ and $g \neq h$. Then there exists $\alpha \in A$ such that $g - h \in K^\alpha \backslash K_\alpha$. Let $X = K^\alpha + g = K^\alpha + h$. $K_\alpha + g$ and $K_\alpha + h$ are cosets of K_α in G which are contained in X. Moreover, $K_\alpha + g \neq K_\alpha + h$. Hence

$$(K_\alpha + g)0_{K^\alpha + g} = (K_\alpha + g)0_X \neq (K_\alpha + h)0_X = (K_\alpha + h)0_{K^\alpha + h},$$

and so $g\psi \neq h\psi$. Thus ψ is one-to-one.

Let G be an *l*-group. Consider $0 < g \in G$, and let M be the intersection of all the maximal convex *l*-subgroups of $G(g)$. M is a normal subgroup of $G(g)$. Moreover, if $0 < a \in M$ then $a < g$. (For let K be a value of $g - a$ in $G(g)$. Then $K \subseteq N$ for some maximal convex *l*-subgroup N of $G(g)$, and $g - a + N = g + N > N$. Thus $g - a \notin N$ and hence $K = N$. Thus $g - a + K > K$ for each value K of $g - a$ in $G(g)$. Hence $g - a > 0$, and $a < g$.) By Lemma 1.6 we have now $x < g$ for all $x \in \bar{M}$; in particular, $\bar{M} \subseteq G(g)$ and $g \notin \bar{M}$. $\bar{M}$ is a normal subgroup of $G(g)$ since M is. Thus by Lemma 2.3 $\bar{M}$ is a normal subgroup of $\overline{G(g)}$.

LEMMA 2.6. *Let G be a normal-valued l-group. If $0 \neq g \in G$ there exists $A, B \in K(G)$ such that $g \in A \backslash B$, B is a normal subgroup of A, and A/B is archimedean.*

PROOF. We assume without loss of generality that $g > 0$. Let M be as in the preceding discussion. Let N be a maximal convex *l*-subgroup of $G(g)$. Since G is normal-valued, $G(g)$ is also, and hence N is a normal subgroup of $G(g)$.

If $0 < x \in \bar{M}$ then $nx < g$ for all positive integers n. Hence if N is a maximal convex *l*-subgroup of $G(g)$, then $n(N + x) = N + nx \leqslant N + g$ for all positive

integers n, and since $G(g)/N$ is archimedean, we conclude $x \in N$. Thus $x \in M$, and hence $M = \bar{M}$.

$G(g)/M$ is l-isomorphic to an l-subgroup of the product of the various $G(g)/N$. Hence $G(g)/M$ is archimedean. $\overline{G(g)}$ is an a^*-extension of $G(g)$ by Lemma 1.5. We have also $M \in K(G)$ and hence $M \in K(\overline{G(g)})$. By Lemma 2.4 $\overline{G(g)}/M$ is an a^*-extension of $G(g)/M$. Thus by Theorem 1.1 $\overline{G(g)}/M$ is archimedean. Now $A = \overline{G(g)}$ and $B = M$ are as desired.

THEOREM 2.7. *Let G be a normal-valued l-group. Then G has an a^*-closure, relative to the variety of normal-valued l-groups.*

PROOF. Let $\{(K^\alpha, K_\alpha)|\alpha \in A\}$ be the set of all pairs of closed convex l-subgroups of G such that K_α is a normal subgroup of K^α and K^α/K_α is archimedean. By Lemmas 2.5 and 2.6 there exists a one-to-one map of G into $\Pi_{\alpha \in A} K^\alpha/K_\alpha$. The remainder of the proof is identical to the proof of the corresponding theorem [4, Theorem 3.6] for the variety of abelian l-groups, and hence will be omitted.

COROLLARY [12]. *Each totally-ordered group has an a-closure.*

PROOF. For totally-ordered groups the notions of a-closure and a^*-closure coincide. Totally-ordered groups are normal-valued, and each a-extension of a totally-ordered group is totally-ordered.

3. Quasi-summands of l-groups.

PROPOSITION 3.1. $K(G)$ *is a complete Brouwerian lattice; the Boolean algebra of pseudocomplements in $K(G)$ is $P(G)$.*

PROOF. It was asserted in [4] that the proof of this result given there under the abelian hypothesis was valid for all l-groups. However, as pointed out to us by S. McCleary, some modification is actually needed in showing $K(G)$ is Brouwerian.

If $\{K_\alpha\}$ is a collection of elements in $K(G)$, then their join in $K(G)$ is $\overline{\bigvee K_\alpha}$, where $\bigvee K_\alpha$ is their join in $C(G)$. Let $K_\alpha, K \in K(G)$. Then $K \cap \overline{\bigvee K_\alpha} = \overline{K \cap \bigvee K_\alpha} = \overline{\bigvee (K \cap K_\alpha)}$, and hence $K(G)$ is Brouwerian. This argument, which uses the fact that $C(G)$ is Brouwerian, was suggested by Rick Ball.

$Q(G)$ will denote the collection of complemented elements in $K(G)$; that is, $Q \in Q(G)$ if and only if $Q \in K(G)$, $Q \cap Q' = 0$, and the join of Q and Q' in $K(G)$ is G. The elements of $Q(G)$ will be called *quasisummands* of G.

PROPOSITION 3.2. $Q(G)$ *is a sublattice of $K(G)$, and a subalgebra of $P(G)$.*

PROOF. This is a direct consequence of the distributivity of $K(G)$; in par-

ticular, $Q \in Q(G)$ implies $Q \in P(G)$ because complements are unique in a distributive lattice.

LEMMA 3.3. *If $Q \in Q(G)$ then Q is a normal subgroup of G.*

PROOF. Q is a normal subgroup of $Q \vee Q'$ since disjoint elements of G commute. Thus by Lemma 2.3 Q is a normal subgroup of $\overline{Q \vee Q'} = G$.

THEOREM 3.4. *Suppose $Q(G) = P(G)$. Then*
 (i) *each a^*-extension of G is representable, and*
 (ii) *G has an a^*-closure relative to the variety of all l-groups.*

PROOF. Suppose H is an a^*-extension of G. Then $Q(H) = P(H)$ and hence by Lemma 3.3 each polar in H is normal. Thus H is representable.

By Theorem 2.7 G has an a^*-closure A, relative to the variety of normal-valued l-groups. Let B be an l-group such that B is an a^*-extension of A. Then B is an a^*-extension of G, and thus B is representable (and hence normal-valued). Thus $A = B$, and the theorem is proved.

