FUNCTIONS WHICH ARE RESTRICTIONS
OF L^p-MULTIPLIERS

BY

MICHAEL G. COWLING

ABSTRACT. Raouf Doss has given a sufficient condition for a measurable function ϕ on a measurable subset Λ of an LCA group Γ to be the restriction (l.a.e.) to Λ of the Fourier transform of a bounded measure, i.e., a Fourier multiplier of type $(1, 1)$. We generalise Doss' theorem, and prove that, if the measurable function ϕ on Λ is approximable on finite subsets of Λ by trigonometric polynomials which are Fourier multipliers of type (p, p) on Γ of norms no greater than C, then ϕ is equal locally almost everywhere to the restriction to Λ of a Fourier multiplier of type (p, p) and norm no greater than C.

Before we prove the main theorem, we shall give some definitions and prove a converse of the theorem for continuous multipliers. By G and Γ, we shall denote dual LCA groups, with Haar measures dx and $d\gamma$ respectively. The group G with the discrete topology is denoted G_d; its dual is $\overline{\Gamma}$, the Bohr compactification of Γ. The continuous injection of Γ into $\overline{\Gamma}$ is denoted β; by $\hat{\beta}$ we denote the injection of G_d onto G. The extended real numbers p and p' satisfy $1 \leqslant p \leqslant \infty$ and $p^{-1} + p'^{-1} = 1$; $L^p(G)$ is the usual Lebesgue space on G. $C(G)$ is the space of bounded continuous functions on G, with the supremum norm; $C_0(G)$ is the space of continuous functions which tend to zero at infinity, and $C_c(G)$ is the space of continuous functions with compact supports; both these spaces are normed as subspaces of $C(G)$. The Fourier transform of a function f is denoted $\hat{f}$; $A(\Gamma)$ is the space of Fourier transforms of functions in $L^1(G)$, with the inherited norm. The subspace of $L^1 \cap A(G)$ of functions whose Fourier transforms have compact supports is denoted $FA_c(G)$.

A (Fourier) multiplier of type (p, p) on Γ is a bounded measurable function ϕ on Γ such that, for any f, g in $C_c(G)$,

(1.1)
$$\left| \int_\Gamma \Phi(\gamma) \hat{f}(\gamma) \hat{g}(\gamma) \, d\gamma \right| \leqslant C \|f\|_p \|g\|_{p'}.$$

We shall identify such functions which are equal locally almost everywhere (l.a.e.)—then the space of (Fourier) multipliers of type (p, p) is a Banach algebra, denoted $M_p(\Gamma)$; the norm of Φ is the least admissible value of C in the inequality (1.1).

It is known that

$$L^\infty(\Gamma) = M_2(\Gamma) \supseteq M_p(\Gamma) \supseteq M_1(\Gamma) = B(\Gamma)$$

($B(\Gamma)$ is the space of Fourier transforms of bounded measures on G, with the inherited norm), and $\|\Phi\|_\infty \leqslant \|\Phi\|_{M_p} \leqslant \|\Phi\|_B$. Consequently, the closure of $A(\Gamma)$ in $M_p(\Gamma)$, denoted $m_p(\Gamma)$, is a subspace of $C_0(\Gamma)$.

The space $M_p(\Gamma)$ is the dual space of the space $A_p(G)$ introduced by A. Figà-Talamanca [5]. $A_p(G)$ is the image under the continuous mapping $f \otimes g \rightarrow f * g$ of the completed projective tensor product $L^p(G) \hat{\otimes} L^{p'}(G)$ into $C_0(G)$. The duality is, when $\Phi \in M_p(\Gamma)$ and $h \in A_p(G)$,

$$(1.2) \qquad\qquad \langle \Phi, h \rangle = \sum_1^\infty \int_\Gamma \Phi(\gamma) \hat{f}_n(\gamma) \hat{g}_n(\gamma) \, d\gamma,$$

where $f_n, g_n \in C_c(G)$, $\Sigma_1^\infty \|f_n\|_p \|g_n\|_{p'} < \infty$, and $h = \Sigma_1^\infty f_n * g_n$. It is known that $C_0(G) = A_1(G) \supseteq A_p(G) \supseteq A_2(G) = A(G)$, and that the corresponding norm inequalities hold. By $B_p(G)$, we denote the algebra of (pointwise) multipliers of $A_p(G)$, and give it the obvious norm. If $f \in A_p(G)$, $\|f\|_{A_p} = \|f\|_{B_p}$ [6], so $A_p(G)$ is a closed ideal of $B_p(G)$.

