SEMIFREE ACTIONS ON HOMOTOPY SPHERES

BY

KAI WANG

ABSTRACT. In this paper, we study the semifree Z_m actions on homotopy sphere pairs. We show that in some cases the equivariant normal bundle to the fixed point set is equivariantly stably trivial. We compute the rank of the torsion free part of the group of semifree actions on homotopy sphere pairs in some cases. We also show that there exist infinitely many semifree Z_{4s} actions on even dimensional homotopy sphere pairs.

0. Introduction. Let G be a compact Lie group. A differentiable group action of G on a differentiable manifold M^n is a homomorphism $\psi\colon G \to \mathrm{Diff}(M)$, where $\mathrm{Diff}(M)$ is the group of diffeomorphisms of M. Let F^k be the submanifold of fixed points. A group action is semifree if the only isotropy subgroups are the trivial subgroup and the group itself. Under these restrictions, the action is linear in some neighborhood of F^k in M^n in the sense that there is an equivariant vector bundle ν normal to F^k in M^n such that the action restricts on each fiber of ν to a linear automorphism. In this paper, we are only interested in the differentiable semifree actions on homotopy sphere pairs, namely, the actions on homotopy spheres such that the fixed point sets are homotopy spheres. The action $G = S^1$ defines a complex structure on ν (the action on ν is just that induced by the complex multiplication). In [2], Browder proved that for a semifree S^1 action on a homotopy sphere pair, the normal bundle of the fixed point set is stably trivial as a complex vector bundle. The first question which interests us is the following:

Problem 1. *Is it true that the equivariant normal bundle to the fixed point set of a semifree Z_m action on a homotopy sphere pair is equivariantly stably trivial?*

If $G = S^1$ or Z_m, it is known (see [2], [6]) that there are infinitely many semifree G actions on odd dimensional homotopy sphere pairs. As to semifree S^1 actions on even dimensional homotopy sphere pairs, there are only finitely many [3].

Problem 2. *Is it ture that there are only finitely many semifree actions of Z_m on even dimensional homotopy sphere pairs?*

There is a natural group structure on the set of semifree actions on the homotopy sphere pairs with the same local representation. The classification scheme has been given by Browder and Petrie [3] and Rothenberg [5]. Through use of this scheme the rank of the torsion free part of the group of semifree S^1 actions (Z_2 actions, respectively) on homotopy sphere pairs has been computed in [3].

Problem 3. *Calculate the rank of the torsion free part of the group of semifree Z_m actions on homotopy sphere pairs.*

It turns out that these **three** problems are related by the exact sequence of Rothenberg (see §1). We show that if $m \not\equiv 0 \pmod 4$, then, at least for a large family of Z_m actions on homotopy sphere pairs, the answer to Problem 1 is positive. This, in turn, yields solutions to Problems 2 and 3. As for the case $m \equiv 0 \pmod 4$, the answers to Problems 1 and 2 are, in general, negative; but we still are able to solve Problem 3, at least for a large family of groups of Z_m actions on homotopy sphere pairs. These solutions are the main results of this paper.

This paper is organized as follows: In §1, we describe the geometric situation, and we quote the results of Browder, Petrie, and Rothenberg. In §2, we will show how to compute the determinant of some matrices that occur in the G-signature formula. In §3, we study the functions $\Phi_k(\theta)$ which occur as coefficients in the G-signature formula. In §4, we prove our main results.

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1. Known results. Let G be a compact Lie group. A G-manifold M^n is a manifold with a fixed G action on it. Let F^k be the submanifold of fixed points. The action is semifree if it acts freely on the complement of F^k. Let ν be the G-equivariant normal bundle to F^k in M^n. The action on each fiber of ν is linear and represents an $(n - k)$-dimensional representation ρ of G. The conjugacy class of this representation is constant on the connected component of F^k. We always assume that F^k is simply-connected. Let $C(G, \rho)$ be the centralizer of $\rho(G)$ in $O(n - k)$. Then there is a reduction of the structural group of the bundle ν from $O(n - k)$ to $C(G, \rho)$, and this in turn induces a reduction of the group of $TM|F^k$ to the connected component

of the identity of $C(G, \rho) \times O(k)$. Note that if F^k is simply-connected, such a reduction always exists. A (G, ρ)-manifold M^n is a G-manifold such that the local representation is equivalent to ρ and which is furnished with a specific reduction of the group of $TM|F^k$ to the identity component of $C(G, \rho) \times O(k)$. Two (G, ρ)-manifolds M^n, N^n are equivalent if there is a G-equivariant diffeomorphism $f: M^n \to N^n$ which preserves the (G, ρ)-orientation. On the set of equivalence classes of (G, ρ)-manifolds we can define a (G, ρ)-oriented connected sum in a manner formally the same as that for the ordinary oriented connected sum. With this in mind, let $\mathcal{S}^n(G, \rho)$ be the set of equivalence classes of semifree (G, ρ)-manifolds M^n such that M^n and F^k are homotopy spheres.

Proposition 1.1 [6]. *For $k \geq 1$, $\mathcal{S}^n(G, \rho)$ under the (G, ρ)-oriented connected sum is an abelian group.*

Let a_1 and a_2 represent elements of $\mathcal{S}^n(G, \rho)$. We say a_1 is h-cobordant to a_2 if there is a semifree (G, ρ)-manifold W^{n+1} which is homotopy equivalent to $S^n \times [0, 1]$ and $F(G; W)$ is homotopy equivalent to $S^k \times [0, 1]$ in such a way that $\partial_i W$ $(i = 1, 2)$ as a (G, ρ)-manifold is equivalent to a_i. Let $\mathcal{C}^n(G, \rho)$ be the set of h-cobordism classes of $\mathcal{S}^n(G, \rho)$. It is routine to check that the (G, ρ)-oriented connected sum preserves h-cobordisms, and hence $\mathcal{C}^n(G, \rho)$ is a quotient group of $\mathcal{S}^n(G, \rho)$. An element of $\bar{\mathcal{R}}^n(G, \rho)$ is an equivalence class of objects, where an object is

