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UNIONS OF HILBERT CUBES

BY

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ABSTRACT. This paper gives a partial solution to the problem whether the union of two Hilbert cubes is a Hilbert cube if the intersection is a Hilbert cube and a Z-set in one of them. Our results imply West's Intermediate Sum Theorem on Hilbert cube factors. Also a technique is developed to obtain Z-sets as limits of Z-sets.

0. Introduction. This paper gives a partial solution to the following problem about the Hilbert cube $Q = [-1, 1]^\infty$:

Question 1. *Is the union of two copies of the Hilbert cube homeomorphic to the Hilbert cube when their intersection is?*

The converse is not the case: let α be an arc in $\{0\} \times Q$ whose complement is not the complement of a point. Then $Q_1 = ([-1, 0] \times Q)/\alpha$, $Q_2 = ([0, 1] \times Q)/\alpha$, and $Q_1 \cup Q_2$ are all homeomorphic to Q (since α can be shown to have the complement of a point in $J \times Q$, where $J = [-1, 0]$, $[0, 1]$ or $[-1, 1]$), whereas $Q_1 \cap Q_2$ is not homeomorphic to Q.

The analogous problem for Hilbert cube factors is solved by West [8], [9]. A Hilbert cube factor is a topological space X such that $X \times Q \cong Q$. This is equivalent to the existence of a space Y such that $X \times Y \cong Q$. In [8] West proved that the union of two Hilbert cube factors is a Hilbert cube factor if the intersection is a Hilbert cube factor and a Z-set in one of them. (This is the Intermediate Sum Theorem.) A general formal definition of Z-set will be given in §1, but for the Hilbert cube Q, a closed subset is a Z-set in Q iff it is equivalent under a space homeomorphism to a subset of an endface. West's result [8] appears here as Theorem 1.4. Observe that from this result it follows immediately that the triod and, more generally, all collapsible polyhedra are Hilbert cube factors.

Using the result of [8], West settles the general question for Hilbert cube

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factors proving that any union of two Hilbert cube factors is a Hilbert cube factor if the intersection is [9].

The earliest result for Hilbert cubes is: the union of two copies Q_1 and Q_2 of the Hilbert cube is a Hilbert cube if the intersection is a copy of the Hilbert cube and is a Z-set in both. This is an easy consequence of the

Homeomorphism Extension Theorem. *Any homeomorphism between two Z-sets in Q can be extended to an autohomeomorphism of Q. Thus we may assume that the intersection is an endface in both, so that the triple $(Q_1, Q_2, Q_1 \cap Q_2)$ is equivalent to $(Q \times [-1, 0], Q \times [0, 1], Q \times \{0\})$ under a homeomorphism on $Q_1 \cup Q_2$, and hence $Q_1 \cup Q_2 \cong Q$.*

In this paper we maintain the condition that $Q_1 \cap Q_2$ is a Z-set in Q_1. However, the requirement that $Q_1 \cap Q_2$ is a Z-set in Q_2 has been relaxed considerably. We use the following notions: a *cap set* for Q is a subset of Q which is equivalent to the *pseudoboundary* $B(Q) = \{x \mid \exists i: |x_i| = 1\}$. A characterization and examples will be given in §1. Furthermore we use *fd cap sets,* a finite-dimensional analogue of cap set. Both cap sets and fd cap sets are widely used in infinite-dimensional topology. The main result of §1 is that the condition that $Q_1 \cap Q_2$ is a Z-set in Q_2 can be replaced by: there is a cap set for $Q_1 \cap Q_2$ which is a countable union of Z-sets in Q_2 (Proposition 1.3). Also we give a simpler proof of West's first result from [8]. T. A. Chapman independently and earlier discovered the inverse-limit type of proof employed here; he had essentially the same (unpublished) proof for the Intermediate Sum Theorem (here Theorem 1.4). In §2 it is shown that, if for a subset $A \subset Q$, $A \cong Q$, there exists an fd cap set for A which is a countable union of Z-sets in Q, then this fd cap set can be 'blown up' to a cap set for A which is a countable union of Z-sets in Q. Therefore the word cap set can be replaced by fd cap set in Proposition 1.3. Combined with a result from [6] this produces Corollary 2.3 to the effect that it is sufficient that $Q_1 \cap Q_2$ has deficiency 1 in Q_2, or that $Q_1 \cap Q_2$ is nowhere dense in Q_2 and $Q_2 \backslash Q_1$ is 0-ULC and 1-ULC.