REMARK. The condition $Q(G) = P(G)$ can be restated in lattice terminology by saying that $K(G)$ is a *Stone lattice*. See [1, p. 130] for a brief discussion and further references on Stone lattices. Theorem 3.11 in this paper gives conditions on G which are equivalent to $Q(G) = P(G)$.

LEMMA 3.5. *Suppose that for each $\lambda \in \Lambda$ G_λ is an l-subgroup of H_λ. Let $G = \Sigma G_\lambda$ and $H = \Sigma H_\lambda$. Then H is an a^*-extension of G if and only if H_λ is an a^*-extension of G_λ for each $\lambda \in \Lambda$.*

PROOF. Suppose H is an a^*-extension of G. Since $H_\lambda \in K(H)$ we have by Proposition 1.3 that H_λ is an a^*-extension of $H_\lambda \cap G = G_\lambda$.

Conversely, suppose H_λ is an a^*-extension of G_λ for all $\lambda \in \Lambda$. If $K \in K(H)$, then $K = \Sigma K \cap H_\lambda = \Sigma K_\lambda$ where $K_\lambda = K \cap H_\lambda$. $K_\lambda \in K(H)$ and hence by Proposition 1.3 $K_\lambda \in K(H_\lambda)$. We have $K \cap G \in C(G)$, and thus $K \cap G = \Sigma K \cap G \cap G_\lambda = \Sigma K \cap G_\lambda = \Sigma K \cap H_\lambda \cap G_\lambda = \Sigma K_\lambda \cap G_\lambda$. If $L \in K(H)$ and $L \cap G = K \cap G$, then $\Sigma L_\lambda \cap G_\lambda = \Sigma K_\lambda \cap G_\lambda$ and hence $L_\lambda \cap G_\lambda = K_\lambda \cap G_\lambda$ for all $\lambda \in \Lambda$. Since H_λ is an a^*-extension of G_λ, we have $L_\lambda = K_\lambda$ (for all λ). Hence $L = \Sigma L_\lambda = \Sigma K_\lambda = K$, and by Theorem 1.9 H is an a^*-extension of G.

LEMMA 3.6. *Suppose $K \in K(G)$ and K is a normal subgroup of G. Let $v \colon G \rightarrow G/K \boxplus G/K'$ be the natural embedding. If $G/K \boxplus G/K'$ is an a^*-extension of Gv, then $(Kv)^{\widetilde{}} = 0 \boxplus G/K'$.*

PROOF. We have $Kv \subseteq 0 \boxplus G/K'$ and hence $(Kv)^{\widetilde{}} \subseteq 0 \boxplus G/K'$. Also $0 \boxplus G/K' = (Av)^{\widetilde{}}$ for some $A \in K(G)$. Since $(Kv)^{\widetilde{}} \subseteq (Av)^{\widetilde{}}$, we have $Kv \subseteq Av$ and

hence $K \subseteq A$. However, since $Av \subseteq 0 \boxplus G/K'$, we have $A \subseteq K$. Thus $A = K$, and $(Kv)^{\widetilde{}} = (Av)^{\widetilde{}} = 0 \boxplus G/K'$.

PROPOSITION 3.7. *For $Q \in K(G)$ the following are equivalent*:

(i) $Q \in \mathcal{Q}(G)$.

(ii) $\overline{Q \vee Q'} = G$.

(iii) *G is an a^*-extension of $Q \vee Q'$*.

(iv) *Q is a normal subgroup of G, and G/Q is an a^*-extension of $(Q' + Q)/Q$, and G/Q' is an a^*-extension of $(Q + Q')/Q'$*.

(v) *Q is a normal subgroup of G, and $G/Q \boxplus G/Q'$ is an a^*-extension of Gv where $v: G \longrightarrow G/Q \boxplus G/Q'$ is the natural embedding*.

PROOF. That (i) is equivalent to (ii), and (iii) implies (ii) are immediate. (ii) implies (iii) by Lemma 1.5. (i)–(iii) imply (iv) by Lemmas 3.3 and 2.4. (iv) implies (v) by Lemmas 3.5 and 2.1.

Finally we show (v) implies (ii). Let $B = \overline{Q \vee Q'}$. By Lemma 3.6 $(Qv)^{\widetilde{}} = 0 \boxplus G/Q'$ and $(Q'v)^{\widetilde{}} = G/Q \boxplus 0$. $(Bv)^{\widetilde{}}$ contains both $(Qv)^{\widetilde{}}$ and $(Q'v)^{\widetilde{}}$. Thus $(Bv)^{\widetilde{}} = G/Q \boxplus G/Q' = (Gv)^{\widetilde{}}$, and hence $B = G$.

$S(G)$ will denote the Boolean algebra of cardinal summands of G.

THEOREM 3.8. *Let V be a variety of l-groups and let $G \in V$.*

(i) *If G is a^*-closed relative to V, then $\mathcal{Q}(G) = S(G)$.*

(ii) *If $Q \in \mathcal{Q}(G)$ and Q is a^*-closed relative to V, then $Q \in S(G)$.*

PROOF. Let $Q \in \mathcal{Q}(G)$. By Proposition 3.7 $G/Q \boxplus G/Q'$ is an a^*-extension of Gv where v is the natural embedding. $(Qv)^{\widetilde{}} = 0 \boxplus G/Q'$ by Lemma 3.6, and $(Qv)^{\widetilde{}}$ is an a^*-extension of Qv by Proposition 1.3. Note $0 \boxplus G/Q', G/Q \boxplus G/Q' \in V$ since $G \in V$. If G is a^*-closed relative to V then $Gv = G/Q \boxplus G/Q'$ and hence $G = Q \boxplus Q'$. If Q is a^*-closed relative to V then $Qv = 0 \boxplus G/Q'$ and hence $G = Q \boxplus Q'$. Thus (i) and (ii) are proved. ($S(G) \subseteq \mathcal{Q}(G)$ is clear.)

DEFINITION. An l-group H is *quasiprojectable* (QP) if $\mathcal{Q}(H) = S(H)$. H is a *QP-hull* for G if H is an a^*-extension of G, H is QP, and if L is an l-subgroup of H such that $G \subseteq L \subseteq H$ and L is QP then $L = H$.

THEOREM 3.9. *If G is an l-group, then G has a unique QP-hull (which we will denote by G^{QP}). If V is a variety of l-groups with $G \in V$, then $G^{QP} \in V$, and if H is an a^*-closure for G relative to V, then H contains a copy of G^{QP}.*

PROOF. If $\{Q_i, i = 1, \ldots, n\}$ is a finite partition in $\mathcal{Q}(G)$, then by Proposition 3.7(iv), Lemma 3.5 and Proposition 2.1, $G/Q'_1 \boxplus \cdots \boxplus G/Q'_n$ is an a^*-extension of Gv where $v: G \longrightarrow G/Q'_1 \boxplus \cdots \boxplus G/Q'_n$ is the natural map. Moreover, v is an embedding since $\bigcap Q'_i = 0$ (since $\mathcal{Q}(G)$ is a subalgebra of $P(G)$). The directed union of a^*-extensions of G is again an a^*-extension of G (see the proof

of [4, Lemma 3.3]). Now the construction in [8], applied to $Q(G)$ instead of $P(G)$, produces a QP-hull for G.

