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FINITE PROJECTIONS IN TENSOR PRODUCT VON NEUMANN ALGEBRAS

BY

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ABSTRACT. The work of Bures, Moore, Takenouchi, Hill and Størmer on the type classification of infinite tensor products of factors is extended to the nonfactor case.

1. **Introduction.** The basic result of this paper is that if a tensor product von Neumann algebra has a nonzero finite projection then it has a nonzero finite tensor product projection.

For tensor products of finitely many algebras this is a result of Sakai [8, Theorem 2]. For tensor products of infinitely many algebras, the first statement of this fact is due to Moore, who proved it for certain restricted tensor products of discrete factors [7, Lemma 5.2]. The restriction made by Moore was removed by Hill [5, Theorem 3.20] and by Takenouchi [10, §3]. In the present work the component algebras are not assumed to be discrete, and they are not assumed to be factors. If the centres are nonatomic a technical complication arises which is studied in the following section.

The basic result leads to a classification of tensor product von Neumann algebras with respect to type. In the factor case the contribution of the present paper is very small (see §4.1 below), although it does include simplifications of two arguments of Takenouchi. In the nonfactor case, it is of interest that the basic result, the existence of finite tensor product projections, plays a role in distinguishing discrete and continuous algebras, not just in distinguishing semifinite and purely infinite algebras.

2. **A lemma on tensor products over commutative algebras.**

2.1. SUBLEMMA. *Let A and B be C^*-algebras and let Z_1 and Z_2 be nonzero isomorphic sub-C^*-algebras of the centres of A and B. We shall identify Z_1 and Z_2 and denote them by Z. Denote by $A \otimes B$ the tensor product of the*

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involutive algebras A and B. Denote by S_Z the set of states of $A \otimes B$ of the form $\mu \otimes \nu$ where μ and ν are states of A and B such that $\mu|Z$ is equal to $\nu|Z$ and is a character of Z. Then the equation

$$p_Z(x)^2 = \sup\{\varphi(y^*x^*xy)\varphi(y^*y)^{-1} \mid y \in A \otimes B, \varphi \in S_Z\}$$

defines a pre-C-algebra seminorm p_Z on $A \otimes B$ which is nonzero on $a \otimes b$ whenever $S_Z(a \otimes b) \neq 0$, and p_Z is the smallest pre-C*-algebra seminorm on $A \otimes B$ with this property. Every state $\varphi \in S_Z$ has norm one with respect to p_Z.*

QUESTION. Must the kernel of p_Z be the linear span of the set $\{az \otimes b - a \otimes zb \mid a \in A, z \in Z, b \in B\}$?

PROOF. The assertion generalizes Theorem 2 of [11], and the proof is similar. We give an outline of the argument for the convenience of the reader.

It is enough to show that every $\varphi \in S_Z$ has norm one with respect to any given pre-C^*-algebra seminorm on $A \otimes B$ which is nonzero on $a \otimes b$ whenever $S_Z(a \otimes b) \neq 0$. By the Kreĭn-Milman theorem it is enough to restrict attention to $\varphi \in S_Z$ of the form $\mu \otimes \nu$ where μ and ν are pure.

Suppose that there exist pure states μ_0 and ν_0 of A and B such that $\mu_0|Z$ equals $\nu_0|Z$ and is a character of Z and such that $\mu_0 \otimes \nu_0$ does not have norm one (with respect to a given pre-C^*-algebra seminorm on $A \otimes B$ with the specified kernel). We shall deduce a contradiction. There exist relatively open sets U and V of pure states of A and B with $\mu_0 \in U$ and $\nu_0 \in V$ such that no $\varphi \in U \otimes V$ has norm one. We may suppose that U and V are invariant under all automorphisms of A and B determined by unitary multipliers, for all such give rise to automorphisms of $A \otimes B$ fixing the given pre-C^*-algebra seminorm. Then by Lemma 8 of [4] there exist two-sided ideals I of A and J of B such that $U^c = I^\perp$ and $V^c = J^\perp$. Choose $a \in I^+$ and $b \in J^+$ such that $\mu_0(a) \neq 0$ and $\mu_0(b) \neq 0$. Then $(\mu_0 \otimes \nu_0)(a \otimes b) \neq 0$, so $a \otimes b$ has nonzero seminorm.

At this point it is possible to deduce a contradiction if A and B are commutative. Indeed, every character of $A \otimes B$ of norm one must be of the form $\mu \otimes \nu$ with $\mu|Z = \nu|Z$, and is not in $U \otimes V$, whence either $\mu(a) = 0$ or $\nu(b) = 0$ and in any case $(\mu \otimes \nu)(a \otimes b) = 0$. This, of course, contradicts the fact that $a \otimes b$ has nonzero seminorm.