1. A preliminary version of the Main Theorem. Proposition 1.3 below is preliminary to Main Theorem 2.2. A closed subset K of a metric space X is a *Z-set in X* if for each $\epsilon > 0$ there exists a map $f: X \to X - K$ such that $d(f, \mathrm{id}_X) < \epsilon$. It is easily seen that any closed subset of a Z-set and any finite or countable closed union of Z-sets is again a Z-set. A stronger form of the Homeomorphism Extension Theorem is: Every homeomorphism between two Z-sets in Q which moves points less than ϵ can be extended to an autohomeomorphism on Q which moves points less than ϵ (see [1], [3]). In the Introduction a cap set of Q was defined as a subset of Q equivalent under an autohomeomorphism of Q to the pseudoboundary $B(Q) = \{x \mid \exists i\colon |x_i| = 1\}$. Now we define (see [2]): A subset M of Q is an *(fd) cap set* for Q if M can be written as $\bigcup_i M_i$, where each M_i is a (finite-dimensional) Z-set in Q, $M_i \subset M_{i+1}$, and such that the following absorption property holds: for all ϵ, j and every (finite-dimensional) Z-set K in Q there is an autohomeomorphism $h\colon Q \to Q$ such that $d(h, \mathrm{id}) < \epsilon$, $h|_{M_j} = \mathrm{id}$ and $h(K) \subset M_i$ for some i. Every closed subset of an (fd) cap set is a Z-set and every two (fd) cap sets are equivalent under a space homeomorphism. Examples for cap sets are: the pseudoboundary

$$B(Q), \qquad M = \{x \mid \sup |x_i| < 1\} = \bigcup_i [-1 + 1/i,\ 1 - 1/i]^\infty,$$

and

$$P = \{x \mid \text{for all but finitely many } i,\ x_i \le 0\}$$

(it is not obvious from the definition that P and $B(Q)$ are cap sets). Examples for fd cap sets are

$$s_f = \{x \in (-1, 1)^\infty \mid \text{for all but finitely many } i,\ x_i = 0\}$$
$$= \bigcup_i \{x \mid \forall_j \le i,\ |x_j| \le 1 - 1/i \text{ and for all } j > i,\ x_j = 0\}$$

and

$$Q_f = \{x \in Q \mid \text{for all but finitely many } i,\ x_i = 0\} = \bigcup_i \{x \mid \forall_j > i,\ x_j = 0\}.$$

We need two results on inverse limits. An onto map $f\colon X \to Y$, where $X \cong Y$ are metric spaces, is a *near-homeomorphism* if f can be uniformly approximated by homeomorphisms.

Theorem A (Morton Brown [4]). *If* $X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} X_3 \xleftarrow{f_3} \cdots$ *is an inverse sequence, where the* X_i *are homeomorphic compact metric spaces and each* f_i *is a near-homeomorphism, then* $\lim_i (X_i, f_i) \cong X_1$.

Lemma B [5]. *Let* (X_i, f_i) *be an inverse sequence of compact subsets of a compact metric space* X *and surjections* $f_i\colon X_{i+1} \to X_i$, *such that for each* i

(a) $X_i \subset X_{i+1}$ *and* $\mathrm{Cl}(\bigcup_i X_i) = X$,

(b) $d(f_i, \mathrm{id}_{X_{i+1}}) \le 2^{-i+1}$,

(c) $\{f_i \circ \cdots \circ f_j\colon X_{j+1} \to X_i \mid j \ge i\}$ *is an equi-uniformly continuous family of functions* (*i.e. for each* $\epsilon > 0$ *there is a* $\delta > 0$ *such that for every* $j > i$ *and every* x *and* y *in* X_{j+1} *with* $d(x, y) < \delta$, $d(f_i \circ \cdots \circ f_j(x), f_i \circ \cdots \circ f_j(y)) < \epsilon$).

Then $X \cong \lim_i (X_i, f_i)$.