If H is QP, and H is an a^*-extension of G, and $\{L_\alpha\}$ is the collection of l-subgroups such that $G \subseteq L_\alpha \subseteq H$ and L_α is QP, then $\bigcap L_\alpha$ is QP (see the proof of [10, Theorem 2.4]). Now uniqueness can be proved as in [10, Theorem 2.6]. Our construction gives $G^{QP} \in V$, and the last statement in the theorem follows from Theorem 3.8.

LEMMA 3.10. *Let G be an l-subgroup of H. Suppose for each $0 < h \in H$ there exists a subset $\{g_\alpha\}$ of G such that $\bigvee_H g_\alpha$ exists and h is a-equivalent to $\bigvee_H g_\alpha$ (i.e., there exists a positive integer n such that $h \leq n \bigvee_H g_\alpha$ and $\bigvee_H g_\alpha \leq nh$). Then H is an a^*-extension of G.*

PROOF. Let $K_1, K_2 \in K(H)$ and suppose $K_1 \cap G = K_2 \cap G$. Let $0 < h \in K_1$. Then h is a-equivalent to $\bigvee_H g_\alpha$ where $0 \leq g_\alpha \in G$; say $h \leq n \bigvee_H g_\alpha$ and $\bigvee_H g_\alpha \leq nh$ where n is some positive integer. Since $\bigvee_H g_\alpha \leq nh$ we have $g_\alpha \in K_1 \cap G = K_2 \cap G$. Thus $\bigvee_H g_\alpha \in K_2$, and hence $h \in K_2$ since $0 \leq h \leq n \bigvee_H g_\alpha$. Thus $K_1 \subseteq K_2$, and similarly $K_2 \subseteq K_1$. By Theorem 1.9, now, H is an a^*-extension of G.

THEOREM 3.11. *The following are equivalent:*
(i) $Q(G) = P(G)$.
(ii) $g'' \in Q(G)$ for all $g \in G$.
(iii) *G is representable, and the P-hull of G is an a^*-extension of G.*
(iv) *G is representable, and the orthocompletion of G is an a^*-extension of G.*
(v) *G is representable, and the SP-hull of G is an a^*-extension of G.*

PROOF. (i) *implies* (ii). Clear.

(ii) *implies* (iii). By Lemma 3.3 g'' is a normal subgroup for all $g \in G$, and hence G is a representable l-group [13]. Now by [3, Theorem 4.2] G^{QP} is a P-group. Since G^{QP} is an a^*-extension of G, G is a large l-subgroup of G^{QP} (by the corollary to Theorem 1.7). Thus G^{QP} contains a copy of G^P, the P-hull of G. Since G^{QP} is an a^*-extension of G, G^P is also (by Proposition 2.1).

(iii) *implies* (iv). By [10, Theorem 2.9] each positive element of the ortho-completion of G is the join of a (disjoint) set of elements in G^P. Thus by Lemma 3.10 and Proposition 2.1 the orthocompletion of G is an a^*-extension of G.

(iv) *implies* (v). This follows from the fact that the orthocompletion of G is an SP-group.

(v) *implies* (i). Let G^{SP} denote the SP-hull of G. We have $P(G^{SP}) = S(G^{SP}) \subseteq Q(G^{SP}) \subseteq P(G^{SP})$. Thus $P(G^{SP}) = Q(G^{SP})$ and hence, since $K(G)$ is isomorphic to $K(G^{SP})$, $P(G) = Q(G)$.

COROLLARY. *If $Q(G) = P(G)$, then the lateral completion of G is an a*-extension of G.*

COROLLARY. *If G is a P-group, an SP-group, or orthocomplete and V is a variety of l-groups, then each a*-closure of G, relative to V, is orthocomplete (and hence a P-group and a SP-group). The conclusion also holds under the weaker hypothesis: $g'' \in Q(G)$ for all $g \in G$.*

PROOF. If G is a P-group, an SP-group, or orthocomplete then, $g'' \in S(G)$ and hence $g'' \in Q(G)$ for all $g \in G$. We prove the corollary assuming only $g'' \in Q(G)$ for all $g \in G$. Let H be an a*-closure for G relative to V. The orthocompletion of H is again in V [10]. By the theorem the orthocompletion of H is an a*-extension of H; thus H is orthocomplete.

COROLLARY. *If G satisfies the conditions of Theorem 3.11, then each a*-closure of G, relative to any variety, contains a copy of the orthocompletion of G.*

PROOF. If H is an a*-closure of G, then H is orthocomplete, and by the corollary to Theorem 1.7 G is large in H. Hence [10] H contains a copy of the orthocompletion of G.

Suppose L is a distributive lattice with 0 and 1, and that $a \in L$ is complemented with complement a'. Let F be the filter in L generated by a, and let J be the ideal in L generated by a'. Then the map $x \mapsto x \vee a$ is a lattice isomorphism of J onto F (as is easily verified).

PROPOSITION 3.12. *If $Q \in Q(G)$, $F \in K(G)$, and $F \supseteq Q$, then $F/Q \in K(G/Q)$.*

PROOF. By Proposition 1.4 each element of $K(G/Q)$ can be written as K/Q where $Q \subseteq K \in K(G)$, and throughout this proof we will assume the elements are in this form.

By Theorem 3.7(iv) G/Q is an a*-extension of $(Q' + Q)/Q$. Thus $K/Q \mapsto K/Q \cap (Q' + Q)/Q = K \cap (Q' + Q)/Q = (K \cap Q') + Q/Q$ is a lattice isomorphism of $K(G/Q)$ onto $K((Q' + Q)/Q)$, and hence $\sigma\colon K(G/Q) \to K(Q')$ by $(K/Q)\sigma = K \cap Q'$ is a lattice isomorphism onto $K(Q')$.

By Proposition 1.3 $K(Q')$ is the (lattice-theoretic) principal ideal of $K(G)$ generated by Q'. Let $\bar{F} = \{K \in K(G) | K \supseteq Q\}$. Then by the paragraph preceding the proposition $K\tau = \overline{K \vee Q'}$ defines a lattice isomorphism of $K(Q')$ onto $\bar{F}$.