In the general case there exist commutative sub-C^*-algebras A_1 of A containing a and Z and B_1 of B containing b and Z. Choose a pure state μ_1 of A_1 such that $\mu_1|Z = \mu_0|Z$ and $\mu_1(a) \neq 0$ (this is possible because $\mu_0(a) \neq 0$), and also choose a pure state ν_1 of B_1 such that $\nu_1|Z = \nu_0|Z$ and $\nu_1(b) \neq 0$. Then by the conclusion in the commutative case, proved in the preceding paragraph, $\mu_1 \otimes \nu_1$ is a state of norm one of $A_1 \otimes B_1$. It is, of course, pure. Extend $\mu_1 \otimes \nu_1$ to a pure state θ of $A \otimes B_1$ of norm one. Then θ has norm one with respect to the largest pre-C^*-algebra norm on $A \otimes B_1$. Hence by Lemma 4 of [11] (which it is not necessary to generalize), $\theta = \mu_2 \otimes \nu_1$ with μ_2 pure. Extend $\mu_2 \otimes \nu_1$ to a pure state θ' of norm one of $A \otimes B$. Then θ' has norm one with respect to the largest pre-C^*-algebra norm on $A \otimes B$, and hence, by Lemma 4 of [11], $\theta' = \mu_2 \otimes \nu_2$ with ν_2 pure. But $\mu_2 \in U$ and $\nu_2 \in V$ since $\mu_2(a) \neq 0$ and $\nu_2(b) \neq 0$; hence, by the construction of U and V, $\mu_2 \otimes \nu_2$ does not have norm one. This, of course, contradicts the construction of $\mu_2 \otimes \nu_2 = \theta'$ to have norm one.

2.2. LEMMA. *Let A and B be commuting C^*-algebras of operators with nonzero intersection Z. Suppose that the kernel of the canonical map β from the tensor product $A \otimes B$ of the algebras A and B onto the algebra R generated by A and B is generated by $\{az \otimes b - a \otimes zb \mid a \in A,\ z \in Z,\ b \in B\}$, so that if $\pi: A \longrightarrow Z$ and $\rho: B \longrightarrow Z$ are Z-linear there exists a unique Z-linear map $\pi \otimes_Z \rho: R \longrightarrow Z$ such that $(\pi \otimes_Z \rho)(ab) = \pi(a)\rho(b)$, $a \in A$, $b \in B$. Then if π and ρ are projections of norm one, so is $\pi \otimes_Z \rho$.*

PROOF. It is enough to show that $\pi \otimes_Z \rho$ has norm one. To show this it is enough to show that $\gamma \circ (\pi \otimes_Z \rho)$ has norm one for each character γ of Z. But if γ is a character of Z, $\gamma \circ (\pi \otimes_Z \rho) \circ \beta = \gamma \circ \pi \otimes \gamma \circ \rho$, so by 2.1 $\gamma \circ (\pi \otimes_Z \rho) \circ \beta$ has norm one on $A \otimes B$ with respect to the preimage by β of the operator norm on R. This says that $\gamma \circ (\pi \otimes_Z \rho)$ has norm one.

2.3. PROBLEM. Let A and B be C^*-algebras, and let C be a C^*-algebra isomorphic to sub-C^*-algebras C_1 of A and C_2 of B, which we shall denote by C. Denote by $A \otimes_C B$ the quotient of the tensor product $A \otimes B$ of the involutive algebras A and B by the two-sided ideal generated by the set $\{ac \otimes b - a \otimes cb \mid a \in A,\ c \in C,\ b \in B\}$, so that if $\pi: A \longrightarrow C$ is a right C-linear map and $\rho: B \longrightarrow C$ is a left C-linear map then there exists a unique complex-linear map $\pi \otimes_C \rho: A \otimes_C B \longrightarrow C$ such that $(\pi \otimes_C \rho)(a \otimes_C b) = \pi(a)\rho(b)$. It is not difficult to show that C is isomorphic to $C \otimes_C C$ (by the map $c_1 \otimes_C c_2 \longrightarrow c_1 c_2$), so that if π and ρ are projections so is $\pi \otimes_C \rho$. If π and ρ are projections of norm one, must $\pi \otimes_C \rho$ also be a projection of norm one, with respect to any pre-C^*-algebra norm on $A \otimes_C B$?

3. The existence of finite tensor product projections.

3.1. *Notation and preliminaries.* Using the notation of Bures [2] for tensor products, we shall write $A = \bigotimes(A_i, \mu_i)$ to mean that μ_i is a normal state of the von Neumann algebra A_i for every index i and that A is the tensor product of the family of von Neumann algebras (A_i) with respect to the family of states (μ_i). This does not specify the Hilbert space on which A acts, but if each μ_i is faithful then the cyclic representation of A defined by the tensor product state $\bigotimes\mu_i$ is faithful.

Although it is not necessary to refer to the Hilbert space on which each A_i acts, we shall do so in order to apply 2.2 conveniently. (This lemma could itself have been formulated more abstractly, but the present formulation seems the least cumbersome.) We shall assume that the action of A_i on the Hilbert space H_i is standard, in the sense that there exists an involutory antiunitary J_i in H_i with the properties that $J_iA_iJ_i = A_i'$ and $J_izJ_i = z^*$ for each $z \in Z_i = A_i \cap A_i'$. The existence in general of such an action was established only relatively recently by work of Tomita (see [12]), but we shall actually need to refer to J_i only in the case that A_i is finite (and even countably decomposable, so that there exists a faithful finite normal trace on A_i), in which case the existence of J_i is classical. We shall denote J_ixJ_i by $j(x)$, $x \in A_i$.

If for each i a projection e_i in A_i is given, we shall denote by $\bigotimes e_i$ the infimum in $\bigotimes(A_i, \mu_i)$ of the finite products of the projections $e_i \otimes 1$. We shall use the easily established fact that a sufficient condition for $\bigotimes e_i$ to be nonzero is that the product of scalars $\Pi\mu_i(e_i)$ be nonzero. The infinite product is of course defined to be the limit of the finite products, or, equivalently, since all the factors belong to the interval $[0, 1]$, the infimum of the finite products. A characterization of the nonvanishing of the product $\Pi\lambda_i$ of nonzero λ_i in the interval $[0, 1]$ that we shall use is the convergence of the sum $\Sigma(1 - \lambda_i)$. (This characterization follows from the inequality $(1 - \lambda) \leqslant -\log \lambda \leqslant (2 \log 2)(1 - \lambda)$, $\lambda \in [\frac{1}{2}, 1]$.)