Proof. Let $(x_i)_i \in \lim (X_i, f_i)$ i.e., for each j, $f_j(x_{j+1}) = x_j$. Because of (b) there is a map $F\colon \lim_i (X_i, f_i) \to X$ which assigns to $(x_i)_i$ its limit. By

(a) this map is onto and by applying (c) in opposite direction it is 1-1.

If the maps f_i in the above lemma do not increase the distance between any two points, then condition (c) of Lemma B is trivially satisfied.

In view of the above, Lemma 1.1 below will be seen to be the core of the proofs of both Proposition 1.3 (the main result of this section) and Theorem 1.4 (West's Intermediate Sum Theorem). The set $Q \times [-1, 1]$ will have the metric

$$d((x, t), (y, s)) = |t - s| + \sum_i 2^{-i} \cdot |x_i - y_i|.$$

Lemma 1.1. *Let* $P_i = \{x \in Q \mid \text{for all } j > i,\ x_j \leq 0\}$. *Let* $\tilde{Q}$ *be a subset of* $Q \times [-1, 0]$ *which contains, for some given* i, $P_i \times \{0\}$ *as a Z-set. Then* $\tilde{Q} \cup (P_i \times [0, 1])$ *is homeomorphic to* Q *and there is a retraction*

$$r_{i-1}: \tilde{Q} \cup (P_i \times [0, 1]) \to \tilde{Q} \cup (P_{i-1} \times [0, 1])$$

which is a near-homeomorphism with distance 2^{-i} *to the identity and does not increase the distance between any two points.*

Proof. Since $\tilde{Q} \cap (P_i \times [0, 1]) = P_i \times \{0\}$ is a Z-set both in $\tilde{Q}$ and in $P_i \times [0, 1]$, we have, by a remark in the Introduction, that $\tilde{Q} \cup (P_i \times [0, 1])$ is homeomorphic to Q. The same is true for $\tilde{Q} \cup (P_{i-1} \times [0, 1])$.

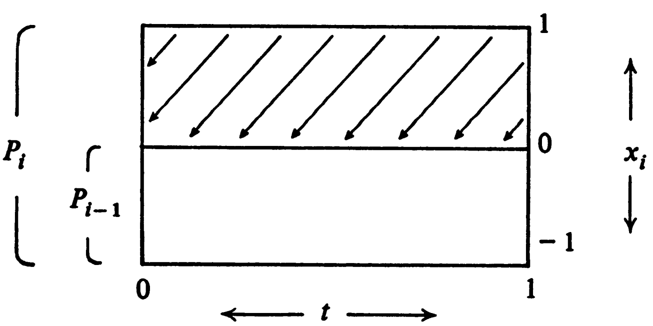
We define the retraction $r_{i-1}: \tilde{Q} \cup (P_i \times [0, 1]) \to \tilde{Q} \cup (P_{i-1} \times [0, 1])$ as follows:

(1) $r_{i-1} | \tilde{Q} \cup (P_{i-1} \times [0, 1])$ is the identity map;

(2) $r_{i-1}(x, t) = ((x_1, x_2, \ldots, x_{i-1}, 0, x_{i+1}, x_{i+2}, \cdots), t - 2^{-i} \cdot x_i)$ if $0 \leq 2^{-i} \cdot x_i \leq t$;

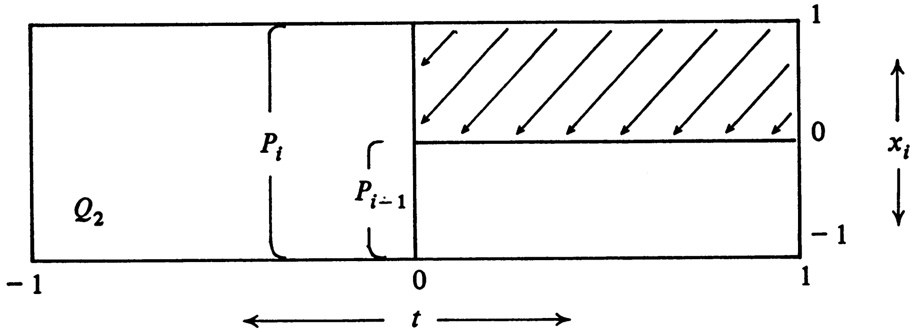
(3) $r_{i-1}(x, t) = ((x_1, x_2, \ldots, x_{i-1}, x_i - 2^i \cdot t, x_{i+1}, x_{i+2}, \cdots), 0)$ if $2^{-i} \cdot x_i \geq t$.