Let $\mu = \sigma\tau$. Then μ is an isomorphism of $K(G/Q)$ onto $\bar{F}$. Since $F \in \bar{F}$ we have $F\mu^{-1} \in K(G/Q)$ and hence there exists $K \in \bar{F}$ with $F\mu^{-1} = K/Q$. But also $K\mu^{-1} = K/Q$ since

$$(K/Q)\sigma\tau = (K \cap Q')\tau = \overline{(K \cap Q) \vee Q} = \overline{(K \vee Q) \cap (Q' \vee Q)}$$
$$= \overline{K \vee Q} \cap \overline{Q' \vee Q} = K \cap G = K.$$

Thus $K = F$, and hence $F/Q = K/Q = F\mu^{-1} \in K(G/Q)$.

We will see in §5 that the hypothesis $Q \in \mathcal{Q}(G)$ in the preceding proposition cannot be weakened to $Q \in \mathcal{P}(G)$.

4. **Cardinal sums, lexicographic sums, and the uniqueness question.** Given a variety V of l-groups with $G \in V$ when does G admit a unique a^*-closure relative to V? It was shown in [4] that each archimedean l-group admits a unique a^*-closure relative to the variety of all l-groups, and that each totally-ordered abelian group (more generally, each abelian l-group in which the special subgroups form a plenary set) admits a unique a^*-closure relative to the variety of abelian l-groups. However, S. Wolfenstein [15] has shown that a totally-ordered abelian group G can admit nonisomorphic a-closures, relative to the variety of all l-groups([1]); these a-closures are actually a^*-closures for G (since for totally-ordered groups the notions of a-extension and a^*-extension coincide). Uniqueness, relative to the variety of all l-groups, was established in [14] for a certain class of l-permutation groups.

We have shown (Theorem 3.4) that if $\mathcal{P}(G) = \mathcal{Q}(G)$ then G admits an a^*-closure relative to the variety of all l-groups, and that each of these a^*-closures is representable. Uniqueness relative to the variety of all l-groups cannot hold in general since totally-ordered groups satisfy $\mathcal{P}(G) = \mathcal{Q}(G)$; however, we have no example contradicting uniqueness relative to the variety of abelian l-groups. We note that if $\mathcal{P}(G) = \mathcal{Q}(G)$ then, by the third corollary to Theorem 3.11, G admits a unique a^*-closure, relative to a variety V, if and only if its orthocompletion does.

Wolfenstein [15] gives an example of an l-group G that is a^*-closed in the variety of representable l-groups, but not a^*-closed in the variety of all l-groups.

If G is normal-valued, then by Theorem 2.7 G admits an a^*-closure relative to the variety of normal-valued l-groups. Again it need not be unique, as the totally-ordered case demonstrates. However, by Theorem 3.9 G admits a unique a^*-closure relative to a variety V if and only if its QP-hull does.

Further results bearing on uniqueness will be established in this section.

LEMMA 4.1. *Let $\{A_\lambda\}$ be a family of l-groups.*

(i) ΠA_λ *is an a^*-extension of ΣA_λ.*

(ii) *If each A_λ is a^*-closed, relative to a variety V of l-groups, then so is* ΠA_λ.

([1]) An earlier example of this phenomenon appeared in [17].

PROOF. (i) Each positive element in ΠA_λ is the join of a set of elements in ΣA_λ. Thus by Lemma 3.10 ΠA_λ is an a^*-extension of ΣA_λ.

(ii) Let $G = \Pi A_\lambda$ and suppose $H \in V$ is an a^*-extension of G. We have $A_\lambda \in S(G)$ and hence $A_\lambda \in Q(G)$. $\bar{A}_\lambda$ is an a^*-extension of A_λ by Proposition 1.3. Thus $A_\lambda = \bar{A}_\lambda \in Q(H)$, and by Theorem 3.8 we have $A_\lambda \in S(H)$. Write $H = A_\lambda \boxplus B_\lambda$. If $g \in \bigcap (B_\lambda \cap G)$, then $g \in (\Sigma A_\lambda)'$, and hence $g = 0$. Thus $(\bigcap B_\lambda) \cap G = 0$, and thus $\bigcap B_\lambda = 0$.

For $h \in H$ write $h = h_\lambda + h^\lambda$ where $h_\lambda \in A_\lambda$ and $h^\lambda \in B_\lambda$. Define σ: $H \rightarrow G$ by $h\sigma = (\_h_\lambda\_)$. σ is one-to-one and the restriction of σ to G is 1_G. Thus $H = G$.

THEOREM 4.2. *Let $\{A_\lambda\}$ be a family of l-groups, and let V be a variety of l-groups. Let B_λ be an a^*-closure for A_λ, relative to V. Then ΠB_λ is an a^*-closure for ΣA_λ, relative to V. Conversely, each a^*-closure for ΣA_λ, relative to V, has this form.*

PROOF. Suppose B_λ is an a^*-closure for A_λ relative to V. Then ΣB_λ is an a^*-extension of ΣA_λ by Lemma 3.5, and thus by Lemma 4.1, ΠB_λ is an a^*-closure for ΣA_λ, relative to V.

Suppose, now, G is an a^*-closure for ΣA_λ relative to V. By Theorem 3.8, $\bar{A}_\lambda \in S(G)$. Write $G = \bar{A}_\lambda \boxplus K_\lambda$. If $T \in V$ is a proper a^*-extension of $\bar{A}_\lambda$, then $T \boxplus K$ is a proper a^*-extension of $G \in V$ by Lemma 3.5. Thus $\bar{A}_\lambda$ is a^*-closed relative to V; moreover, by Proposition 1.3, $\bar{A}_\lambda$ is an a^*-extension of A_λ.

We have $(\bigcap K_\lambda) \cap \Sigma A_\lambda = 0$. Thus $g \longmapsto (\_g_\lambda\_)$, where $g = g_\lambda + g^\lambda$ with $g_\lambda \in \bar{A}_\lambda$, $g^\lambda \in K_\lambda$, is an embedding of G into $\Pi \bar{A}_\lambda$. Thus without loss of generality we have $\Sigma A_\lambda \subseteq G \subseteq \Pi \bar{A}_\lambda$. By the first part of the theorem we have that $\Pi \bar{A}_\lambda$ is an a^*-closure for ΣA_λ. Thus $G = \Pi \bar{A}_\lambda$.