If A_i is finite and μ_i is faithful (a situation which will predominate), then there exists a unique trace on A_i which coincides with μ_i on the centre Z_i of A_i. We shall denote this trace by τ_i; necessarily τ_i is faithful, normal and finite. There exists a unique positive operator h_i affiliated with A_i and of finite trace (with respect to τ_i) such that $\mu_i = \tau_ih_i$ (i.e. $\mu_i(x) = \tau_i(h_ix)$ for all $x \in A_i$); h_i is called the Radon-Nikodým derivative of μ_i with respect to τ_i. The case that h_i is bounded and invertible will be dominant in the considerations that follow.

If μ_i is a faithful normal state of A_i then (whether A_i is finite or not) there exists a unique projection π_i of norm one from A_i onto Z_i, the centre of A_i, such that $\mu_i = \mu_i \circ \pi_i$. This is easily established, using the commutative case of the preceding Radon-Nikodým theorem. Necessarily π_i is normal and is Z_i-linear.

The preceding notation will be used without comment in what follows. We shall also use the notation introduced in 2.2, in the case $A = A_i$ and $B = A_i'$.

An additional notation that we shall use (following Moore [6]) is $|x|_c$, where x is an operator and $c > 0$ is real, to denote $\inf\{|x|, c\}$.

3.2. THEOREM. *Let $A = \bigotimes(A_i, \mu_i)$. Suppose that there exists a nonzero finite projection in A. Then there exists for each i a nonzero projection $e_i \in A_i$ such that $\bigotimes e_i$ is nonzero and finite.*

PROOF. By the result of Sakai referred to in the second paragraph of 1, if e is a purely infinite projection of A_i then $e \otimes 1$ is a purely infinite projection of A. Hence the largest semifinite projection in A is contained in the tensor product of the largest semifinite projections of the algebras A_i. (The complement of this projection is a sum of purely infinite central projections and is therefore purely infinite.) It is thus seen to be sufficient to consider the case that each A_i is semifinite.

If each A_i is semifinite then the largest semifinite projection in A is, by Sakai's result and the fact that the tensor product of two semifinite von Neumann algebras is semifinite, contained for each finite set of indices F in $\{A_i \otimes 1 \mid i \notin F\}''$. Since the intersection of these algebras over all finite sets F is the scalars, A is in this case either semifinite or purely infinite. Since by hypothesis A has a nonzero finite projection, the assumption that each A_i is semifinite, which we shall make henceforth, entails that A also is semifinite.

If for each i a nonzero projection f_i in A_i is specified, with $\mu_i(f_i) \neq 0$, and if $\bigotimes f_i$ is nonzero, then it is sufficient to prove the conclusion of the theorem after the replacement of each A_i by $f_iA_if_i$, of each μ_i by $\mu_i(f_i)^{-1}\mu_i|f_iA_if_i$, and of A by $(\bigotimes f_i)A(\bigotimes f_i)$. Since such projections f_i can be chosen so that each $f_iA_if_i$ is finite, each $\mu_i|f_iA_if_i$ is faithful, and each h_i is bounded and invertible, we see that it is enough to prove the theorem in the case that each A_i is finite, each μ_i is faithful, and each h_i is bounded and invertible.

In this case, the theorem follows from 3.3, 3.5, 3.6, 3.7 (here projections e_i are constructed with $\bigotimes e_i \neq 0$), and 3.8.

3.3. LEMMA. *Let $A = \bigotimes(A_i, \mu_i)$. Assume that each A_i is finite and that each μ_i is faithful. Then if A is semifinite,*

$$\sum \mu_i(1 - |\pi_i(h_i^{it})|) < \infty \quad \textit{for all real } t.$$

PROOF (Moore [7]). Fix a faithful semifinite normal trace τ on A, set $\bigotimes \mu_i = \mu$, and denote by h the Radon-Nikodým derivative of μ with respect to τ (so that $\mu = \tau h$). For each finite subset F of indices set $\bigotimes_{i \notin F}(A_i, \mu_i) = A_F$ and $\bigotimes_{i \notin F}\mu_i = \mu_F$, and denote by π_F the unique projection of norm one onto the centre of A_F such that $\mu_F = \mu_F \circ \pi_F$. We have $A_\emptyset = A$ and $\mu_\emptyset = \mu$, and we may write $\pi_\emptyset = \pi$. For each F fix a faithful semifinite normal trace τ_F on A_F. (We may choose $\tau_\emptyset$ to be τ, but if $F \neq \emptyset$ then the restriction of τ to A_F may fail to be semifinite.) Denote by h_F the Radon-Nikodým derivative of μ_F with respect to τ_F (so that $\mu_F = \tau_F h_F$). Then by the essential uniqueness of the trace on A, for each F there exists an operator z_F affiliated with the centre of A such that $(\bigotimes_{i \in F} h_i) \otimes h_F = z_F h$.