In a picture (we show only the x_i- and t-coordinates):



Obviously $d(r_{i-1}, \mathrm{id}) = 2^{-i}$. To show that r_{i-1} is a near-homeomorphism we remark that the pair $(\tilde{Q}, P_i \times \{0\})$ is homeomorphic to $(P_i \times [-1, 0], P_i \times$

$\{0\}$) because of the Homeomorphism Extension Theorem for Z-sets. Thus we get the following picture (again only showing the x_i- and t-coordinates):



It is easily seen from this picture that r_{i-1} can be approximated by homeomorphisms, also of the form $r'_{i-1} \times \text{id}$, where r'_{i-1} is defined on the x_i- and t-coordinates and id on the other coordinates.

Finally, it is left as an exercise to the reader that r_i does not increase distances.

Lemma 1.2. *Let $A \subset Q_2$ be homeomorphic to Q. Then any homeomorphism $i: A \to Q \times \{0\}$ can be extended to an embedding of Q_2 into $Q \times [-1, 0]$.*

Proof. Let $h: Q_2 \to Q \times [-1, 0]$ be an arbitrary embedding of Q_2 onto a Z-set. By the Homeomorphism Extension Theorem for $Q \times [-1, 0]$ there is a homeomorphism $h': Q \times [-1, 0] \to Q \times [-1, 0]$ which extends $i \circ h^{-1}: h(A) \to i(A)$. Then $h' \circ h$ is an embedding of Q_2 into $Q \times [-1, 0]$ which is an extension of i.

We are now ready to prove the preliminary version of the Main Theorem:

Proposition 1.3. *If X is $Q_1 \cup Q_2$, where $Q_1 \cong Q_2 \cong Q_1 \cap Q_2$ and $Q_1 \cap Q_2$ is a Z-set in Q_1 such that there is a cap set for $Q_1 \cap Q_2$ which is a countable union of Z-sets in Q_2, then X is homeomorphic to Q.*

Proof. We may write $Q_1 = Q \times [0, 1]$, where $Q_1 \cap Q_2$ is identified with $Q \times \{0\}$. By Lemma 1.2, we may also write $Q_2 = \tilde{Q} \subset Q \times [-1, 0]$. Furthermore we may assume that the cap set postulated in the proposition is the set $P \times \{0\} = \bigcup_i P_i \times \{0\}$. We form the inverse sequence

$$\tilde{Q} \cup (P_1 \times [0, 1]) \xleftarrow{r_1} \tilde{Q} \cup (P_2 \times [0, 1]) \xleftarrow{r_2} \tilde{Q} \cup (P_3 \times [0, 1]) \cdots$$

with r_i as in Lemma 1.1. This inverse sequence satisfies all requirements of Lemma B and therefore shows that $Q_1 \cup Q_2$ is homeomorphic to

$\lim_i((\tilde{Q} \cup (P_i \times [0, 1])), r_i)$, whose inverse limit is a Hilbert cube according to Theorem A.

West's Intermediate Sum Theorem [8] can be considered as a corollary to Proposition 1.3, but instead we give a separate proof that bypasses the notion of cap set.

Theorem 1.4 (**West** [8]). *The union of two Hilbert cube factors* X_1 *and* X_2, *whose intersection is a Hilbert cube factor and a Z-set in* X_1, *is a Hilbert cube factor.*

Proof. Set $X_i' = X_i \times Q$, $i = 1, 2$. Since $X_1 \cap X_2$ is a Z-set in X_1, the definition of Z-set gives immediately that also $X_1' \cap X_2'$ is a Z-set in X_1', whereas X_1', X_2', and $X_1' \cap X_2'$ are Hilbert cubes. Therefore we can identify $(X_1', X_1' \cap X_2')$ with the pair $(Q \times [0, 1], Q \times \{0\})$. Applying Lemma 1.2, we can identify X_2' with a set $\tilde{Q} \subset Q \times [-1, 0]$ which contains $Q \times \{0\}$ and such that $X_1' \cap X_2'$ corresponds to $Q \times \{0\}$.