COROLLARY. *With the notation of the theorem, suppose G is an l-subgroup of ΠA_λ such that $\Sigma A_\lambda \subseteq G \subseteq \Pi A_\lambda$. Then ΠB_λ is an a^*-closure for G relative to V, and each a^*-closure of G, relative to V, has this form. In particular, if each A_λ admits a unique a^*-closure relative to V, then so does G.*

PROOF. ΠA_λ is an a^*-extension of ΣA_λ by Lemma 3.10. Thus G is an a^*-extension of ΣA_λ. Hence each a^*-closure of G, relative to V, is an a^*-closure of ΣA_λ, relative to V. For the converse note that $\Sigma A_\lambda \subseteq G \subseteq \Pi A_\lambda \subseteq \Pi B_\lambda$.

COROLLARY. *Let V be a variety of l-groups. Suppose $\{Q_\lambda\}$ is a disjoint collection of quasisummands of $G \in V$ such that $\bigcap Q'_\lambda = 0$, and each Q_λ admits a unique a^*-closure relative to V. Then G admits a unique a^*-closure relative to V.*

PROOF. Without loss of generality (by Theorem 3.9) we may assume each Q_λ is actually a summand of G. Since $\bigcap Q'_\lambda = 0$ we can identify G with an l-subgroup of ΠQ_λ. Now $\Sigma Q_\lambda \subseteq G \subseteq \Pi Q_\lambda$ and the previous corollary applies.

LEMMA 4.3. *Let V be a variety of l-groups with $G \in V$, and suppose $C \in C(G)$ is a*-closed, relative to V. Then $C \in K(G)$. Moreover, if $H \in V$ is an a*-extension of G, then $C \in K(H)$.*

PROOF. $C \in K(G)$ by Lemma 1.5. Let $H \in V$ be an a*-extension of G. By Propositions 1.2 and 1.3 $\bar{\bar{C}}$ is an a*-extension of C. Thus $C = \bar{\bar{C}} \in K(H)$.

THEOREM 4.4. *Let A be the variety of abelian l-groups, and let $G \in A$. If $C \in C(G)$ is such that both C and G/C are a*-closed relative to A, then G is a*-closed relative to A.*

PROOF. Let $H \in A$ be an a*-extension of G. By Lemmas 4.3 and 2.4, we conclude that H/C is an a*-extension of G/C, and hence $H = G$.

REMARK. The commutativity restriction in the preceding theorem cannot be completely discarded since it is critical to our proof that C be a normal subgroup of H. There is, in fact, an example in [15] of a representable l-group G that possesses a minimal prime subgroup C such that both G and G/C are a*-closed relative to the variety of all l-groups but G is not.

$N(G) = \{g \in G|\ |g|$ is the sum of finitely many nonunits of $G\}$ is called the *lex-kernel* of G. $N(G)$ is a closed, prime, normal convex l-subgroup of G. If $C \in C(G)$, then $C \supseteq N(G)$ or $C \subseteq N(G)$; moreover $N(G)$ is the least prime subgroup of G that has this property. (We admit the possibility $N(G) = G$.) See [11, pp. 2.24–2.25, 5.13] for proofs of these statements.

Let $C \in C(G)$. If $C \supseteq N(G)$, then C is a closed prime subgroup of G [7]. If $C \subseteq N(G)$, then $\bar{C} \in K(G)$ and $\bar{C} \subseteq N(G)$. The closed prime subgroups of G are the finite-meet irreducible elements of $K(G)$ [4, Proposition 1.4]. Thus $N(G)$ is the least finite-meet irreducible element of $K(G)$ that is comparable to all elements of $K(G)$. In particular, the lex-kernel of G is distinguishable in $K(G)$. Hence we have proved the following lemma.

LEMMA 4.5. *If H is an a*-extension of G, then $N(H) = N(G)^{\approx}$.*

It follows now that if $N(G)$ and $G/N(G)$ are a*-closed relative to some variety V of l-groups with $G \in V$, then G is a*-closed relative to V. (See the proof of Theorem 4.4 and the remark following it.)

G is a *lex-extension* of M if M is a prime subgroup of G and $g \in G\backslash M$ implies either $g > x$ for all $x \in M$ or $g < x$ for all $x \in M$. Equivalently [11, Theorem 2.7], $N(G) \subseteq M$. Thus G is a lex-extension of M if and only if M is a finite-meet

irreducible element of $K(G)$ and M is comparable to each element of $K(G)$. Hence, if G is a lex-extension of M, and H is an a^*-extension of G, then H is a lex-extension of $\widetilde{M}$.

Henceforth in this section the symbol $\oplus$ will designate group direct sum (and not cardinal sum). G is a *direct lex-extension of M by T* if G is a lex-extension of M and T is a subgroup of G such that $G = M \oplus T$. In this case, T is a totally-ordered subgroup of G.

Let G be a direct lex-extension of M by T, and let $K \in K(G)$. Then $K \supseteq M$ or $K \subseteq M$. If $K \supseteq M$, then $K = M \oplus K \cap T$ and $K \cap T \in K(T)$. (Since T is totally-ordered, each convex subgroup of T is closed.) On the other hand, if $K \subseteq M$, then since $M \in K(G)$ we have by Proposition 1.3 that $K \in K(M)$.

LEMMA 4.6. *Let G be a direct lex-extension of $M \in C(G)$ by T. Suppose L is an a^*-extension of M, and S is an a^*-extension of T. Then $H = L \oplus S$ (lexicographically ordered) is an a^*-extension of G.*

PROOF. Suppose $K_1, K_2 \in K(H)$ and $K_1 \cap G = K_2 \cap G$. If $K_1 \subset L \subseteq K_2$, then since $K_1, L \in K(L)$ we have $K_1 \cap G = K_1 \cap M \subset L \cap M = L \cap G \subseteq K_2 \cap G$, contradicting $K_1 \cap G = K_2 \cap G$. Thus either $K_1, K_2 \subseteq L$ or $K_1, K_2 \supseteq L$. If $K_1, K_2 \subseteq L$, then $K_1, K_2 \in K(L)$ and $K_1 \cap M = K_1 \cap G = K_2 \cap G = K_2 \cap M$, and hence $K_1 = K_2$. Now suppose $K_1, K_2 \supseteq L$. Then $K_1 = L \oplus (K_1 \cap S), K_2 = L \oplus (K_2 \cap S)$ and $K_1 \cap S, K_2 \cap S \in K(S)$. Now $(K_1 \cap S) \cap T = K_1 \cap S \cap G = K_2 \cap S \cap G = (K_2 \cap S) \cap T$. Hence $K_1 \cap S = K_2 \cap S$, and thus $K_1 = K_2$. By Theorem 1.9 H is an a^*-extension of G.

If G is an abelian l-group, then its divisible hull G^d is an a-extension of G (and hence an a^*-extension of G). Thus if G is a lex-extension of M, then G^d is a lex-extension of $\widetilde{M} = M^d$.