(a) a (G, ρ)-oriented homotopy sphere Σ^n,

(b) a (G, ρ)-oriented imbedding $\psi: S^k \times R^{n-k} \to \Sigma^n$
such that G acts freely on $\Sigma^n - \psi(S^k \times 0)$ and G acts on $S^k \times R^{n-k}$ by $g(x, y) = (x, \rho(g)y)$. An equivalence of two objects (Σ_1^n, ψ_1), (Σ_2^n, ψ_2) is an equivariant diffeomorphism $d: \Sigma_1^n \to \Sigma_2^n$ that preserves the (G, ρ)-orientation and such that $\psi_2 = \psi_1 \cdot d$. $\bar{\mathcal{R}}^n(G, \rho)$ is a group under the (G, ρ)-oriented connected sum. Let b_i $(i = 1, 2)$ represent an element of $\bar{\mathcal{R}}^n(G, \rho)$. We say b_1 is h-cobordant to b_2 is there is a (G, ρ)-manifold W^{n+1} which is homotopy equivalent to $S^n \times [0, 1]$ and a (G, ρ)-oriented imbedding $\psi: S^k \times [0; 1] \times R^{n-k} \to W^{n+1}$ such that G acts freely on $W^{n+1} - \psi(S^k \times [0, 1] \times 0)$ and G acts on $S^k \times [0, 1] \times R^{n-k}$ by $g(x, t, y) = (x, t, \rho(g)y)$ in such a way that $\partial_i W^{n+1}$ $(i = 1, 2)$ as an element of $\bar{\mathcal{R}}^n(G, \rho)$ is equivalent to b_i. Let $\mathcal{R}^n(G, \rho)$ be the set of h-cobordism classes of $\bar{\mathcal{R}}^n(G, \rho)$. It is easy to check that $\mathcal{R}^n(G, \rho)$ is a group under the (G, ρ)-oriented connected sum. There is a map $\omega: \mathcal{R}^n(G, \rho) \to \mathcal{C}^n(G, \rho)$ defined in the obvious way. Let $\gamma: \mathcal{C}^n(G, \rho) \to \Gamma_k + \pi_{k-1}(C(G, \rho))$ be defined as follows. Since the elements of $\pi_{k-1}(C(G, \rho))$ classify the equivariant bundles over S^k, the map γ assigns to each element in $\mathcal{C}^n(G, \rho)$ its fixed point set and the equivariant normal bundle.

Theorem 1.2 (Rothenberg [5]). *The following sequence*

$$\cdots \longrightarrow \mathcal{R}^n(G, \rho) \xrightarrow{\omega} \mathcal{C}^n(G, \rho) \xrightarrow{\gamma} \Gamma_k + \pi_{k-1}(C(G, \rho)) \xrightarrow{\mu} \mathcal{R}^{n-1}(G, \rho) \longrightarrow \cdots$$

is exact, where the map μ is defined in [5].

Theorem 1.3 (Rothenberg). (i) *If n is even,*

$$\mathcal{R}^n(G, \rho) \otimes \mathbf{C} \cong R_{n,m} \oplus \pi_{k-1}(O(n-k)) \otimes \mathbf{C},$$

where $R_{n,m} \subset \mathbf{C}[Z_m]$, the complex group ring, is the subspace generated by $\{g^i + (-1)^n g^{m-i}\}$ $(i = 1, 2, \ldots, [m/2])$; and

(ii) $\mu \otimes \mathbf{C} \cong \Delta \oplus \psi \otimes \mathbf{C}$, *where μ is the map in Theorem 1.2, and ψ: $\pi_{k-1}(C(Z_m, \rho)) \rightarrow \pi_{k-1}(O(n-k))$ is the map induced by the inclusion with Δ defined as follows. For $f: S^{k-1} \rightarrow C(Z_m, \rho)$, let η be the vector bundle over S^k with f as characteristic map. Let Z_m act on η via ρ. Then $\Delta(f) = \sum_{j=1}^m \sigma(S(\eta), g^j)g^j$, where σ is the Atiyah-Singer invariant* [1].

These results were also proved by Browder and Petrie (which were announced in [3]) in the special case where the equivariant normal bundles are equivariantly trivial. The general case appeared in unpublished work of Rothenberg.

Remark 1.4. The rank $R_{n,m}$, which has been computed in [7], is given as follows:

$$\text{rank } R_{n,m} = \begin{cases} (m-1)/2 & \text{if } m \text{ is odd,} \\ m/2 & \text{if } m \text{ is even and } n \text{ is even,} \\ (m-2)/2 & \text{if } m \text{ is even and } n \text{ is odd.} \end{cases}$$

2. Computation of determinants. In this section we show how to compute the determinants of some matrices which will be useful in §3.

Let $m > 2$ and x be positive integers such that $(x, m) = 1$. We define $\alpha_m(x)$ by $0 < \alpha_m(x) < m/2$ and $\alpha_m(x) \pm x \equiv 0 \pmod{m}$ if $m \not\equiv 0 \pmod{4}$, and $0 < \alpha_m(x) < m/4$ and $\alpha_m(x) \pm x \equiv 0 \pmod{m/2}$ if $m \equiv 0 \pmod{4}$. It is easy to see that $\alpha_m(x)$ is well defined. Let

$$\Lambda(m) = \begin{cases} \{k | 0 < k < m/2 \text{ and } (k, m) = 1\} & \text{if } m \not\equiv 0 \pmod 4, \\ \{k | 0 < k < m/4 \text{ and } (k, m) = 1\} & \text{if } m \equiv 0 \pmod 4. \end{cases}$$

Proposition 2.1. $\Lambda(m)$ *is an abelian group with multiplication $x \circ y = \alpha_m(xy)$.*

Proof. Note that $(m, m-1) = 1$. If $m \not\equiv 0 \pmod 4$, $\Lambda(m)$ with the multi-

plication $\alpha_m(xy)$ may be identified in the obvious way with $Z_m^\times/\{1, m-1\}$, where $Z_m^\times$ is the group of units of Z_m. If $m \equiv 0 \pmod 4$, it is easy to see that $(m, m/2 - 1) = (m, m/2 + 1) = 1$. $\Lambda(m)$ with the multiplication $\alpha_m(xy)$ may be identified with $Z_m^\times/\{1, m/2 - 1, m/2 + 1, m - 1\}$. In either case, α_m: $Z_m^\times \to \Lambda(m)$ is the projection map. $\square$

By the fundamental theorem of finite abelian groups, there exist cyclic subgroups $\Lambda_j(m)$ $(j = 1, \ldots, s)$ of $\Lambda(m)$ such that $\Lambda(m)$ is a direct product $\Pi_{j=1}^s \Lambda_j(m)$. Let $e_j \in \Lambda_j(m)$ be generators that are fixed once and for all. Let t_j be the order of $\Lambda_j(m)$. For each $x \in \Lambda(m)$, there are well defined $\phi_j(x) \in Z_{t_j}$ such that $x = \alpha_m(\Pi_{j=1}^s e_j^{\phi_j(x)})$. The following lemma is obvious.