We show that $(X_1' \cup X_2') \times Q$ is homeomorphic to Q: we form an inverse sequence $(Y_i, r_i')_i$ where $Y_i = (\tilde{Q} \times Q) \cup (Q \times [0, 1] \times P_i)$ and P_i is defined as in Lemma 1.1. The bonding maps r_i' are the identity on $\tilde{Q} \times Q$; on the term $Q \times [0, 1] \times P_i$, r_{i-1}' is identified as $\mathrm{id}_Q \times r_{i-1}$, where r_{i-1} is as in Lemma 1.1. Since P_i is a Z-set in Q, also $Q \times \{0\} \times P_i$ is a Z-set in $\tilde{Q} \times Q$. Therefore it follows, by a similar argument as in Lemma 1.1, that r_i' is a near-homeomorphism and that $Y_i \cong Q$. Furthermore, r_i' does not increase distances. Thus the sequence $(Y_i, r_i')_i$ satisfies all the conditions of Lemma B and therefore

$$(X_1 \cup X_2) \times Q \cong (X_1' \cup X_2') \times Q \cong \lim(Y_i, r_i') \cong Q.$$

Remark. As mentioned in the Introduction, West uses this result (for which he needed a much longer proof) to prove in [9] the general statement on Q-factors.

2. **Reduction to fd cap sets.** In this section we will describe how to blow up an fd cap set for $Q_1 \cap Q_2$, consisting of Z-sets in Q_2, to a cap set for $Q_1 \cap Q_2$, consisting of Z-sets in Q_2. Before proceeding to this, we need some additional definitions and results:

A set $K \subset Q = [-1, 1]^\infty$ has *deficiency* k if K projects onto a point in at least k coordinates; K has *infinite deficiency* if K projects onto a point in infinitely many coordinates. A subset of Q is topologically equivalent to a closed subset of infinite deficiency iff it is a Z-set in Q [1]. It follows from our definition of Z-set (which is closely related to that given in Toruńczyk [7, Theorem 2(a)(ii)], but which is different from Anderson's original definition given in [1]), that the collection of nonempty Z-sets forms a G_δ in 2^Q, the hyperspace of Q. The proof goes as follows: let $\mathcal{Z}_i = \{K \in 2^Q |$

there exists an $f\colon Q \to Q - K$ with $d(f, \mathrm{id}_Q) < 1/i\}$. Obviously, $\mathcal{Z}_i$ is open for every i, and $\bigcap_i \mathcal{Z}_i$ is exactly the collection of all Z-sets.

The definition of cap set can be modified somewhat: a countable increasing union $\bigcup_i M_i$ of Z-sets is always a cap set if for every ϵ there exists a map $h\colon Q \to M_i$ for some i with $d(h, \mathrm{id}_Q) < \epsilon$ and if moreover for every i, $M_i \cong Q$ and M_i is a Z-set in M_{i+1}. This can be seen as follows: Since $M_i \cong Q$, every map into M_i can be approximated by an embedding onto a Z-set in M_i. So assume that $j \geq 1$ and a Z-set $K \subset Q$ are given. Let for some $i > j$ and h, $h\colon Q \to M_i$ have distance less than $\epsilon/4$ to the identity. Let $d(g, h) < \epsilon/4$ and g embeds Q in a Z-set in M_i. Then $d(g, \mathrm{id}) < \epsilon/2$. Applying the Homeomorphism Extension Theorem for M_i, we can extend $(g|M_j)^{-1}\colon g(M_j) \to M_j$ to an autohomeomorphism $f\colon M_i \to M_i$ such that $d(f, \mathrm{id}) < \epsilon/2$. Notice that $d(g, \mathrm{id}) < \epsilon/2$ iff $d(g^{-1}, \mathrm{id}) < \epsilon/2$ and also that $d(g, \mathrm{id}) < \epsilon/2$ and $d(f, \mathrm{id}) < \epsilon/2$ together imply that $d(f \circ g, \mathrm{id}) < \epsilon$. Thus $f \circ g\colon Q \to M_i$ has distance less than ϵ to the identity and leaves M_j pointwise fixed. Applying the Homeomorphism Extension Theorem again, but this time to Q, we extend $f \circ g|K \cup M_j$ to the desired homeomorphism $F\colon Q \to Q$ with $d(F, \mathrm{id}) < \epsilon$ and $F|M_j = \mathrm{id}$.