THEOREM 4.7. *Let A be the variety of abelian l-groups, and let $G \in \mathsf{A}$. Suppose G is a lex-extension of M, and that H is an a^*-closure for G relative to A. Then H is a direct lex-extension of an a^*-closure of M, relative to A, by the a^*-closure of G/M, relative to A.*

PROOF. We assume without loss of generality that G is divisible. Hence G is a rational vector space, and M is a subspace of G. Thus $G = M \oplus T$ for some subspace T of G. T is totally-ordered and hence [4, Corollary 3.12] admits a unique a^*-closure relative to A. Of course, T is isomorphic to G/M.

Since H is an a^*-extension of G, H is a lex-extension of $\widetilde{M}$. H is divisible since H^d is an a^*-extension of H. If $0 \lessdot t \in T \cap \widetilde{M}$, then $t = \bigvee_H h_\alpha$ where $0 \leqslant h_\alpha \in \widetilde{M}_\alpha$. Let $m_\alpha \in M$ with $h_\alpha \leqslant m_\alpha$. Then, since $t \in G\backslash M$ and $2m_\alpha \in M$, we have $2m_\alpha < t$, whence $2h_\alpha < t$ and $h_\alpha < \frac{1}{2}t$. This contradicts $t = \bigvee h_\alpha$.

Thus $T \cap \bar{\bar{M}} = 0$. $\bar{\bar{M}}$ is a subspace of the rational vector space H. Hence there is a subspace S of H such that $H = \bar{\bar{M}} \oplus S$ and $T \subseteq S$.

Thus H is a direct lex-extension of $\bar{\bar{M}}$ by S. Since H is a a*-closed relative to A, both $\bar{\bar{M}}$ and S are also, by Lemma 4.6. Furthermore, $\bar{\bar{M}}$ is an a*-extension of M by Propositions 1.2 and 1.3.

It remains only to show that S is an a*-extension of T. If $K \in K(S)$, then (as is easily verified) $\bar{\bar{M}} \oplus K \in K(H)$ and $(\bar{\bar{M}} \oplus K) \cap G = M \oplus (K \cap T)$. Suppose $K_1, K_2 \in K(S)$ and $K_1 \cap T = K_2 \cap T$. Then $M \oplus (K_1 \cap T) = M \oplus (K_2 \cap T)$, and since H is an a*-extension of G, $\bar{\bar{M}} \oplus K_1 = \bar{\bar{M}} \oplus K_2$. Since H is a direct lex-extension of $\bar{\bar{M}}$ by S, and $K_1, K_2 \in K(S)$, we conclude $K_1 = K_2$. Thus by Theorem 1.10 S is an a*-extension of T.

COROLLARY. *Suppose the abelian l-group G is a lex-extension of M. If M admits a unique a*-closure, relative to the variety of abelian l-groups, then so does G.*

5. An example. *All l-groups in this section shall be assumed to be abelian.* Consider the following infinite partially ordered set Λ:

$$\Lambda = \begin{array}{c} \bullet\ 0 \\ | \\ \bullet\ \bullet\ \bullet\ \bullet \quad \cdots \\ 1\ 2\ 3\ 4 \end{array}$$
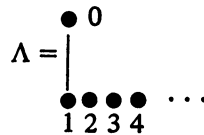
Let $V = V(\Lambda, \mathbf{R})$. We can think of V as consisting of all sequences of real numbers (indexed by the set of nonnegative integers). For $b = (b_i) \in V$ we have $b \geqslant 0$ if and only if $b_0 > 0$ and $b_i \geqslant 0$ for all $i \geqslant 2$, or $b_i \geqslant 0$ for all $i \geqslant 0$.

Let $G = \{b \in V | b_i = b_0 \text{ for all but finitely many } i\}$. V is an l-group, and G is a large l-subgroup of V. The reader is referred to [4] for background on this section. We investigate some properties of G. (For further properties, see [5].)

(1) *V is the lateral completion of G, and V is not an a*-extension of G* (cf. the first corollary to Theorem 3.11).

PROOF. G is a large l-subgroup of V, and V is laterally complete. Suppose H is an l-subgroup of V such that $G \subseteq H \subseteq V$ and H is laterally complete. Since $\Sigma_{i=1}^{\infty} \mathbf{R}_i \subseteq G$, we have $\Pi_{i=1}^{\infty} \mathbf{R}_i \subseteq H$, and hence, since G contains the constant sequences $H = V$.

Let $K_1 = \{b \in V | b_i = 0 \text{ if } i \neq 1\}$ and $K_2 = \{b \in V | b_i = 0 \text{ for all } i \geqslant 2\}$. Then $K_1, K_2 \in K(V)$ and $K_1 \cap G = K_1 = K_2 \cap G$. Thus V is not an a*-extension of G.

(2) *There exists an l-subgroup X of G and $Y \in K(G)$ such that $Y \cap X \notin K(X)$.*

PROOF. Take $X = \{g \in G | g_1 = 0\}$ and $Y = \Sigma_{i=2}^{\infty} R_i$. Then $Y \in K(G)$ and $Y = Y \cap X$. However, the closure of Y in X is X. Thus $Y \cap X \notin K(X)$.

(3) *Let $\Gamma(G)$ denote the collection of all regular subgroups of G. Then $\Gamma(G)$ is order-isomorphic to Λ, and each element of $\Gamma(G)$ is closed.*

PROOF. For fixed $n \geq 1$, let $G_n = \{g \in G | g_n = 0\}$. Then G_n is a regular subgroup of G (for all $n \geq 1$), and so is $\Sigma_{i=1}^{\infty} R_i$. Moreover, the reader can verify that these are the only elements of $\Gamma(G)$. We have $G_n \in K(G)$ for all $n \geq 1$, since, in fact, $G_n \in P(G)$. It is easily verified that $\Sigma_{i=1}^{\infty} R_i \in K(G)$.

(4) $C(G) = K(G)$.

PROOF. This follows from (3) since each element of $C(G)$ is the intersection of elements in $\Gamma(G)$.

Note that the constant sequence 1 is an element of G and each element of $\Gamma(G)$ is a value of 1. Thus we have another example of an abelian l-group in which each convex l-subgroup is closed but which is not finite-valued (cf. [4, p. 322]).

(5) *G is a-closed.*

PROOF. Let H be an a-extension of G. Then by [11, Lemma 4.11] without loss of generality we can assume H is an l-subgroup of V (since $\Gamma(G) \cong \Lambda$) such that $G \subseteq H \subseteq V$. Let $0 < h \in H$. There exists $0 < g \in G$ such that $ng \geq h$ and $nh \geq g$ for some positive integer n. Note that $g_0 = 0$ if and only if $h_0 = 0$. If $g_0 = h_0 = 0$, then g has finite support (i.e., all but finitely many g_i are 0), and hence h must also; thus $h \in \Sigma_{i=1}^{\infty} R_i \subseteq G$.