Hence, for each real t, in succession:

$$\left(\bigotimes_{i\in F} h_i^{it}\right)\otimes h_F^{it} = z_F^{it}h^{it};$$

$$\left(\bigotimes_{i\in F} \pi_i(h_i^{it})\right)\otimes \pi_F(h_F^{it}) = \pi(z_F^{it}h^{it});$$

$$\left(\bigotimes_{i\in F} |\pi_i(h_i^{it})|\right)\otimes |\pi_F(h_F^{it})| = |\pi(h^{it})|;$$

$$\left(\prod_{i\in F} \mu_i(|\pi_i(h_i^{it})|)\right)\mu_F(|\pi_F(h_F^{it})|) = \mu(|\pi(h^{it})|);$$

$$\prod_{i\in F} \mu_i(|\pi_i(h_i^{it})|) \geqslant \mu(|\pi(h^{it})|).$$

Therefore the sum $\Sigma\mu_i(1 - |\pi_i(h_i^{it})|)$ is finite at least for all t such that $\mu(|\pi(h^{it})|) \neq 0$, and since π is ultrastrongly continuous this means at least for all t in an interval about 0. This is all that we shall need, but since for any t there exists a finite set K such that $\pi_K(h_K^{it}) \neq 0$ (there exists $u \in (\bigotimes_{i\in K} A_i) \otimes 1$ for some finite K such that $\pi(uh^{it}) \neq 0$, and then $\pi_K(h_K^{it}) \neq 0$ because h^{it} is a central unitary multiple of $(\bigotimes_{i\in K} h_i^{it}) \otimes h_K^{it}$), the sum is finite for all t.

3.4. REMARK. The converse implication to that of 3.3 is also true, as is shown by the sequel.

If the predual of $A = \bigotimes(A_i, \mu_i)$ is separable, this may also be shown as in [9, p. 814]. First, observe that for each t the condition $\Sigma\mu_i(1 - |\pi_i(h_i^{it})|) < \infty$ is equivalent to the innerness of the modular automorphism σ_t^μ defined in [12], where $\mu = \bigotimes\mu_i$ (cf. the paragraph which follows). If σ_t^μ is inner for every t, then by Theorem 0.1 of [6] the one-parameter group $t \longmapsto \sigma_t^\mu$ is inner, in the sense of [12], whence by Theorem 14.1 of [12], A is semifinite. (Note that this method is not applicable if the predual of A is not separable; a purely infinite factor with only inner modular automorphisms is constructed in 1.5 of [3].)

It is perhaps worth remarking that for arbitrary unitaries $u_i \in A_i$, the condition $\Sigma\mu_i(1 - |\pi_i(u_i)|) < \infty$ is equivalent to the existence (in $\bigotimes(A_i, \mu_i)$) of the tensor product $\bigotimes v_i^* u_i$, where v_i is (any) unitary in the centre of A_i such that $\pi_i(u_i) = v_i|\pi_i(u_i)|$. Existence in $\bigotimes(A_i, \mu_i)$ of $\bigotimes v_i^* u_i$ of course means strong convergence of the finite products of the elements $v_i^* u_i \otimes 1$.

3.5. LEMMA. *Let $A = \bigotimes(A_i, \mu_i)$. Assume that each A_i is finite and that each μ_i is faithful. Then*

$$\sum\mu_i(1 - |\pi_i(h_i^{it})|) < \infty \Longleftrightarrow \sum\mu_i(\pi_i \otimes_{Z_i} j \circ \pi_i \circ j(|h_i^{it} j(h_i^{it}) - 1|^2)) < \infty.$$

PROOF (cf. [9, p. 816]). First, it follows from the inequalities

$$1 - \lambda \leqslant 1 - \lambda^2 = (1 - \lambda)(1 + \lambda) \leqslant 2(1 - \lambda), \quad 0 \leqslant \lambda \leqslant 1,$$

that

$$\sum \mu_i(1 - |\pi_i(h_i^{it})|) < \infty \Longleftrightarrow \sum \mu_i(1 - |\pi_i(h_i^{it})|^2) < \infty.$$

Then, for each i,

$$\begin{aligned}\pi_i \otimes_{Z_i} j \circ \pi_i \circ j(|h_i^{it} j(h_i^{it}) - 1|^2) &= \pi_i \otimes_{Z_i} j \circ \pi_i \circ j(2 - h_i^{it} j(h_i^{it}) - h_i^{-it} j(h_i^{-it})) \\ &= 2 - \pi_i(h_i^{it}) j \circ \pi_i(h_i^{it}) - \pi_i(h_i^{-it}) j \circ \pi_i(h_i^{-it}) \\ &= 2 - \pi_i(h_i^{it})\pi_i(h_i^{-it}) - \pi_i(h_i^{-it})\pi_i(h_i^{it}) \\ &= 2(1 - |\pi_i(h_i^{it})|^2).\end{aligned}$$

3.6. LEMMA. *Let $A = \bigotimes(A_i, \mu_i)$. Assume that each A_i is finite and that each μ_i is faithful. Assume that each h_i is bounded and invertible. Then, for each real $c > 0$,*

$$\sum \mu_i(\pi_i \otimes_{Z_i} j \circ \pi_i \circ j(|h_i^{it} j(h_i^{it}) - 1|^2)) < \infty$$
$$\Longleftrightarrow \sum \mu_i(\pi_i \otimes_{Z_i} j \circ \pi_i \circ j(|h_i j(h_i^{-1}) - 1|_c^2)) < \infty.$$