Lemma 2.1. *Let $A \subset Q$, $A \cong Q$ and suppose A contains an fd cap set $F = \bigcup_i F_i$ such that each F_i is a Z-set in Q. Then A contains a cap set $M = \bigcup_i M_i$ such that each M_i is a Z-set in Q.*

Proof. Write $A = [-1, 1]^\infty$ with metric $d(x, y) = \Sigma_i\, 2^{-i} \cdot |x_i - y_i|$ and for each i, $F_i = \{x|$ for all $j > i\colon x_j = 0\}$. Let p_i denote projection onto the ith coordinate and p_i' projection onto F_i. We will construct a family $\{M_i\}_i$ with for each i, $M_i \supset F_i$, $M_i \cong Q$ and M_i is a Z-set in M_{i+1} as well as in A and Q. The maps p_i' and the remarks above show together that $M = \bigcup_i M_i$ will be a cap set as desired.

First, remark that the collection of nonempty closed subsets of A which are Z-sets both in A and in Q form a G_δ in 2^A, too. So write this collection as $\mathcal{Z}_A = \bigcap_i \mathcal{O}_i$, where each $\mathcal{O}_i$ is an open subset of 2^A. Since F_1 is a Z-set both in A and in Q, there exists an open neighborhood O_1 of F_1 such that any closed set K with $F_1 \subset K \subset O_1$ belongs to $\mathcal{O}_1$. Expand $F_1 = F_1^{(1)}$ to a set $F_1^{(2)} \subset F_3 \cap O_1$ of the form $F_1^{(2)} = \{(x_1, 0, x_2, 0, 0, \cdots)|\, |x_2| \leq c_2\}$. Choose $O_2 \subset \mathrm{Cl}\,(O_2) \subset O_1$ such that O_2 is an open neighborhood of $F_1^{(2)}$ and such that for every closed set K with $F_1^{(2)} \subset K \subset O_2$, $K \in \mathcal{O}_2$. Again choose $F_1^{(3)} \subset F_5 \cap O_2$ of the form $\{(x_1, 0, x_2, 0, x_3, 0, 0, \cdots\}|\, |x_2| \leq c_2$ and $|x_3| \leq c_3\}$. Repeat this construction for all n. Let $M_1 = \mathrm{Cl}\,(\bigcup_n F_1^{(n)}) = \{x|\, x_{2i-1} \leq c_i$ and $x_{2i} = 0$, $i = 2, 3, 4, \cdots\}$. Now for all n, $F_1^{(n)} \subset M_1 \subset \mathrm{Cl}\,(O_{n+1}) \subset O_n$, hence $M_1 \in \bigcap_n \mathcal{O}_n$, thus M_1 is a Z-set both in A and in Q. Obviously $M_1 \cong Q$.

Next we give the construction of M_2, which is similar to the general inductive step. Since M_2 will be deficient only in the coordinates divisible by 2^2, we shall be able to prove that M_1 is a Z-set in M_2, M_1 being deficient in all odd coordinates.