Lemma 2.2. *For $x, y \in \Lambda(m)$, $\phi_j(\alpha_m(xy)) \equiv \phi_j(x) + \phi_j(y) \pmod{t_j}$.*

Let r be a positive integer, which will be fixed throughout this section; and let x be a positive integer such that $(x, m) = 1$. We define $\delta_m(x)$ as follows:

Case i. $m \not\equiv 0 \pmod 4$,

$$\delta_m(x) \equiv \begin{cases} 0 \pmod 2 & \text{if } x - [x/m]m < m/2, \\ r \pmod 2 & \text{otherwise.} \end{cases}$$

Case ii. $m \equiv 0 \pmod 4$,

$$\delta_m(x) \equiv \begin{cases} 0 & \pmod 2 \quad \text{if } 0 < x - [x/m]m < m/4, \\ r - 1 & \pmod 2 \quad \text{if } m/4 < x - [x/m]m < m/2, \\ 1 & \pmod 2 \quad \text{if } m/2 < x - [x/m]m < 3m/4, \\ r & \pmod 2 \quad \text{if } 3m/4 < x - [x/m]m < m. \end{cases}$$

From now on, in order to simplify notation, we shall write $\Lambda = \Lambda(m)$, $\alpha(x) = \alpha_m(x)$, etc., when m is understood.

Lemma 2.3. *For positive integers a, b such that $(a, m) = (b, m) = 1$, $\delta(a\alpha(b)) \equiv \delta(ab) + \delta(b) \pmod 2$.*

Proof. *Case i.* $m \not\equiv 0 \pmod 4$. Observe that $\alpha(x) \equiv x$ or $-x \pmod m$ according as $\delta(x) \equiv 0 \pmod 2$, or $r \pmod 2$, respectively. Thus $a\alpha(b) \equiv ab$ or $-ab \pmod m$ according as $\delta(b) \equiv 0 \pmod 2$ or $r \pmod 2$, respectively. The lemma is now clear.

Case ii. $m \equiv 0 \pmod 4$. The same method applies in this case. $\square$

Let $\{\Lambda_j\}$, $j = 1, \ldots, t$, be the maximal family of factors of $\Lambda = \Pi_{j=1}^s \Lambda_j$ that have the property that either $t_j \not\equiv 0 \pmod 4$ or $\delta(e_j^{t_j}) \equiv 0 \pmod 2$. Let $\Lambda^* = \Pi_{j=1}^t \Lambda_j$. We will define a map $\psi: \Lambda^* \to Z_4$ as follows: For $j = 1, \ldots, t$, let $\psi(e_j)$ be a solution, which will be fixed once and for all, of

(2.4)
$$t_j \psi(e_j) \equiv 2\delta(e_j^{t_j}) \quad (\text{mod } 4).$$

Then define

(2.5)
$$\psi(\alpha(e_j^k)) \equiv k\psi(e_j) + 2\delta(e_j^k) \quad (\text{mod } 4).$$

Lemma 2.6. $\psi: \Lambda_j \to Z_4$ *is well defined.*

Proof. Since $\alpha(e_j^{t_j+k}) = \alpha(e_j^k)$,

$$\psi(\alpha(e_j^{t_j+k})) \equiv (k + t_j)\psi(e_j) + 2\delta(e_j^{t_j+k}) \quad (\text{mod } 4),$$

$$\equiv k\psi(e_j) + 2\delta(e_j^{t_j}) + 2\delta(e_j^{t_j+k}) \quad (\text{mod } 4)$$

by (2.4),

$$\equiv k\psi(e_j) + 2\delta(e_j^k \alpha(e_j^{t_j})) \quad (\text{mod } 4)$$

by Lemma 2.3,

$$\equiv k\psi(e_j) + 2\delta(e_j^k) \quad (\text{mod } 4),$$

$$\equiv \psi(\alpha(e_j^k)) \quad (\text{mod } 4),$$

as required. $\square$

For $x \in \Lambda^*$, $x = \alpha(\prod_{j=1}^t x_j)$, $x_j \in \Lambda_j$. Define

(2.7)
$$\psi(x) \equiv \sum_{j=1}^t \psi(x_j) + 2\delta\left(\prod_{j=1}^t x_j\right) \quad (\text{mod } 4).$$

It is obvious that $\psi: \Lambda^* \to Z_4$ is well defined.