PROOF (Moore [7]). To simplify notation we restate the lemma in a more abstract form: If for each i a positive form φ_i is given on the C^*-algebra B_i and an element $x_i = x_i^*$ of B_i is given, then, for each real $c > 0$,

$$\sum \varphi_i(|1 - \exp itx_i|^2) < \infty \text{ for all real } t \Longleftrightarrow \sum \varphi_i(|1 - \exp x_i|_c^2) < \infty.$$

Before proving this equivalence, we note that by the inequality

$$(\log 2)|1 - \exp \lambda| \leqslant |\lambda| \leqslant (2 \log 2)|1 - \exp \lambda| \quad \text{for } |1 - \exp \lambda| \leqslant \tfrac{1}{2},$$

the convergence of the sum $\Sigma\varphi_i(|1 - \exp x_i|_c^2)$ is, for $0 < c \leqslant \frac{1}{2}$, equivalent to the convergence of the sum $\Sigma\varphi_i(|x_i|_c^2)$. Moreover, if either of these sums is convergent for a single real $c > 0$, it is convergent for all such c. So it is enough to prove that

$$\sum \varphi_i(|1 - \exp itx_i|^2) < \infty \text{ for all real } t \Longleftrightarrow \sum \varphi_i(|x_i|_c^2) < \infty.$$

By writing each x_i as $x_i' + x_i''$ where $x_i' \in e_i'B_ie_i'$ and $x_i'' \in e_i''B_ie_i''$ with e_i' and e_i'' orthogonal projections in the bidual of B_i and $|x_i'| \geqslant ce_i'$, $|x_i''| \leqslant c$, and replacing φ_i by $\varphi_i' + \varphi_i''$ where $\varphi_i' = e_i'\varphi_ie_i'$ and $\varphi'' = e_i''\varphi_ie_i''$, we may suppose that each x_i satisfies either $|x_i| \geqslant c$ or $|x_i| \leqslant c$.

We have for each real t,

$$|1 - \exp itx_i|^2 = 2((tx_i)^2/2! - (tx_i)^4/4! + \cdots),$$

so if t is small enough that $t^2x_i^2 \leqslant 1$, then

$$(tx_i)^2 - (tx_i)^4/12 \leqslant |1 - \exp itx_i|^2 \leqslant (tx_i)^2.$$

Ad $\Rightarrow$. We have $1 - t^2x_i^2/12 \geqslant 1 - t^2c^2/12$ if $|x_i| \leqslant c$, so if t is small enough that $1 - t^2c^2/12 > 0$, then

$$\sum_{|x_i|\leqslant c} \varphi_i(|1 - \exp itx_i|^2) < \infty \Rightarrow \sum_{|x_i|\leqslant c} \varphi_i(|x_i|_c^2) < \infty.$$

We must show that

$$\sum_{|x_i|\geqslant c} \varphi_i(|1 - \exp itx_i|^2) < \infty \text{ for all real } t \Rightarrow \sum_{|x_i|\leqslant c} \|\varphi_i\| < \infty.$$

Since the map $t \longmapsto \Sigma_{|x_i|\geqslant c}\varphi_i(|1 - \exp itx_i|^2)$ is lower semicontinuous, if it is everywhere finite it must be bounded on some interval. Denote by I an interval of nonzero length $|I|$ such that for some $M > 0$,

$$\sum_{|x_i|\geqslant c} \varphi_i(|1 - \exp itx_i|^2) \leqslant M \quad \text{for all } t \in I.$$

Then

$$\sum_{|x_i|\geqslant c} \varphi_i\left(\int_I |1 - \exp itx_i|^2\, dt\right) \leqslant M|I|.$$

It is now enough to show that for some $m > 0$,

$$m \leqslant \int_I |1 - \exp itx_i|^2\, dt \quad \text{for all } i \text{ with } |x_i| \geqslant c.$$

It is enough to show that for some $m > 0$,

$$m \leqslant \int_I |1 - \exp its|^2\, dt \quad \text{for all real } s \text{ with } |s| \geqslant c.$$

This holds since the function $s \longmapsto \int_I |1 - \exp its|^2\, dt$ is continuous, nonzero, and by the Riemann-Lebesgue lemma has limit $2|I|$ as $s \longrightarrow \infty$.

Ad $\Leftarrow$. (This implication will not be needed.) Fix t. Choose $c > 0$ so that $t^2c^2 \leqslant 1$. From the inequality immediately preceding the proof of $\Rightarrow$,

$$\sum_{|x_i|\leqslant c} \varphi_i(|x_i|_c^2) < \infty \Rightarrow \sum_{|x_i|\leqslant c} \varphi_i(|1 - \exp itx_i|^2) < \infty.$$

Also (independently of the choice of c),

$$\sum_{|x_i|\geqslant c} \varphi_i(|x_i|_c^2) < \infty \Rightarrow \sum_{|x_i|\geqslant c} \|\varphi_i\| < \infty \Rightarrow \sum_{|x_i|\geqslant c} \varphi_i(|1 - \exp itx_i|^2) < \infty.$$

3.7. LEMMA. *Let $A = \bigotimes(A_i, \mu_i)$. Assume that the hypotheses of* 3.6 *are satisfied. Suppose that the second of the equivalent conditions of* 3.6 *is satisfied. Then it is possible to choose for each i a projection e_i in the von Neumann algebra generated by h_i such that $e_i h_i j(e_i h_i^{-1}) \leqslant 8$, in such a way that $\Sigma\mu_i(1 - e_i) < \infty$.*

PROOF. For each $n \in \mathbf{Z}$ denote by e_i^n the spectral projection of h_i corresponding to the half-open interval $[2^n, 2^{n+1}[$.