$$C(G) = B_1(G) \supseteq B_p(G) \supseteq B_2(G) = B(G),$$

and again the corresponding norm inequalities hold. If $1 < p < \infty$, it is known that $B_p(G)$ is the dual space of $m_p(\Gamma)$ [7]. The duality is, for $h \in B_p(G)$ and $\Phi \in m_p(\Gamma)$,

$$\langle \Phi, h \rangle = \lim_{n \to \infty} \hat{\Phi}_n * h_v(0),$$

where $\Phi_n \in A(\Gamma)$ and $\|\Phi_n - \Phi\|_{M_p} \rightarrow 0$ as $n \rightarrow \infty$, and h_v is the reflection of h: $h_v(x) = h(-x)$. This duality coincides with that of (1.2) when $\Phi \in m_p(\Gamma)$ and $h \in A_p(G)$. We note that one can (and we shall) define $\langle \Phi, h \rangle$ meaningfully for larger sets than we have done.

Our first theorem concerns the local behaviour of multipliers, generalizing Theorem 1 of [2]. Let K be a compact subset of Γ, and let $m_p(K)$ be the algebra of restrictions to K of elements of $m_p(\Gamma)$, defined to be the quotient $m_p(\Gamma)$ modulo the ideal $i(K)$ of $m_p(\Gamma)$-functions which vanish on K. Consider the mapping $\beta^*: m_p(\overline{\Gamma}) \rightarrow M_p \cap C(\Gamma)$ defined by the rule $\beta^*: \phi \rightarrow \phi \circ \beta$; Lohoué [8] and Saeki [9] have shown that β^* is norm-decreasing. Further, if $\phi \in m_p(\overline{\Gamma})$, there exist ϕ_n in $A(\overline{\Gamma})$ such that $\|\phi_n - \phi\|_{M_p(\overline{\Gamma})} \rightarrow 0$, so $\psi \cdot \beta^* \phi_n \in A(\Gamma)$ and $\|\psi \cdot \beta^* \phi_n - \psi \cdot \beta^* \phi\|_{M_p(\underline{\Gamma})} \rightarrow 0$, where $\psi \in A(\Gamma)$ and ψ takes the value one on K, i.e., for any ϕ in $m_p(\overline{\Gamma})$ there exists ϕ' in $m_p(\Gamma)$ such that $\beta^* \phi$ and ϕ' are equal on K. Finally, if $\phi \in i(\beta(K))$, then $\beta^* \phi$ vanishes on K. Therefore β^* induces a quotient map denoted β_q^*, from $m_p(\beta(K))$ into $m_p(K)$. It is easily seen that β_q^* is one-to-one and decreases norms.

Let $B_p(K)$ be the annihilator of $i(K)$ in $B_p(G)$. Consider the mapping $\hat{\beta}^*$:

$B_p(G) \rightarrow B_p(G_d)$, defined by the rule $\hat{\beta}^*: f \rightarrow f \circ \hat{\beta}$, and its restriction to $B_p(K)$. Lohoué [8] has shown that $\hat{\beta}^*$ is an isometry; further, it is readily checked that, if $f \in B_p(G)$ and $\phi \in m_p(\overline{\Gamma})$, $\langle\phi, \hat{\beta}^* f\rangle = \langle\beta^*\phi, f\rangle$ (where we interpret $\beta^*\phi$ as $\lim_{n\to\infty}\beta^*\phi_n$, where $\phi_n \in A(\Gamma)$ and $\lim_{n\to\infty}\|\phi - \phi_n\|_{M_p} = 0$). Therefore the restriction of $\hat{\beta}^*$ to $B_p(K)$ maps into $B_p(\beta(K))$, and is the adjoint map of the quotient map β_q^* (in the cases where $q = 1$ or ∞, the restriction map extends uniquely to the adjoint map of β_q^* and this adjoint map is an isometry). A monomorphism whose adjoint is an isometry is itself an isometry [4, VI. 6]; we have therefore proved:

THEOREM 1. *The algebra of restrictions of $m_p(\Gamma)$-multipliers to the compact subset K of Γ is isomorphic and isometric to the algebra of restrictions of $m_p(\overline{\Gamma})$-multipliers to $\beta(K)$.*

COROLLARY. *An $m_p(\Gamma)$-multiplier ϕ may be approximated on finite subsets of Γ by trigonometric polynomials whose norms, as multipliers of type (p, p) on Γ, are bounded by the (p, p)-multiplier norm of ϕ.*

This follows from a theorem of Lohoué [8] and Saeki [9], which states that, for $\Phi \in M_p \cap C(\overline{\Gamma})$, $\|\Phi \circ \beta\|_{M_p} = \|\Phi\|_{M_p}$. We may extend this corollary to any continuous multiplier ϕ using several results of Lohoué [8]. Let Λ be a finite subset of Γ; denote by δ the continuous map of Γ_d onto Γ and by $\hat{\delta}$ the injection of G into $\overline{G}$, then $\phi \circ \delta \in M_p(\Gamma_d)$, and further, there exists a net of measures $(\mu_\alpha)_{\alpha \in A}$ supported in finite subsets of $\hat{\delta}(G)$ such that, for every $f \in A_p(\overline{G})$, $\lim_A\langle\hat{\mu}_\alpha, f\rangle = \langle\phi \circ \delta, f\rangle$ and $\|\hat{\mu}_\alpha\|_{M_p} \leqslant \|\phi \circ \delta\|_{M_p}$. If we consider $f \in \delta^{-1}(\Lambda)$, we see that $\lim_A\langle\hat{\mu}_\alpha, \delta^{-1}(\gamma)\rangle = \phi(\gamma)$ for $\gamma \in \Lambda$, so defining ϕ_α by the rule $\phi_\alpha \circ \delta = \hat{\mu}_\alpha$, we have a net $(\phi_\alpha)_{\alpha \in A}$ of trigonometric polynomials on Γ such that $\lim_A \phi_\alpha(\gamma) = \phi(\gamma)$ for all $\gamma \in \Lambda$ and $\|\phi_\alpha\|_{M_p} \leqslant \|\phi\|_{M_p}$; the corollary therefore holds for continuous multipliers.

The author [1] has shown also that the algebra of restrictions of $M_p \cap C(\Gamma)$-functions to the compact subset K of Γ is isomorphic and isometric to the algebra of restrictions of $M_p \cap C(\overline{\Gamma})$-functions to $\beta(K)$. To reproduce the proof would obscure the essence of this paper, namely, Theorem 2, which follows directly. Theorem 2 is a generalisation of the main theorem of Doss' paper [3], to which this paper owes much.

THEOREM 2. *Let ϕ be a measurable function on the measurable subset Λ of Γ. If ϕ can be uniformly approximated on finite subsets of Λ by trigonometric polynomials t such that $\|t\|_{M_p} \leqslant C$, then ϕ is the restriction to Λ of a function Φ in $M_p(\Gamma)$ such that $\|\Phi\|_{M_p} \leqslant C$.*

PROOF. The theorem is trivial if Λ is locally null, so we shall assume the contrary; further, we observe that the hypotheses of the theorem imply that ϕ is

bounded (by C). Because $M_p(\Gamma)$ is the dual space of $A_p(G)$, it suffices to show that, if f is in $L^\infty(\Gamma)$ and vanishes off the compact subset Λ_0 of Λ,

$$(2.1) \qquad \left| \int_\Lambda d\gamma\, \phi(\gamma) f(\gamma) \right| \leqslant C \|\hat{f}\|_{A_p}(G).$$