Lemma 2.8. *For* $j = 1, \ldots, t$,

$$\psi(\alpha(e_j^{p+q})) \equiv \psi(\alpha(e_j^p)) + \psi(\alpha(e_j^q)) + 2\delta(\alpha(e_j^p)\alpha(e_j^q)) \quad (\text{mod } 4).$$

Proof. By (2.5),

$$\psi(\alpha(e_j^{p+q})) \equiv (p + q)\psi(e_j) + 2\delta(e_j^{p+q}) \quad (\text{mod } 4),$$

$$\psi(\alpha(e_j^p)) \equiv p\psi(e_j) + 2\delta(e_j^p) \quad (\text{mod } 4),$$

$$\psi(\alpha(e_j^q)) \equiv q\psi(e_j) + 2\delta(e_j^q) \quad (\text{mod } 4).$$

Therefore

$$\psi(a(e_j^{p+q})) - (\psi(a(e_j^p)) + \psi(a(e_j^q))) \equiv 2\delta(e_j^p) + 2\delta(e_j^q) + 2\delta(e_j^{p+q}) \quad (\text{mod } 4),$$

$$\equiv 2\delta(e_j^p) + 2\delta(e_j^p a(e_j^q)) \quad (\text{mod } 4)$$

by Lemma 2.3,

$$\equiv 2\delta(a(e_j^p)a(e_j^q)) \quad (\text{mod } 4)$$

by Lemma 2.3 again, as required. □

Proposition 2.9. *For $x, y \in \Lambda^*$, $\psi(a(xy)) \equiv \psi(x) + \psi(y) + 2\delta(xy) \,(\text{mod } 4)$.*

Proof. Let $x = a(\Pi_{j=1}^t x_j), y = a(\Pi_{j=1}^t y_j)$. Then

$$\psi(a(xy)) = \psi\left(a\left(\prod_{j=1}^t x_j y_j\right)\right)$$

$$\equiv \sum_{j=1}^t \psi(a(x_j y_j)) + 2\delta\left(\prod_{j=1}^t a(x_j y_j)\right) \quad (\text{mod } 4)$$

by (2.7),

$$\equiv \sum_{j=1}^t (\psi(x_j) + \psi(y_j) + 2\delta(x_j y_j)) + 2\delta\left(\prod_{j=1}^t a(x_j y_j)\right) \quad (\text{mod } 4)$$

by Lemma 2.8; while

$$\psi(x) + \psi(y) \equiv \sum_{j=1}^t (\psi(x_j) + \psi(y_j)) + 2\delta\left(\prod_{j=1}^t x_j\right) + 2\delta\left(\prod_{j=1}^t y_j\right) \quad (\text{mod } 4).$$

Consequently,

$$\psi(a(xy)) - (\psi(x) + \psi(y))$$

$$\equiv 2\sum_{j=1}^t \delta(x_j y_j) + 2\delta\left(\prod_{j=1}^t a(x_j y_j)\right) + 2\delta\left(\prod_{j=1}^t x_j\right) + 2\delta\left(\prod_{j=1}^t y_j\right) \quad (\text{mod } 4).$$

By Lemma 2.3,

$$2\delta\left(\prod_{j=1}^t a(x_j y_j)\right) = 2\delta\left(\left(\prod_{j=2}^t a(x_j y_j)a(x_1 y_1)\right)\right)$$

$$\equiv 2\delta\left(\left(x_1 y_1 \prod_{j=2}^t a(x_j y_j)\right) + \delta(x_1 y_1)\right) \quad (\text{mod } 4);$$

and by successive applications of Lemma 2.3,

$$\equiv 2\delta\left(\prod_{j=1}^{t} x_j y_j\right) + 2 \sum_{j=1}^{t} \delta(x_j y_j) \quad \text{(mod 4)}.$$

Thus

$$\psi(\alpha(xy)) - (\psi(x) + \psi(y)) \equiv 2\delta\left(\prod_{j=1}^{t} x_j\right) + 2\delta\left(\prod_{j=1}^{t} y_j\right) + 2\delta\left(\prod_{j=1}^{t} x_j y_j\right) \quad \text{(mod 4)},$$

$$\equiv 2\delta\left(\prod_{j=1}^{t} x_j\right) + 2\delta\left(\prod_{j=1}^{t} x_j \alpha\left(\prod_{j=1}^{t} y_j\right)\right) \quad \text{(mod 4)}$$

by Lemma 2.3,

$$\equiv 2\delta\left(\alpha\left(\prod_{j=1}^{t} x_j\right)\alpha\left(\prod_{j=1}^{t} y_j\right)\right) \quad \text{(mod 4)}$$

by Lemma 2.3 again,

$$\equiv 2\delta(xy),$$

as required. □

Theorem 2.10. *For each* $p \in \Lambda^*$, *let* a_p *be any complex number corresponding to* p, *then*

$$\det\left((-1)^{\delta(hk)} a_{\alpha(hk)}\right)_{h,k \in \Lambda^*} = \pm \prod_{p \in \Lambda^*}\left(\sum_{q \in \Lambda^*} a_q i^{\psi(q)}\left(\prod_{j=1}^{t} \omega_j^{\phi_j(p)\phi_j(q)}\right)\right),$$

where ω_j *is a primitive* t_j*th root of unity.*

Proof. Let $r(h)$ be defined by the equation $\alpha(hr(h)) = 1$. Regard the a_p's as indeterminants, and let

$$A = \left((-1)^{\delta(hk)} a_{\alpha(hk)}\right)_{h,k \in \Lambda^*, k \neq 1}.$$

Fix $p \in \Lambda^*$, and add $i^{\psi(q)}(\prod_{j=1}^{t} \omega_j^{\phi_j(p)\phi_j(q)})$ times the qth column to the first column. Then

$$\det\left((-1)^{\delta(hk)}a_{\alpha(hk)}\right)_{h,k\in\Lambda^*}$$

$$=\det\left(\left.\begin{array}{c}\vdots\\[2pt]\displaystyle\sum_{q\in\Lambda^*}a_{\alpha(hq)}i^{\psi(q)+2\,\delta(hq)}\left(\prod_{j=1}^{t}\omega_j^{\phi_j(p)\phi_j(q)}\right)\\[2pt]\vdots\end{array}\right|A\right)_{h\in\Lambda^*}.$$

We now compute

$$\sum_{q\in\Lambda^*}a_{\alpha(hq)}i^{\psi(q)+2\,\delta(hq)}\left(\prod_{j=1}^{t}\omega_j^{\phi_j(p)\phi_j(q)}\right)$$

$$=i^{\psi(\tau(h))}\left(\prod_{j=1}^{t}\omega_j^{\phi_j(\tau(h))\phi_j(p)}\right)\left(\sum_{q\in\Lambda^*}a_{\alpha(hq)}i^{\psi(\alpha(hq))}\left(\prod_{j=1}^{t}\omega_j^{\phi_j(p)\phi_j(\alpha(hq))}\right)\right)$$

$$=i^{\psi(\tau(h))}\left(\prod_{j=1}^{t}\omega_j^{\phi_j(\tau(h))\phi_j(p)}\right)\left(\sum_{q\in\Lambda^*}a_q i^{\psi(q)}\left(\prod_{j=1}^{t}\omega_j^{\phi_j(p)\phi_j(q)}\right)\right).$$