For each i,

$$\sum_{|m-n|>1} e_i^m j(e_i^n) \leqslant |h_i j(h_i^{-1}) - 1|_1^2.$$

Hence by the hypothesis that $\Sigma\mu_i(\pi_i \otimes_{Z_i} j \circ \pi_i \circ j(|h_i j(h_i^{-1}) - 1|_1^2)) < \infty$,

$$\sum_i \sum_{|m-n|>1} \mu_i(\pi_i \otimes_{Z_i} j \circ \pi_i \circ j(e_i^m j(e_i^n))) < \infty.$$

This may be rewritten as

$$\sum_i \sum_{|m-n|>1} \mu_i(\pi_i(e_i^m)\pi_i(e_i^n)) < \infty.$$

Since $\Sigma_{m,n}\mu_i(\pi_i(e_i^m)\pi_i(e_i^n)) = 1$, it follows that

$$\sum_i \mu_i\left(1 - \sum_m \pi_i(e_i^m)\pi_i(e_i^{m-1} + e_i^m + e_i^{m+1})\right) < \infty.$$

For each i denote by $1 - p_i$ the largest projection in Z_i $(= A_i \cap A_i')$ such that $(1 - p_i)\pi_i(e_i^m) \leqslant 1/4$, all $m \in \mathbf{Z}$. Then for each i,

$$\sum_m (1 - p_i)\pi_i(e_i^m)\pi_i(e_i^{m-1} + e_i^m + e_i^{m+1}) \leqslant \frac{3}{4} \sum_m (1 - p_i)\pi_i(e_i^m) = \frac{3}{4}(1 - p_i);$$

$$\sum_m \mu_i(\pi_i(e_i^m)\pi_i(e_i^{m-1} + e_i^m + e_i^{m+1})) \leqslant \frac{3}{4}\mu_i(1 - p_i) + \mu_i(p_i) \leqslant 1.$$

Hence

$$\sum(1 - (3\mu_i(1 - p_i)/4 + \mu_i(p_i))) < \infty;$$

that is,

$$\frac{1}{4}\sum \mu_i(1 - p_i) < \infty.$$

In other words, if a finite number of indices i are neglected then the tensor product projection $\bigotimes p_i$ is nonzero. Hence, replacing A_i by $p_i A_i p_i$, μ_i by $\mu_i(p_i)^{-1}\mu_i | p_i A_i p_i$, and A by $(\bigotimes p_i)A(\bigotimes p_i)$, we see that it is enough to prove the lemma assuming that each p_i is 1.

Under this assumption, there exists for each fixed i a family of projections

(q_k) in Z_i with sum 1 and for each k an integer $m(k)$ such that $\pi_i(\Sigma_k q_k e_i^{m(k)}) \geqslant$ ¼. For each $n \in \mathbf{Z}$ set $\Sigma_k q_k e_i^{m(k)+n} = f_i^n$. Then

$$\sum_{|n|>1} f_i^0 j(f_i^n) \leqslant |h_i j(h_i^{-1}) - 1|_1^2.$$

Hence as in the second paragraph,

$$\sum_i \sum_{|n|>1} \mu_i(\pi_i(f_i^0)\pi_i(f_i^n)) < \infty,$$

whence by the property $\pi_i(f_i^0) \geqslant$ ¼,

$$\sum_i \sum_{|n|>1} \mu_i(f_i^n) \leqslant 4 \sum_i \sum_{|n|>1} \mu_i(\pi_i(f_i^0)\pi_i(f_i^n)) < \infty.$$

It follows that if we set $f_i^{-1} + f_i^0 + f_i^1 = e_i$, then the e_i satisfy the requirements of the conclusion of the lemma.

3.8. Lemma. *Let $A = \bigotimes(A_i, \mu_i)$. Assume that the hypotheses of* 3.6 *are satisfied. Suppose that the second of the equivalent conditions of* 3.6 *is satisfied. Suppose that for each i, $h_i j(h_i^{-1}) \leqslant 8$. Then A is finite.*

Proof. From $h_i j(h_i^{-1}) \leqslant 8$ follows $|h_i j(h_i^{-1}) - 1| \leqslant 7$. Therefore by the second of the equivalent conditions of 3.6, with $c = 49$,

$$\sum \mu_i(\pi_i \otimes_{Z_i} j \circ \pi_i \circ j((h_i j(h_i^{-1}) - 1)^2)) < \infty.$$

With $x \in A_i$, we calculate as follows:

$$\begin{aligned}\mu_i(\pi_i \otimes_{Z_i} j \circ \pi_i \circ j(xj(h_i^{-1}))) &= \mu_i(\pi_i(x)\pi_i(h_i^{-1})) \\ &= \mu_i(\pi_i(\pi_i(x)h_i^{-1})) = \mu_i(\pi_i(x)h_i^{-1}) \\ &= \tau_i(\pi_i(x)) = \mu_i(\pi_i(x)) = \mu_i(x).\end{aligned}$$

We make use of this with h_i and h_i^2 in place of x to obtain

$$\mu_i(\pi_i \otimes_{Z_i} j \circ \pi_i \circ j((h_i j(h_i^{-1}) - h_i)(h_i - 1))) = 0,$$

from which follows, by the Pythagorean equation,

$$\mu_i(\pi_i \otimes_{Z_i} j \circ \pi_i \circ j((h_i j(h_i^{-1}) - 1)^2)) \geqslant \mu_i(\pi_i \otimes_{Z_i} j \circ \pi_i \circ j((h_i j(h_i^{-1}) - h_i)^2)).$$