For the Hahn-Banach theorem then shows that there is a multiplier Φ in $M_p(\Gamma)$ such that $\|\Phi\|_{M_p} \leqslant C$ and

$$\int_\Lambda d\gamma\, \Phi(\gamma) f(\gamma) = \int_\Lambda d\gamma\, \phi(\gamma) f(\gamma)$$

for any f in $L^\infty(\Gamma)$ vanishing off some compact subset of Λ. It follows that $\phi = \Phi$ (locally almost everywhere) on Λ. For convenience, we shall extend ϕ to a function ϕ on Γ by taking ϕ to be zero off Λ.

A key observation to be made is that, if f is in $L^\infty(\Gamma)$ and vanishes off the compact subset Λ_0 of Λ, then

$$\int_{\Lambda_0} d\gamma\, \phi(\gamma) f(\gamma) = (\phi f)\check{\ }(0) \doteq \int dx\, (\phi f)\check{\ }(x) u(x),$$

where u is a nonnegative $A_c(G)$-function with support V "close" to 0 in G, and u also satisfies the condition that $\int_V dx\, u(x) = 1$. In fact, if V is "small enough",

$$\int_{\Lambda_0} d\gamma\, \phi(\gamma) f(\gamma) \doteq \int dx\, (\phi f)\check{\ }(x) \delta(x) u(x)$$

for any δ in Λ_0, since, as x tends to 0 in G, $\delta(x)$ tends to 1 uniformly for δ in the compact set Λ_0. This approximation is needed for technical reasons which will become apparent later.

By Theorem 1 (the case where $p = 1$ and β is the injection of G into its Bohr compactification $\bar{G}$), u may be written as an absolutely convergent trigonometric series $\Sigma_1^\infty b_n \delta_n$ on an arbitrarily large compact set K_0. We have then that

$$\int_{\Lambda_0} d\gamma\, \phi(\gamma) f(\gamma) \doteq \int_{K_0} dx\, (\phi f)\check{\ }(x) \delta(x) \sum_1^\infty b_n \delta_n(x)$$

$$= \int_{K_0} dx\, (\phi f)\check{\ }(x) \sum_1^\infty b_n (\delta + \delta_n)(x),$$

$(\delta + \delta_n)$ being the group theoretic sum of δ and δ_n.

If it happened that $(\phi f)\check{\ }$ vanished off K_0, we could write, using the inversion theorem,

$$\int_{\Lambda_0} d\gamma\, \phi(\gamma) f(\gamma) \doteq \sum_1^\infty b_n \phi(\delta + \delta_n) f(\delta + \delta_n) \doteq \sum_1^N b_n \phi(\delta + \delta_n) f(\delta + \delta_n)$$

for large N, because $\Sigma_1^\infty |b_n| < \infty$. In the formal proof below, the function $(\phi \cdot f)$ is regularised to $(\phi f) * k$, where $\hat{k}$ is supported in K_0, in order to validate the use of the inversion theorem.

Since ϕ can be approximated on the finite subset $\{\delta + \delta_1, \ldots, \delta + \delta_N\} \cap \Lambda$ of Λ by trigonometric polynomials t with $\|t\|_{M_p}$ no greater than C,

$$\left|\int_{\Lambda_0} d\gamma \phi(\gamma) f(\gamma)\right| \doteqdot \left|\sum_1^N b_n \phi(\delta + \delta_n) f(\delta + \delta_n)\right|$$

(2.2)
$$\doteqdot \left|\sum_1^N b_n t(\delta + \delta_n) f(\delta + \delta_n)\right|$$

$$\leqslant C \left\|\sum_1^N b_n f(\delta + \delta_n)(\delta + \delta_n)\right\|_{B_p(G)}$$

The essence of the formal proof is the estimation of

$$\left\|\sum_1^N b_n f(\delta + \delta_n)(\delta + \delta_n)\right\|_{B_p}.$$

All we really need to show is that, for any small positive ϵ,

(2.3)
$$\left\|\sum_1^N b_n f(\delta + \delta_n)(\delta + \delta_n)\right\|_{B_p(G)} \leqslant \|\hat{f}\|_{A_p(G)} + \epsilon,$$

for some δ in Λ_0 because then (2.1) can be established. The proof of this inequality involves regularisation of f to $f * k$, where k is in $FA_c(\Gamma)$.