Thus $\sum_{q\in\Lambda^*}a_q i^{\psi(q)}(\prod_{j=1}^{t}\omega_j^{\phi_j(p)\phi_j(q)})$ is a factor of $\det\left((-1)^{\delta(hk)}a_{\alpha(hk)}\right)_{h,k\in\Lambda^*}$. It is easy to see that, in general, for $p'\neq p$,

$$\sum_{q\in\Lambda^*}a_q i^{\psi(q)}\left(\prod_{j=1}^{t}\omega_j^{\phi_j(p)\phi_j(q)}\right)\neq\sum_{q\in\Lambda^*}a_q i^{\psi(q)}\left(\prod_{j=1}^{t}\omega_j^{\phi_j(p')\phi_j(q)}\right).$$

Thus

$$\det\left((-1)^{\delta(hk)}a_{\alpha(hk)}\right)_{h,k\in\Lambda^*}=c\prod_{p\in\Lambda^*}\left(\sum_{q\in\Lambda^*}a_q i^{\psi(q)}\left(\prod_{j=1}^{t}\omega_j^{\phi_j(p)\phi_j(q)}\right)\right)$$

for some constant c, which is easily seen to be ± 1. $\square$

Let $\operatorname{Im}(x+iy)=y$, where x and y are real.

Proposition 2.11. *Suppose for all $q\in\Lambda^*$, the a_q are real. Then if $p\in\Lambda^*$,*

$$\operatorname{Im}\sum_{q\in\Lambda^*}a_q i^{\psi(q)}\left(\prod_{j=1}^{t}\omega_j^{\phi_j(p)\phi_j(q)}\right)$$

$$=\sum_{q\in\Lambda^*;\,q<\tau(q)}(a_q-(-1)^{\psi(\tau(q))}a_{\tau(q)})\left(\operatorname{Im}\left(i^{\psi(q)}\prod_{j=1}^{t}\omega_j^{\phi_j(p)\phi_j(q)}\right)\right).$$

Proof. Let $A(q) = i^{\psi(q)}\prod_{j=1}^{t}\omega_j^{\phi_j(p)\phi_j(q)}$, and let $\overline{A(q)}$ be the complex conjugate of $A(q)$. It is easy to check that $\overline{A(q)} = (-1)^{\psi(q)}A(r(q))$. Thus

$$\sum_{q \in \Lambda^*} a_q A(q) - \sum_{q \in \Lambda^*} a_q \overline{A(q)}$$

$$= \sum_{q < \tau(q)} a_q A(q) + \sum_{q < \tau(q)} a_{\tau(q)}A(\tau(q))$$

$$- \sum_{q < \tau(q)} (-1)^{\psi(q)}a_q A(\tau(q)) - \sum_{q < \tau(q)} (-1)^{\psi(q)}a_{\tau(q)}A(q)$$

$$= \sum_{q < \tau(q)} a_q(A(q) - (-1)^{\psi(q)}A(\tau(q))) - (-1)^{\psi(\tau(q))}$$

$$\cdot \sum_{q < \tau(q)} a_{\tau(q)}(A(q) - (-1)^{\psi(\tau(q))}A(\tau(q)))$$

$$= \sum_{q < \tau(q)} (a_q - (-1)^{\psi(\tau(q))}a_{\tau(q)})(A(q) - \overline{A(q)}). \quad \square$$

Remark 2.12. If r is even, then $\delta(x) \equiv 0 \pmod 2$ for all $x \in \Lambda$. Hence $\Lambda^* = \Lambda$, and $\psi(x) = 0$ for all $x \in \Lambda$.

Remark 2.13. If m is an odd prime, then Λ is cyclic.

Remark 2.14. If Λ is cyclic and r is even, then

$$\det(a_{\alpha(hk)})_{h,k \in \Lambda} = \pm \prod_{p=1}^{\lambda}\left(\sum_{q=1}^{\lambda} a_q \omega^{pq}\right),$$

where $\lambda = |\Lambda|$ and ω is a primitive λth root of unity.

The following example shows the necessity of restricting the definition of ψ to Λ^*.

Example 2.15. If $m = 17$ and r is odd, there exists no map $\psi: \Lambda \to Z_4$ such that Proposition 2.9 holds.

Proof. Suppose there were $\psi: \Lambda \to Z_4$ such that Proposition 2.9 holds. Let $e \in \Lambda$ be a generator, then

$$\sum_{x \in \Lambda} \psi(\alpha(xe)) \equiv \sum_{x \in \Lambda} \psi(x) + 8\psi(e) + 2\sum_{x \in \Lambda} \delta(xe) \pmod 4.$$

Consequently,

$$\sum_{x \in \Lambda} \delta(xe) \equiv 0 \pmod 2.$$

But

$$\sum_{x \in \Lambda} \delta(xe) \equiv \sum_{t=0}^{7} \delta(\alpha(e^t)e) \equiv \sum_{t=0}^{7} (\delta(e^{t+1}) + \delta(e^t)) \equiv \delta(e^8) \quad (\text{mod } 2).$$

Thus $\delta(e^8) \equiv 0$ (mod 2). It can be easily checked that if e is a generator of Λ, $e^8 \equiv -1$ (mod 17). Hence $\delta(e^8) \equiv 1$ (mod 2). This contradiction demonstrates the nonexistence of ψ. $\square$

3. **Coefficients in the G-signature formula.** Let the functions $\Phi_k(\theta)$, $\theta \in [0, \pi]$, be defined by the equation

$$\mathfrak{M}_\theta = \prod_j \left(\frac{\tanh(i\theta/2)}{\tanh((x_j + i\theta)/2)} \right) = \sum_k \Phi_k(\theta)c_k + \text{decomposable terms},$$

where c_k is the kth Chern class.

Lemma 3.1.