We conclude that

$$\sum \mu_i(\pi_i \otimes_{Z_i} j \circ \pi_i \circ j((h_i j(h_i^{-1}) - h_i)^2)) < \infty.$$

Let us now show that $h_i \geqslant 1/8$. It follows from the fact that $\tau_i | Z_i$ is equal to $\mu_i | Z_i$ that $\tau_i(ph_i) = \tau_i(p)$ for any central projection p in A_i. Hence, if

$ph_i \leqslant 1$ for such a p, then $ph_i = 1$, so there is a spectral projection f_i of h_i with central support 1 such that $f_i h_i \geqslant f_i$. The inequality $h_i \geqslant 1/8$ then follows from the condition $h_i j(h_i^{-1}) \leqslant 8$.

Since $h_i j(h_i^{-1}) - h_i = h_i(j(h_i^{-1}) - 1)$, we now have

$$\sum \mu_i(\pi_i \otimes_{Z_i} j \circ \pi_i \circ j((j(h_i^{-1}) - 1)^2)) < \infty;$$

that is, $\Sigma\mu_i((1 - h_i^{-1})^2) < \infty$. Since $(1 - h_i^{-1/2})^2 \leqslant (1 - h_i^{-1})^2$, we have $\Sigma\mu_i((1 - h_i^{-1/2})^2) < \infty$.

Since $\mu_i(h_i^{-1}) = \tau_i(1) = 1$ for each i, it follows that the tensor product vector $\bigotimes h_i^{-1/2}$ exists in the tensor product Hilbert space $\bigotimes(H_{\mu_i}, 1)$, where H_{μ_i} is the completion of A_i in the inner product defined by μ_i, so that the element 1 of A_i is also an element of H_{μ_i}. Since this tensor product Hilbert space is also the Hilbert space H_μ, where $\mu = \bigotimes\mu_i$, it follows that the vector $\bigotimes h_i^{-1/2}$ defines a state on A. Since this state is a normal tensor product state which on each $A_i \otimes 1$ agrees with $\tau_i \otimes 1$, it is a faithful normal trace, and A is finite.

4. The type classification of tensor products of von Neumann algebras.

4.1. *Comments.* The type classification of tensor products of finite families of von Neumann algebras, in terms of the types of the component algebras, is now well understood—the last step was taken by Sakai, who in [8] showed that the largest semifinite projection in a tensor product (of a finite family) is the tensor product of the largest semifinite projections in the component algebras. (The same statement is also true with "semifinite" replaced by "finite", or "discrete".)

The classification into types of tensor products of infinite families of von Neumann algebras is not this straightforward, but a simple consequence of Sakai's result (together with a "zero-one" law) is that the largest semifinite projection in a tensor product of an arbitrary family of von Neumann algebras is either the tensor product of the largest semifinite projections in the component algebras or is zero. (The same situation also holds with "semifinite" replaced by "finite" or "discrete".) It is therefore sufficient just to describe conditions for a tensor product of semifinite (resp. finite, discrete) von Neumann algebras to be again semifinite (resp. finite, discrete).

For factors the type classification problem has already been almost completely solved. Let us collate Theorems 4.2, 4.4 and 4.5 below with the literature.

Condition 4.5(ii)′ for a tensor product of discrete von Neumann algebras to be discrete is, in the factor case, due to Bures (it is the condition in Proposition 5.3 of [1]).

Condition 4.4(ii) for a tensor product of finite von Neumann algebras to be finite is also, in the factor case, due to Bures (it is the condition in Theorem 4.3 of [2]). Earlier work on tensor products of discrete finite factors was done by Bures [1] and Moore [7].

Condition 4.2(iv) for a tensor product of semifinite von Neumann algebras to be semifinite was formulated by Moore (see Lemma 3.2 of [7] and the neighbouring discussion). It is basic in the later work of Takenouchi, Hill, and Størmer. Moore's proof of necessity is the only one known. Sufficiency (for various classes of factors) was shown via condition 4.2(ii)″ by Moore [7, Lemma 5.2], Hill [5, Theorem 3.20] and Takenouchi [10, §3]. The same route is followed here. Størmer [9] used an entirely different method, assuming separable predual, which is also applicable when the centre is nontrivial–see 3.4 above. Størmer's method bypasses 4.2(ii)″, but it should be noted that 4.2(ii)″ is used in the proof of 4.5 (in the nonfactor case), and also to obtain 4.2(iii).

Conditions 4.2(v) and 4.2(vi) were first formulated by Takenouchi, although related conditions had been considered by Moore in a restricted case, and the proof of the equivalence of these conditions is by means of a lemma of Moore [7, Lemma 3.5]. Takenouchi showed that 4.2(iv) implies 4.2(v), for tensor products of discrete factors [10, §2]. In [9] Størmer rewrote Takenouchi's proof for tensor products of continuous factors, and also proved the converse implication. Here (in 3.5) we repeat Størmer's proof of the converse, and reverse it to give a proof of 4.2(iv) ⇒ 4.2(v) slightly simpler than Takenouchi's. Takenouchi proved the implication 4.2(v) ⇒ 4.2(ii)″. Here we show that it is easier to deduce 4.2(ii)″ from 4.2(vi).