Now suppose $g_0 \neq 0$. Then there exists $r \in R$ such that $rg_0 = h_0$. Let $(rg - h) \vee 0 = b = (b_i) \in H$. There exists $0 \leq f \in G$ such that $mb \leq f$ and $mf \leq b$ for some positive integer m. We have $b_0 = 0$, and thus $f_0 = 0$, and $b \in \Sigma_{i=1}^{\infty} R_i \subseteq G$. Similarly, $(h - rg) \vee 0 \in G$. Thus $rg - h \in G$, and hence $h \in G$. Thus $G = H$.

Let $A = \{b \in V | \text{ the sequence } b = (b_i) \text{ converges to } b_0\}$. A is an l-subgroup of V and $G \subseteq A$.

(6) *A is an a*-extension of G.*

PROOF. We show that the hypotheses of Lemma 3.10 are satisfied. Let $0 < a \in A$. If $a_0 = 0$, then $a_i \geq 0$ for all i, and a is the join of a (disjoint) subset of G. If $a_0 > 0$, let $b \in V$ be such that $b_i = 0$ if $a_i = 0$, and $b_i = 1$ otherwise.

Since (a_i) converges to $a_0 > 0$, only finitely many of the b_i are 0, and hence $b \in G$. Also because (a_i) converges to $a_0 > 0$, b is a-equivalent to a. Hence by Lemma 3.10 A is an a^*-extension of G.

Thus G is not a^*-closed even though G is an a-closed vector lattice and $C(G) = K(G)$.

(7) *If H is an l-subgroup of V, and H is an a^*-extension of G, then $H \subseteq A$.*

PROOF. Let $0 < x \in H$ be such that $x_0 > 0$. Let $\perp$ denote the polar operation in H. If $x_i = 0$ for infinitely many i, then $x^{\perp\perp} \cap G \subseteq \Sigma_{i=1}^{\infty} R_i$; but since H is an a^*-extension of G, we have $x^{\perp\perp} = (x^{\perp\perp} \cap G)^{\widetilde{}}$, which is impossible since $x \notin (x^{\perp\perp} \cap G)^- = (x^{\perp\perp} \cap G)^{\widetilde{}}$. Thus $x_i > 0$ for all but finitely many i.

Now suppose (by way of contradiction) there exists $0 < h \in H \backslash A$. Let $t \in V$ be such that $t_i = h_i - h_0$. Then $t \in H \backslash A$ and $t_0 = 0$. Since (t_i) does not converge to 0, there exists $n > 0$ such that $n|t_i| \geq 1$ for infinitely many i. Let $b_i = |t_i|$ and $b = (b_i)$. Then $b = |t| \in H$. Let $c_i = 1$ for all i, and let $c = (c_i)$. We have $(c - nb) \vee 0 \in H$. Also $(c - nb) \vee 0 = (x_i)$, where x_i is the larger of 0 and $1 - n|t_i|$. Thus $x_0 = 1$ and infinitely many x_i are 0, contradicting what was established in the preceding paragraph. Thus $H \subseteq A$.

(8) *A is the unique a^*-closure of G.*

PROOF. Let H be an a^*-extension of G. Since the divisible hull of H is an a^*-extension of H, and hence of G, there will be no loss of generality in assuming H is divisible.

Since each element of $\Gamma(G)$ is in $K(G)$, it follows from [4, Proposition 1.4] that $\Delta = \{\bar{K} | K \in \Gamma(G)\}$ is a plenary subset of $\Gamma(H)$. Now by [4, Lemma 3.9] there exists an l-isomorphism σ of H into V such that $\sigma|_G$ is the identity on G. It follows, thus, from (6) and (7) that A is the unique a^*-closure of G.

Let $P = \{a \in A | a_0 = a_1 = 0\}$ and $K = \{a \in A | a_0 = 0\}$. Let $'$ denote the polar operation in A.

(9) $P' = \{a \in A | a_i = 0 \text{ for all } i \neq 1\}$, $P = P''$, *and* $\overline{P \vee P'} = K$. *Thus P and P' are polars in A, but not quasisummands of A.*

PROOF. Let $x_i = 1$ if $i = 1$, and $x_i = 0$ otherwise. Then $x = (x_i) \in A$, and $x' = P$ and $x'' = P'$. It is clear that $P \vee P' = K$, and not difficult to show $K \in K(A)$. Thus $\overline{P \vee P'} = K$.

(10) *P is not a^*-closed. Neither is A/P' even though both A and P' are a^*-closed.*

PROOF. Note $P \cong K$, and K is not a^*-closed since by Lemma 3.9 $\Pi_{i=1}^{\infty} R_i$

is an a^*-extension of K. (In fact, $\Pi_{i=1}^\infty R_i$ is the a^*-closure of K.)

$P' \cong \mathbf{R}$ is a^*-closed, and A is also by (8). However, $A/P' \cong \{a \in A | a_1 = 0\}$ $\cong$ {convergent sequences in $\Pi_{i=1}^\infty R_i$}. Thus the a^*-closure of A/P' is l-isomorphic to $\Pi_{i=1}^\infty R_i$, and A/P' is not a^*-closed.

The pathology in (10) cannot occur for quasisummands in view of Theorem 3.7 and Lemma 3.5.

(11) *We have* $K \in K(A)$, $P' \in P(A)$, *and* $K \supsetneq P'$. *However,* $K/P' \notin K(A/P')$ (*cf. Proposition* 3.12).

PROOF. As in the proof of (10), we can view A/P' as the l-group of all convergent sequences in $\Pi_{i=1}^\infty R_i$. Thus K/P' consists of those sequences in $\Pi_{i=1}^\infty R_i$ which converge to 0. Clearly, $K/P' \notin P(A/P')$. Thus by Theorem 1.1 (since A/P' is archimedean) we have $K/P' \notin K(A/P')$.

(12) *K is a large archimedean l-subgroup of A, and $K \in K(A)$. Furthermore, A is minimal with respect to being an a^*-closed l-group containing K as an l-subgroup. However, A is not archimedean, and, moreover, A does not contain an a^*-closure for K.*

PROOF. It has previously been noted, or can easily be verified, that K is a large, archimedean l-subgroup of A, $K \in K(A)$, and A is not archimedean. Suppose H is an l-subgroup of A such that $K \subseteq H \subseteq A$ and H is a^*-closed. K is not a^*-closed; hence there exists $0 < h \in H \backslash K$. Now A is an a-extension of H, since $h_0 > 0$ and each $a \in A$ converges to a_0. Hence by (6) we have $A = H$. Thus the minimality of A is established, and also we have that A does not properly contain an a^*-closure for K. A is not itself an a^*-closure for K since K is archimedean and A is not.