(i)
$$\Phi_{2r-1}(\theta) = (-1)^r \frac{i}{\sin^{2r-1}\theta} P(\cos^2\theta),$$

(ii)
$$\Phi_{2r}(\theta) = (-1)^{r+1} \frac{\cos\theta}{\sin^{2r}\theta} Q(\cos^2\theta),$$

where P and Q are some polynomials with nonnegative coefficients.

Proof. By a result of F. Hirzebruch [4],

$$\sum_{j=0}^{\infty} (-1)^j \Phi_j(\theta)z^j = 1 - z \frac{d}{dz} \left(\frac{\tanh(i\theta/2)}{\tanh((z + i\theta)/2)} \right)$$

$$= 1 + z \operatorname{csch}(z + i\theta).$$

The lemma can now be proved easily by induction.

The following corollaries follow easily by elementary calculations.

Corollary 3.2. (i) $\Phi_r(\pi - \theta) = (-1)^{r-1}\Phi_r(\theta)$,

(ii) $\Phi_r(\pi + \theta) = -\Phi_r(\theta)$,

(iii) $\Phi_r(2\pi - \theta) = (-1)^r\Phi_r(\theta)$.

Corollary 3.3. (i) $\Phi_{4k}(\theta)$ *is monotonic increasing on* $[0, \pi]$;

(ii) $\Phi_{4k+1}(\theta)i$ *is monotonic decreasing on* $[0, \pi/2]$, *and monotonic increasing on* $[\pi/2, \pi]$;

(iii) $\Phi_{4k+2}(\theta)$ *is monotonic decreasing on* $[0, \pi]$;

(iv) $\Phi_{4k+3}(\theta)i$ *is monotonic increasing on* $[0, \pi/2]$, *and monotonic decreasing on* $[\pi/2, \pi]$;

(v) $|\Phi_r(\theta)|$ *is monotonic decreasing on* $[0, \pi/2]$ *and monotonic increasing on* $[\pi/2, \pi]$.

Corollary 3.4. *For each integer* $m > 2$, $|\Phi_r(2\pi/m)| < 2^{-r}(\sec(\pi/m))|\Phi_r(\pi/m)|$.

Let $m > 2$ be an integer.

Corollary 3.5. *There exist integers* s *and* t *such that* $s \neq t$, $0 < s$, $t < m/2$, *and* $(s, m) = (t, m) = 1$ *and such that* $\Phi_r(2s\pi/m) = (-1)^{r-1}\Phi_r(2t\pi/m)$ *if and only if* $m \equiv 0 \pmod 4$.

Proof. It is easy to see that $2s + 2t = m$, so m is even. Since $(s, m) = (t, m) = 1$, s, t must be odd. Hence $m \equiv 0 \pmod 4$.

Conversely, suppose $m = 4k$, and choose $0 < s < 2k$ such that $(s, m) = 1$. Then s is odd and $(s, k) = 1$. There exist p, q such that $ps + qk = 1$. So $(2p - q)k - p(2k - s) = 1$. Thus $((2k - s), k) = 1$. It is obvious that $2k - s$ is odd, $2k - s \neq s$, and $\Phi_r(2s\pi/m) = (-1)^{r-1}\Phi_r(2(2k - s)\pi/m)$. □

Consider the following matrix

$$\widetilde{\Phi}_r = \left(\Phi_r\left(\frac{2hk\pi}{m}\right)\right)_{h,k \in \Lambda},$$

where Λ is defined in §2. $\widetilde{\Phi}_r$ plays the dominant role in the G-signatures of cyclic group actions on homotopy sphere pairs.

For any positive integer x such that $(m, x) = 1$, let $\alpha(x)$, $\delta(x)$ be defined as in §2. It is easy to see that

$$\Phi_r\left(\frac{2hk\pi}{m}\right) = (-1)^{\delta(hk)}\Phi_r\left(\frac{2\alpha(hk)\pi}{m}\right),$$

where h, $k \in \Lambda$. As a special case of Theorem 2.10 we have the following

Theorem 3.6. *Let* Λ^*, $\psi(q)$, $\phi_i(q)$, *and* ω_i *be defined as in* §2. *Then*

$$\det\left(\Phi_r\left(\frac{2hk\pi}{m}\right)\right)_{h,k \in \Lambda^*} = \pm \prod_{p \in \Lambda^*}\left(\sum_{q \in \Lambda^*}\Phi_r\left(\frac{2q\pi}{m}\right)i^{\psi(q)}\left(\prod_{j=1}^{t}\omega_j^{\phi_j(p)\phi_j(q)}\right)\right).$$

Corollary 3.7. *Let* $\lambda = |\Lambda^*|$. *If* $r \geq \ln((\lambda - 1)\sec(\pi/m))/\ln 2$, *then*

$$\det(\Phi_r(2hk\pi/m))_{h,k \in \Lambda^*} \neq 0.$$

Proof. It is easy to see that if b_i, β_i $(i = 0, \ldots, n)$ are complex num-

bers such that $|b_0| > \sum_{j=1}^n |b_j|$ and $|\beta_i| = 1$ $(i = 0, \dots, n)$, then $\sum_{i=0}^n b_i \beta_i \neq 0$. Thus Corollary 3.7 is an easy consequence of Corollary 3.4 and Theorem 3.6. □

For small m's, we may compute directly to get

Corollary 3.8. $\det (\Phi_r(2hk\pi/m))_{h,k \in \Lambda^*} \neq 0$ *for* $m \leq 12$.

Remark 3.9. Based on these observations, it is reasonable to conjecture that $\det (\Phi_r(2hk\pi/m))_{h,k \in \Lambda^*} \neq 0$. In fact, Rothenberg conjectured in his talk at the Second Conference on Transformation Groups that $\det \tilde{\Phi}_r \neq 0$ for each odd prime m.

Remark 3.10. The first estimation of r for a fixed odd prime m for which $\det \tilde{\Phi}_r \neq 0$ was given by Schultz [8]. His method can also be applied for m which are not necessarily odd primes. However, it appears to us that Schultz's method cannot be used to show that $\det \tilde{\Phi}_r \neq 0$ for all m and all r. Our approach is different from his and is, perhaps, a hopeful way to prove the conjecture.