4.2. THEOREM. *Let* $A = \bigotimes(A_i, \mu_i)$ (*see* 3.1). *Suppose that each* A_i *is semifinite. Then the following conditions are equivalent.*

(i) *A is semifinite.*

(ii) *A has a nonzero finite tensor product projection.*

(ii)′ *A has a finite tensor product projection of central support* 1.

(ii)″ *A has a finite tensor product projection $\bigotimes e_i$ of central support* 1 *such that for each i the unitary* $1 - 2e_i$ *is in the centre of the set of unitaries u such that* $u^*\mu_i u = \mu_i$.

(iii) *A has a faithful normal semifinite tensor product trace*; *that is, a faithful normal semifinite trace τ such that there exist faithful normal traces τ_i of A_i for each i with* $\tau(\bigotimes a_i) = \Pi\tau_i(a_i) < \infty$ *for elements* $\bigotimes a_i$ ($a_i \geqslant 0$) *generating a strongly dense ideal of A.*

If each μ_i is faithful, then with notation as in 3.1, *these conditions are equivalent to the following.*

(iv) $\Sigma\mu_i(1 - |\pi_i(h_i^{it})|) < \infty$ *for all real t.*

(v) $\Sigma\mu_i(\pi_i \otimes_{Z_i} j \circ \pi_i \circ j(|1 - \Delta_i^{it}|^2)) < \infty$ *for all real t, where Δ_i is the modular operator of μ_i* ($\Delta_i = h_i j(h_i^{-1})$).

(vi) $\Sigma\mu_i(\pi_i \otimes_{Z_i} j \circ \pi_i \circ j(|1 - \Delta_i|_c^2)) < \infty$ *for some* (*and hence for all*) $c > 0$.

PROOF. (iii) ⇒ (i) and (ii)″ ⇒ (ii)′ ⇒ (ii) ⇒ (i) are clear. (Cf. the second paragraph of 4.1.) (i) ⇒ (iv) ⟺ (v) ⟺ (vi) ⇒ (ii) was established in proving 3.2.

(i) ⇒ (ii)$''$. The proof of 3.2 showed that (ii)$''$ holds with "of central support 1" replaced by "nonzero". This distinction is unimportant because of the fact that if for each α the tensor product $\bigotimes e_i^\alpha$ is finite, and if for each i the e_i^α have mutually orthogonal central supports, then the tensor product $\bigotimes(\Sigma_\alpha e_i^\alpha)$ is finite also.

(i) ⇒ (iii). The proof of 3.2 showed that (iii) holds with "faithful" replaced by "nonzero". The situation is closely similar to that in the preceding paragraph. (Replace e_i^α by τ_i^α, a normal semifinite trace, and in place of "finite" read "semifinite".)

4.3. REMARK. It should be noted that although condition 4.2(v) can be rewritten as $\Sigma|1 - \bar{\mu}_i(\Delta_i^{it})| < \infty$, where $\bar{\mu}_i = \mu_i \circ (\pi_i \otimes_{Z_i} j \circ \pi_i \circ j)$ is a state of the C^*-algebra B_i generated by A_i and A_i', so that it is equivalent to the existence in $\bigotimes(H_{\bar{\mu}_i}, \xi_{\bar{\mu}_i})$ of $\bigotimes\Delta_i^{it}$, this condition is not the same as the existence in $\bigotimes(H_{\mu_i}, \xi_{\mu_i})$ of $\bigotimes\Delta_i^{it}$. Indeed, since $\Delta_i^{it}\xi_{\mu_i} = \xi_{\mu_i}$, the latter always holds.

4.4. THEOREM. *Let $A = \bigotimes(A_i, \mu_i)$. Suppose that each A_i is finite. For each i denote by τ_i the unique trace of A_i agreeing on the centre of A_i with μ_i. Then the following conditions are equivalent.*

(i) *A is finite.*

(ii) *$\bigotimes\tau_i$ exists as a normal state of A.* (*For equivalent statements of this see* [2, 4.1].)

PROOF. (ii) ⇒ (i). (ii) implies that A has a nonzero finite central projection, whence the largest finite central projection in A is 1.

(i) ⇒ (ii). If A is finite then there exists a centre-valued trace on A fixing each element of the centre. Denote this centre-valued trace by θ; then $(\bigotimes\mu_i) \circ \theta = \bigotimes\tau_i$.

4.5. THEOREM. *Let $A = \bigotimes(A_i, \mu_i)$. Suppose that each A_i is discrete. Then the following conditions are equivalent.*

(i) *A is discrete.*

(ii) *A has a nonzero abelian tensor product projection.*

(ii)$'$ *A has an abelian tensor product projection $\bigotimes e_i$ of central support* 1, *such that for all but finitely many i, e_i satisfies the condition stated in* 4.2(ii)$''$ (*which may be described as belonging to the centre of the invariance algebra of μ_i*).

PROOF. (ii)$'$ ⇒ (ii) ⇒ (i) is clear. (Cf. the second paragraph of 4.1.)

(i) ⇒ (ii)$'$. By (i) ⇒ (ii)$''$ of 4.2 there exists a finite tensor product projection $\bigotimes f_i$ of central support 1 such that for each i, f_i belongs to the centre of the invariance algebra of μ_i. Then $(\bigotimes f_i)A(\bigotimes f_i)$ is both finite and discrete, and hence all except finitely many f_i must be abelian. If f_i is abelian set $f_i = e_i$. If f_i is not abelian choose any abelian projection in A_i of central support 1 and denote this by e_i. The e_i then satisfy the requirements of (ii)$'$.

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