6. **Some other extensions of l-groups.** The difficulty of the uniqueness question for a^*-closures and other considerations suggest that it may be worthwhile to consider certain extensions that are similar to, but perhaps different from, a^*-extensions. None of the extensions considered below have the simplicity of formulation that the notion of a^*-extension has; moreover, each seems to have its own drawbacks.

We assume throughout this section that all l-groups are abelian. (The reader can generalize, if desired.)

If the hypotheses of Lemma 3.10 are satisfied let us write GRH and say H is a #-*extension* of G. Consider the smallest transitive, inductive relation containing R on the class of abelian l-groups. If the pair (G, H) belongs to this relation, we will say H is an $a^\#$-*extension* of G. Since the relation of being an a^*-extension is transitive and inductive (Lemmas 2.1 and 2.2), we obtain from Lemma 3.10 that

each $a^{\#}$-extension of G is an a*-extension. It follows now from the existence of a*-closures (relative to the variety of abelian l-groups) that each l-group has an $a^{\#}$-closure. ($a^{\#}$-*closure* is defined in the natural way.)

If G is archimedean or if G is totally-ordered (more generally, if the special subgroups of G form a plenary subset of G), then the (unique, by [4]) a*-closure of G is the unique $a^{\#}$-closure of G. In fact, in each of these cases the a*-closure of G is a #-extension of G^d, the divisible hull of G.

Let G be an l-subgroup of H. Wolfenstein has shown that H is an a-extension of G if and only if the following condition is satisfied: Given $0 < h \in H$ there exists $0 < g \in G$ such that C is a value of g in G if and only if $\widetilde{C}$ is a value of h in H. It seems plausible that a suitable weakening of this condition might produce something resembling the notion of a*-extension.

Suppose (i) G is an l-subgroup of H such that given $0 < h \in H$ there exists $0 < g \in G$ such that if C is a value of g in G then $\widetilde{C}$ is a value of h in H. It can be shown that if (i) is satisfied then G is large in H; however, it is possible that H is nonarchimedean and G is archimedean. (Take G to be the l-group K of §5, and H to be the G of §5.)

This last pathology indicates that (i) should be strengthened. Let us demand further (ii) that if $a \in G$ and $0 < a < h$ then we can choose $g \in G$ with $0 < a < g$ such that if C is a value of g in G then $\widetilde{C}$ is a value of h in H. Now it can be shown that H is archimedean whenever G is; and that if G is archimedean with a basis, $\Sigma \mathbf{R}_i \subseteq G \subseteq \Pi \mathbf{R}_i$, then $\Pi \mathbf{R}_i$ is the closure of G, obtained by working with extensions satisfying (i) and (ii). Also, if the special subgroups of G form a plenary subset Δ of $\Gamma(G)$, and we take $H = V(\Delta, \mathbf{R})$, then (i) and (ii) are satisfied.

Clearly, there are too many unanswered questions here for us to conclude that this modification of Wolfenstein's condition is useful, or, indeed, that we have chosen the best modification.

If the special subgroups form a plenary subset Δ of $\Gamma(G)$, then $V(\Delta, \mathbf{R})$ is the unique a*-closure of G; however, in the other cases that $\Gamma(G)$ possesses a minimal plenary subset, $V(\Delta, \mathbf{R})$ is not an a*-extension of G. This suggests that our notion of a*-extension is too restrictive. One way of relaxing it is the following: H is an N-*extension* of G if (1) G is a large l-subgroup of H, and (2) $C \longmapsto C \cap G$ is a one-to-one map of the closed prime subgroups of H onto those of G. The analogues of Lemmas 2.1 and 2.2 are satisfied; hence we have the notion of an N-*closure*.

THEOREM 6.1. *If* $\Gamma(G)$ *has a minimal plenary subset* Δ, *then* $V(\Delta, \mathbf{R})$ *is the unique* N-*closure of* G.

PROOF. Let Δ be the minimal plenary subset of $\Gamma(G)$. (If $\Gamma(G)$ has a min-

imal plenary subset, then it has only one.) Δ is the set of closed regular subgroups of G. For $\delta \in \Delta$ let H^δ, H_δ be closed prime subgroups of H such that $H^\delta \cap G = G^\delta$ and $H_\delta \cap G = G_\delta$. Then $(\bigcap H_\delta) \cap G = \bigcap (H_\delta \cap G) = \bigcap G_\delta = 0$, and since G is large in H, we have $\bigcap H_\delta = 0$. If H_γ is a regular subgroup of H and $H_\gamma \supseteq H_\delta$ for some $\delta \in \Delta$, then H_γ is closed and $H_\gamma \cap G \supseteq H_\delta \cap G$, whence $H_\gamma \cap G = G_\gamma \in \Delta$. Thus $\{H_\delta \,|\, \delta \in \Delta\}$ is a minimal plenary subset of $\Gamma(H)$.

Without loss of generality G and H are divisible. There exists a v-isomorphism $\sigma: G \longrightarrow V(\Delta, \mathbf{R})$, and by [4, Lemma 3.9] σ extends to a v-isomorphism of H into $V(\Delta, \mathbf{R})$. Moreover, $V(\Delta, \mathbf{R})$ is an N-extension of G. It follows that $V(\Delta, \mathbf{R})$ is the unique N-closure of G. (This proof has assumed a familiarity with the results and notation of Chapters 4 and 5 of [11].)

Unfortunately, an abelian l-group G (even an archimedean l-group) need not admit an N-closure. (Of course, many l-groups have no proper closed prime subgroups at all.)

QUESTIONS. (1) Does every l-group admit an a^*-closure, relative to the variety of all l-groups? (Rick Ball [16] has recently given an affirmative answer to this question, using permutation group techniques and some of the theory in §2 of this paper.)

(2) Does each abelian l-group admit a unique a^*-closure, relative to the variety of abelian l-groups?

(3) Is each a^*-extension of a normal-valued l-group normal-valued? (It is proved in [15] that each a-extension of a normal-valued l-group is normal-valued. Also, Holland and McCleary have recently shown that if G is a completely distributive normal-valued l-group, then G has an a^*-closure H in the variety of all l-groups and H is necessarily completely distributive and normal-valued.)

(4) If G is a^*-closed, relative to the variety of abelian l-groups, is G a vector lattice?

(5) If H is $a^\#$-closed, is H a^*-closed?

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