The regularisation of ϕf and f necessitates that we obtain integral estimates for $(\phi f)\check{}(0)$ and $\|\Sigma_1^N b_n f(\delta + \delta_n)(\delta + \delta_n)\|_{B_p}$. Rather than prove (2.2) and (2.3) directly, we are obliged to first establish that

$$\int_{\Lambda_0} d\delta \left|\int_{\Lambda_0} d\gamma \phi(\gamma) f(\gamma) - \sum_1^N b_n \phi(\delta + \delta_n) f(\delta + \delta_n)\right| < \epsilon$$

and

$$\int_{\Lambda_0} d\delta \left\|\sum_1^N b_n f(\delta + \delta_n)(\delta + \delta_n)\right\|_{B_p(G)} \leqslant \mu(\Lambda_0) \|\hat{f}\|_{A_p(G)} + \epsilon,$$

where $\mu(\Lambda_0)$ is the Haar measure of Λ_0. From these "randomised" estimates, we deduce the "pointwise" estimates (2.2) and (2.3) by measure theoretic arguments.

Lemma 1 shows how to construct a particular trigonometric series $\Sigma b_n \delta_n$. Lemma 2 establishes the integral estimates indicated above. This proof relies on the fact that the polynomial $\Sigma b_n \delta_n$ of Lemma 1 can be chosen with sufficiently weak dependence on ϕ and f to permit regularisation of f and ϕf.

LEMMA 1. *Let G_1 be an LCA group of the form $R^a \oplus T^b \oplus D$, where a and b are nonnegative integers and D is a discrete group. Let V_0 be a neighborhood of 0 in G_1 which is a direct sum of compact symmetric neighborhoods of 0 in the R and T summands occurring in G_1, and of the neighbourhood $\{0\}$ of 0 in D. Denote by u the function $m(V_0)^{-2} \chi_{V_0} * \chi_{V_0}$ on G_1, where m and $*$*

represent the Haar measure and convolution on G_1 respectively. Then

    (a) $u \geqslant 0$,

    (b) *u vanishes outside* $V_0 + V_0$,

    (c) $\int_{G_1} dx\, u(x) = 1$.

To any compact set K_0 containing $V_0 + V_0$, we can associate a trigono-metric series q on G_1: $q = \Sigma_1^\infty b_n \gamma_n$ (γ_n in the dual group Γ_1 of G_1) such that

    (d) $\Sigma_1^\infty |b_n| = m(V_0)^{-1}$,

    (e) $u(x) = \Sigma_1^\infty b_n \gamma_n(x)$ $\forall x \in K_0$.

Further, denoting by $\bar{m}$ and $\divideontimes$ the Haar measure and convolution on the Bohr compactification $\bar{G}_1$ of G_1, we can write the unique continuous extension $\bar{q}$ of the almost periodic function q on G_1 to $\bar{G}_1$ in the following manner:

    (f) $\bar{q} = m(V_0)^{-1} \bar{m}(U)^{-1} \chi_U \divideontimes \chi_U$,

where U is a closed symmetric subset of $\bar{G}_1$ such that

    (g) $\int_{K_0} dx\, \chi_U(\bar{y} + x) \leqslant m(V_0)$ $\forall \bar{y} \in \bar{G}_1$;

here we regard G_1 as a dense subgroup of $\bar{G}_1$.

PROOF. *Case* (i). $G_1 = T$. The group T is its own Bohr compactification. Let U be the set V_0 and let $\Sigma_1^\infty b_n \gamma_n$ be the Fourier series of u. Evidently u, $\Sigma_1^\infty b_n \gamma_n$ and U satisfy the conditions of the lemma.