4. The main results. For a positive integer $m \neq 2$, let $\Lambda' = \{k | 0 < k < m/2, (k, m) = 1\}$. Let $(\Sigma^{2n}, F^{2r}; Z_m)$ be a semifree Z_m action on a homotopy sphere pair. The equivariant normal bundle ν to F^{2r} in Σ^{2n} splits equivariantly into a Whitney sum $\nu = \bigoplus_{j \in \Lambda'} \nu_j$ so that each factor ν_j is invariant under the Z_m action, and the restriction of the action to each fiber of ν_j is just complex multiplication by $\exp(2j\pi i/m)$. Let g be a generator of Z_m. By the Atiyah-Singer G-signature Theorem [1, 6.12], we have

$$\text{Sign}(g^h, \Sigma^{2n})$$
$$= 2^r \prod_{k \in \Lambda'} (i \tan(hk\pi/m))^{-n_k} \prod_{k \in \Lambda'} \left(\sum_j \Phi_j \left(\frac{2hk\pi}{m} \right) c_j(\nu_k) \right) \cap [F^{2r}]$$

for $h \in \Lambda'$ and where $n_k = \dim_C \nu_k$. Note that since F^{2r} is a suspension, products of positive dimensional classes are zero. Hence

$$\text{Sign}(g^h, \Sigma^{2n}) = K_h \left(\sum_{k \in \Lambda'} \Phi_r \left(\frac{2hk\pi}{m} \right) c_r(\nu_k) \right) \cap [F^{2r}],$$

where $h \in \Lambda'$ and K_h are constant depending only on h.

By definition, $\text{Sign}(g^h, \Sigma^{2n})$ is defined on the Z_m module $H^n(\Sigma^{2n}, Q)$, which is zero. So $\text{Sign}(g^h, \Sigma^{2n}) = 0$ for all $h \in \Lambda'$. If $m \not\equiv 0 \pmod 4$, $\Lambda' = \Lambda$, where Λ is as in §2. If $\det \tilde{\Phi}_r \neq 0$, where $\tilde{\Phi}_r$ is as in §3, it is easy to see that $c_r(\nu_k) = 0$ for all $k \in \Lambda$. Since the stable complex vector bundles

over spheres are determined by their Chern classes, the ν_j are stably trivial as complex vector bundles. Thus we have proved the following

Theorem 4.1. *For $m \neq 2$ and $m \not\equiv 0 \pmod 4$, let $(\Sigma^{2n}, F^{2r}; Z_m)$ be a semifree Z_m action on a homotopy sphere pair. If $\det \widetilde{\Phi}_r \neq 0$, then the equivariant normal bundle ν to F^{2r} in Σ^{2n} is equivariantly stably trivial.*

Remark 4.2. We say a Z_m, $m \neq 2$, equivariant bundle is trivial if each equivariant factor is trivial as a complex vector bundle.

Remark 4.3. This result has also been discovered and proved by Schultz [8] by a different argument.

Let $\rho: Z_m \to U(n-r)$ be a fixed unitary fixed point free representation of complex dimension $n - r$. If $m \not\equiv 0 \pmod 4$, and $\det \widetilde{\Phi}_r \neq 0$, one sees, as an easy consequence of Theorem 4.1, that the image of γ in Theorem 1.2 is Γ_{2r}, a finite group. It has also been proved in [5] that $\mathcal{R}^{2n}(Z_m, \rho)$ is finite. Hence we have

Theorem 4.4. *For $m \neq 2$ and $m \not\equiv 0 \pmod 4$, let $\rho: Z_m \to U(n-r)$ be a fixed unitary fixed point free representation of complex dimension $n - r$. If $\det \widetilde{\Phi}_r \neq 0$, then $\mathcal{C}^{2n}(Z_m, \rho)$ is finite.*

As an easy consequence of Theorems 1.2, 1.3 and 4.4, we have

Corollary 4.5. *Let m, ρ be as in Theorem 4.4. If $\det \widetilde{\Phi}_r \neq 0$, then*

$$\text{rank } \mathcal{C}^{2n-1}(Z_m, \rho) \otimes \mathbf{C} = \text{rank } R_{n,m} + \text{rank } \pi_{2r-1}(O(2n-2r)) \otimes \mathbf{C}$$

$$- \text{rank } \pi_{2r-1}(C(Z_m, \rho)) \otimes \mathbf{C},$$

where $\text{rank } R_{n,m}$ is as in Remark 1.4; and $\rho = \Sigma_{j \in \Lambda} n_j t^j$,

$$\text{rank } \pi_{2r-1}(C(Z_m, \rho)) \otimes \mathbf{C} = \text{Card}\{n_j | n_j \geq r\}.$$

Remark 4.6. Corollary 3.7 shows that for fixed m and r even and sufficiently large, $\det \widetilde{\Phi}_r \neq 0$.

In the rest of this section, we will consider the case $m \equiv 0 \pmod 4$. Let $m = 4s$. If $n \geq 3r$, we consider $\rho = n_1 t + n_2 t^{2s-1}$ for $n_1 \geq r$, $n_2 \geq r$. Then $C(Z_m, \rho) \cong U(n_1) \times U(n_2)$, and $\pi_{2r-1}(U(n_1) \times U(n_2)) = Z \oplus Z$. The map Δ: $\pi_{2r-1}(U(n_1) \times U(n_2)) \to C[Z_m - 1]$ in Theorem 1.3 can be computed explicitly as follows: By [1, 6.15], $\Delta(f_1, f_2)(g^{2s}) = \text{Sign}((S^{2r})^2) = 0$, and

$$\Delta(f_1, f_2)(g^h) = K_h\left(\Phi_r\left(\frac{2h\pi}{4s}\right)(f_1^* c_r + (-1)^{r-1} f_2^* c_r)\right) \cap [S^{2r}],$$

where $h \neq 2s$ and $f_i^* c_r$ is the Chern class of the complex vector bundle induced by the map $f_i \colon S^{2r-1} \to U(n_i)$ $(i = 1, 2)$. It is clear that $\dim_C \ker \Delta = 1$. Note that if r is odd, the map Ψ in Theorem 1.3 vanishes. Hence we have, by Theorem 1.3,