Since $F_2 \cup M_1$ is a Z-set, there is an open neighborhood O_2' of $F_2 \cup M_1$ such that for any closed set K with $F_2 \cup M_1 \subset K \subset O_2'$, $K \in \mathcal{O}_2$. Expand $F_2 = F_2^{(2)}$ to $F_2^{(3)} \subset F_3 \cap O_2'$ such that $F_2^{(3)} \supset F_1^{(2)} = M_1 \cap F_3$, where $F_2^{(3)}$ is of the form $F_2^{(3)} = \{(x_1, x_2, x_3, 0, 0, \cdots) \mid |x_3| \le \phi_{2,3}(x_1, x_2)\}$ with $\phi_{2,3}$: $[-1, 1] \times [-1, 1] \to (0, 1]$ continuous and strictly positive. Choose $O_3' \subset \mathrm{Cl}(O_3') \subset O_2'$ such that for every closed set K with $F_2^{(3)} \cup M_1 \subset K \subset O_3'$, $K \in \mathcal{O}_3$. Since we want deficiency in all coordinates divisible by 4, we skip the fourth coordinate and expand $F_2^{(3)}$ to $F_2^{(4)} \subset F_5 \cap O_3'$ such that $F_2^{(3)} \supset F_1^{(3)} = M_1 \cap F_5$ and $F_2^{(4)}$ is of the form $F_2^{(4)} = \{(x_1, x_2, x_3, 0, x_4, 0, 0, \cdots) \mid |x_3| \le \phi_{2,3}(x_1, x_2)$ and $|x_4| \le \phi_{2,4}(x_1, x_2, x_3)\}$ with $\phi_{2,4}$ continuous and strictly positive. Going on in this fashion we obtain a family $(F_2^{(n)})_{n=2}^{\infty}$ and set $M_2 = \mathrm{Cl}(\bigcup_n F_2^{(n)})$. As before one can show that M_2 is a Z-set both in A and in Q. Using Lemma B, it is straightforward to write M_2 as an inverse limit of finite-dimensional cubes, thus showing that M_2 is homeomorphic to Q. Also, one can specify a coordinatization for M_2 such that M_1 has infinite deficiency in M_2 and therefore is a Z-set in M_2. The inductive step is completely analogous to the construction of M_2, with M_n containing F_n and M_{n-1} and being deficient in all coordinates divisible by 2^n.

It follows from the remarks at the beginning of this section that we have obtained a cap set $M = \bigcup_n M_n$ for A which is a countable union of Z-sets in Q.

Main Theorem 2.2. *If* $X = Q_1 \cup Q_2$, *where* $Q_1 \cong Q_2 \cong Q_1 \cap Q_2 \cong Q$; $Q_1 \cap Q_2$ *is a* Z-*set in* Q_1 *and contains an fd cap set which is a countable union of* Z-*sets in* Q_2 *then* $X \cong Q$.

Proof. Combine 1.3 and 2.1.

The potential usefulness of this reduction to fd cap sets appears in the following corollary (the definition of i-ULC is given in [6]):

Corollary 2.3. *If* $X = Q_1 \cup Q_2$ *where* $Q_1 \cong Q_2 \cong Q_1 \cap Q_2 \cong Q$; $Q_1 \cap Q_2$ *is a* Z-*set in* Q_1 *and either has deficiency* 1 *in* Q_2 *or* $Q_2 \setminus Q_1$ *is* 0-*ULC and* 1-*ULC and dense in* Q_2, *then* $X \cong Q$.

Proof. According to 2.4 and 2.5 of [6], in these two cases every closed finite-dimensional subset of $Q_1 \cap Q_2$ is a Z-set in Q_2.

The above corollary is not a very sharp result: we need only one fd cap set for $Q_1 \cap Q_2$ consisting of Z-sets in Q_2, whereas in the two cases listed every fd cap set for $Q_1 \cap Q_2$ would do.

Question 2. *If* $A \subset Q$, $A \cong Q$, *does there exist an fd cap set for* A *which is a countable union of Z-sets in* Q?

Question 2a. *If* $A \subset Q$, $A \cong Q$, *does* A *contain a copy of* Q *which is a Z-set in* Q?

From a positive answer to Question 2 it would follow that $Q_1 \cup Q_2 \cong Q$ if $Q_1 \cong Q_2 \cong Q_1 \cap Q_2 \cong Q$ and $Q_1 \cap Q_2$ is a Z-set in Q_1. Another possible line of investigation is trying to relax the conditions on the embedding of $Q_1 \cap Q_2$ in Q_1. In particular:

Question 3. *If* $Q_1 \cong Q_2 \cong Q_1 \cap Q_2 \cong Q$ *and* $Q_1 \cap Q_2$ *contains an fd cap set which is a countable union of Z-sets both in* Q_1 *and in* Q_2, *then is* $Q_1 \cup Q_2$ *homeomorphic to* Q?

Question 3a. *If* $Q_1 \cong Q_2 \cong Q_1 \cap Q_2 \cong Q$ *and* $Q_1 \cap Q_2$ *contains fd cap sets* M_1 *and* M_2 *such that* M_i *is a countable union of Z-sets in* Q_i, $i = 1$, 2, *then is* $Q_1 \cup Q_2$ *homeomorphic to* Q?

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