Case (ii). $G_1 = R$. Take an interval $[-I/2, I/2]$ containing K_0 in its interior, and identify this interval with the quotient group R/IZ. The injection β of R into $\bar{R}$ induces a homeomorphic isomorphism β_0 of R/IZ onto the quotient group $\bar{R}/H$, where H is the closure in $\bar{R}$ of $\beta(IZ)$. We denote by π_0 and π the projections of R onto R/IZ and $\bar{R}$ onto $\bar{R}/H$ respectively; then the following diagram commutes:

$$
\begin{array}{ccc}
R & \xrightarrow{\;\beta\;} & \bar{R} \\
\pi_0 \downarrow & & \downarrow \pi \\
R/IZ & \xrightarrow{\;\beta_0\;} & \bar{R}/H
\end{array}
$$

Define U to be the subset $\pi^{-1}\beta_0\pi_0(V_0)$, and let q be the trigonometric series $\Sigma_1^\infty b_n \gamma_n$ on R such that $\bar{q} = m(V_0)^{-1} \bar{m}(U)^{-1} \chi_U \divideontimes \chi_U$.

We observe that an equivalent expression for U is $\chi_U = \Sigma_{x \in IZ} T_x \chi_{V_0}$, whence, for any y in R

$$\int_{K_0} dx\, \chi_U(x + y) \leqslant m(V_0).$$

Proof of the inequality (g) of the lemma is similar.

The series q is independent of the particular choice of Haar measure on $\bar{R}$. For the moment, we shall assume that the Haar measure of $[-I/2, I/2]$ is equal to the total mass of R/IZ, that β_0 is measure preserving, and that the Haar measures of $\bar{R}$ and $\bar{R}/H$ are also equal. Then $m(V_0) = \bar{m}(U)$, and for any x in R,

$$q(x) = \overline{q}(\beta(x)) = m(V_0)^{-2}\chi_U \stackrel{*}{*} \chi_U(\beta(x))$$

$$= m(V_0)^{-2}[\chi_{\beta_0\pi_0(V_0)} * \chi_{\beta_0\pi_0(V_0)}] \circ \pi(\beta(x))$$

$$= m(V_0)^{-2}[\chi_{\beta_0\pi_0(V_0)} * \chi_{\beta_0\pi_0(V_0)}](\beta_0\pi_0(x))$$

$$= m(V_0)^{-2}[\chi_{\pi_0(V_0)} * \chi_{\pi_0(V_0)}](\pi_0(x)).$$

We conclude that q is periodic, with period I, and that

$$q(x) = m(V_0)^{-2}\chi_{V_0} * \chi_{V_0}(x), \quad x \in [-I/2, I/2).$$

The function q and set U therefore satisfy the conditions of the lemma.

Case (iii). $G_1 = D$. Choose a neighbourhood W of 0 in $\overline{D}$ such that W meets the finite set $\beta(K_0 - K_0)$ in just the point 0. Let U be a compact symmetric neighbourhood of 0 in $\overline{D}$ such that $U - U \subseteq W$. Take $\overline{q}$ to be the function on D given by the formula:

$$\overline{q} = m(V_0)^{-1}\overline{m}(U)^{-1}\chi_U \stackrel{*}{*} \chi_U.$$

By the construction of U, (g) is satisfied, for if $\overline{y}_1$ and $\overline{y}_2$ both lie in $\beta(K_0) \cap (\overline{x} + U)$ for some $\overline{x}$ in $\overline{D}$, then

$$\overline{y}_1 - \overline{y}_2 \in \beta(K_0 - K_0) \cap (U - U) = \{0\}.$$

Since V_0 is just the point $\{0\}$ of D,

$$\int_{K_0} dx\, \chi_U(x + \overline{y}) = m(K_0 \cap \beta^{-1}(U - \overline{y})) \leqslant m(\{0\}) = m(V_0).$$

Moreover, $\text{supp}(\overline{q}) \subseteq W$ since U is symmetric, and so $q(x) = 0$ for all x in $K\backslash\{0\}$. Further, $q(0) = m(V_0)^{-1}$. Now u is just the function $m(V_0)^{-2}\chi_{V_0} * \chi_{V_0}$ on D, i.e. $u = m(V_0)^{-1}\chi_{V_0}$. The conditions of the lemma are therefore satisfied.