Theorem 4.7. *There exist infinitely many inequivalent semifree Z_{4s} actions on S^{2n} with fixed point set S^{2r} for $n \geq 3r$ and r odd.*

If r is even, $n \geq 5r$ and $s \geq 3$, let $\rho = n_1 t + n_2 t^{s-1} + n_3 t^{s+1} + n_4 t^{2s-1}$ for $n_i \geq r$ $(i = 1, 2, 3, 4)$. A similar computation shows that $\dim_C \ker \Delta = 2$ and $\dim_C (\Psi | \ker \Delta) = 1$. Hence we have, by Theorem 1.3,

Theorem 4.8. *There exist infinitely many inequivalent semifree Z_{4s} actions on S^{2n} with fixed point set S^{2r} for $n \geq 5r$, $s \geq 3$ and r even.*

Remark 4.9. This is the first known family of infinitely many semifree actions on even dimensional spheres. It should be noted that the previous attempt to prove this sort of result which was announced in [9] is false.

Remark 4.10. Let ρ be the representation as in the proof of Theorem 4.7 (Theorem 4.8, respectively). By the proof of Theorem 4.7 (Theorem 4.8, respectively), $\dim_C \ker \mu \otimes C = 1$. By Theorem 1.2, $\dim_C \operatorname{Im} \gamma \otimes C = 1$ where $\gamma \colon C^{2n}(Z_{4s}, \rho) \to \Gamma_{2k} + \pi_{2k-1}(C(Z_{4s}, \rho))$ is the classifying map of the equivariant normal bundles to the fixed point sets of semifree Z_{4s} actions on homotopy sphere pairs. Thus, in general, the equivariant normal bundle to the fixed point set of a semifree Z_{4s} action on a homotopy sphere pair is not equivariantly stably trivial. Nevertheless, we have the following:

Theorem 4.11. *Let $(\Sigma^{2n}, F^{2r}; Z_{4s})$ be a semifree Z_{4s} action on a homotopy sphere pair. If $\det \tilde{\Phi}_r \neq 0$, then the normal bundle of F^{2r} in Σ^{2n} is stably trivial as a complex (real, respectively) vector bundle if r is odd (even, respectively), where $\tilde{\Phi}_r$ is as in §3.*

Proof. Note that $\Phi_r(2h(2s-k)\pi/4s) = (-1)^r \Phi_r(2hk\pi/4s)$ and $\Lambda' = \{k \mid k \in \Lambda \text{ or } 2s - k \in \Lambda\}$, where Λ is as in §2. Observe that since F^{2r} is a homotopy sphere, the only nonzero classes are in dimension $2r$, and, since F^{2r} is a suspension, products of positive dimensional classes are zero. Hence

$$\operatorname{Sign}(g^h, \Sigma^{2n}) = K_h \left(\sum_{k \in \Lambda'} \Phi_r \left(\frac{2hk\pi}{4s} \right) c_r(\nu_k) \right) \cap [F^{2r}]$$

$$= K_h \left(\sum_{k \in \Lambda} \Phi_r \left(\frac{2hk\pi}{4s} \right) c_r(\nu_k \oplus (-1)^{r-1} \nu_{2s-k}) \right) \cap [F^{2r}].$$

By the argument used to prove Theorem 4.1, it follows that the direct sums $\nu_k \oplus (-1)^{r-1}\nu_{2s-k}$ ($k \in \Lambda$) are stably trivial as complex vector bundles. If r is odd,

$$\nu = \bigoplus_{k \in \Lambda'} \nu_k = \bigoplus_{k \in \Lambda} (\nu_k \oplus \nu_{2s-k}).$$

Thus ν is stably trivial as a complex vector bundle. If r is even, ν_k is stably equivalent to ν_{2s-k}, and

$$\nu = \bigoplus_{k \in \Lambda'} \nu_k = 2 \bigoplus_{k \in \Lambda} \nu_k.$$

Note that $\pi_{4j}(0) = Z_2$ or 0. Thus ν is stably trivial as a real vector bundle. $\square$

Remark 4.12. The above argument also shows that for a semifree Z_{4s} action on a homotopy sphere pair (Σ^{2n}, F^{2r}), with a local representation $\rho = \Sigma_{j \in \Lambda'} n_j t^j$ such that either $n_j < r$ or $n_{2s-j} < r$ for $j \in \Lambda'$, the equivariant normal bundle to F^{2r} is equivariantly stably trivial under the assumption $\det \widetilde{\Phi}_r \neq 0$.

Finally we can prove the following theorem by an argument similar to the one we used to prove Corollary 4.5.

Theorem 4.13. *If* $\det \widetilde{\Phi}_r \neq 0$, *then* $\operatorname{rank} \mathcal{C}^{2n}(Z_{4s}, \rho) \otimes \mathbf{C} = r(\rho)$ *and*

$$\operatorname{rank} \mathcal{C}^{2n-1}(Z_{4s}, \rho) \otimes \mathbf{C} = \operatorname{rank} R_{n,4s} + \operatorname{rank} \pi_{2r-1}(O(2n-2r)) \otimes \mathbf{C}$$

$$- \operatorname{rank} \pi_{2r-1}(C(Z_{4s}, \rho)) \otimes \mathbf{C} + r(\rho),$$

where

$$r(\rho) = \operatorname{Card}\{n_j | n_j \geq r\} - \operatorname{Card}\{n_j | n_j \geq r, \; n_{2s-j} \geq r\}.$$

Remark 4.14. Again Corollary 3.7 shows for fixed m and r even and sufficiently large, $\det \widetilde{\Phi}_r \neq 0$.

Remark 4.15. Our Theorems 4.1, 4.11 and Remark 4.10 to Theorems 4.7 and 4.8 settle Problem 1 for a large class of cases. Theorems 4.4, 4.7 and 4.8 do the same to Problem 2. Finally, Corollary 4.5 to Theorem 4.4 and Theorem 4.13 are related to Problem 3.

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DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK AT BUFFALO, AMHERST, NEW YORK 14226