Case (iv). $\overline{G}_1 = R^a \oplus T^b \oplus D$. We view $\overline{G}_1$ as the direct sum $\overline{R}^a \oplus T^b \oplus \overline{D}$. Any compact neighbourhood K_0 of 0 in G_1 can be contained in another of the form

$$[(1 - I)/2, (I - 1)/2)^a \oplus T^b \oplus K_1$$

in G_1. Construct U, q and u for each of the summands, using this latter neighbourhood, and identifying $[-I/2, I/2)$ with R/IZ in each of the R-summands. Take u and q on G_1 and U in $\overline{G}_1$ to be the products of those u, q and U constructed in the summands.

Our next lemma is the core of the theorem. As we have already explained, the result claimed can be deduced quite easily from the integral estimates obtained.

LEMMA 2. *Let Γ be an LCA group with dual G. Let ϕ and f be bounded measurable functions on Γ, with f vanishing off the compact subset Λ_0 of Γ. Suppose ϵ is a given, arbitrarily small, positive quantity. Then, for a certain polynomial $\Sigma_1^N b_n \gamma_n$, the following estimates hold:*

$$(2.4) \qquad \int_{\Lambda_0} d\gamma \left| \sum_1^N b_n \phi(\gamma + \gamma_n) f(\gamma + \gamma_n) - (\phi f)\hat{\ }(0) \right| < \epsilon$$

and

$$(2.5) \qquad \int_{\Lambda_0} d\gamma \left\| \sum_1^N b_n f(\gamma + \gamma_n)(\gamma + \gamma_n) \right\|_{B_p(G)} \leqslant \mu(\Lambda_0) \|\hat{f}\|_{A_p(G)} + \epsilon,$$

where μ is the Haar measure on Γ.

PROOF. Extend Λ_0 to a compact neighbourhood Λ_1 of 0 in Γ, and let Γ_1 be the open subgroup of Γ generated by Λ_1. We take the Haar measure on Γ_1 to be the restriction to Γ_1 of that on Γ. Let H_1 be the annihilator of Γ_1 in G, and take G_1 to be the quotient group G/H_1. From the structure theorem for compactly generated LCA groups, G_1, which is the dual of Γ_1, is of the form $R^a \oplus T^b \oplus D$. If γ is in Γ_1, then γ is constant on cosets of H_1, as are $\hat{f}$ and $(\phi f)\hat{\ }$, since f and ϕf are supported in Γ_1. Thus elements of $\Gamma_1, \hat{f}$, and $(\phi f)\hat{\ }$ naturally define functions on G_1: we shall deal largely with these functions.

Choose a neighbourhood V_0 in G_1 of the form described in Lemma 1, such that $m(V_0) < 1$ and, if u is the function $m(V_0)^{-2} \chi_{V_0} * \chi_{V_0}$ of Lemma 1, then

$$(2.6) \qquad \left| \int_{G_1} dx\, u(x) \gamma(x) (\phi f)\hat{\ }(x) - (\phi f)\hat{\ }(0) \right| < \delta \qquad \forall \gamma \in \Lambda_0,$$

where δ is a small quantity whose choice will be explained later. This is possible because $(\phi f)\hat{\ }$ is continuous and Λ_0 is compact.

Choose k in $FA_c(\Gamma)$ such that

$$(2.7) \qquad\qquad \|(\phi f) * k - \phi f\|_1 < \delta\, m(V_0) < \delta$$

and

$$(2.8) \qquad\qquad \|f * k - f\|_1 < \delta\, m(V_0) < \delta;$$

recall that $m(V_0) < 1$. Enlarge the support K_1 of $\hat{k}$ in G_1 to a compact neighbourhood K_2 of 0 in G_1 which contains $V_0 + V_0$. From the theorem on finitely generated abelian groups, K_2 generates a subgroup of G_1 of the form $R^a \oplus T^b \oplus (K \oplus Z^c)$, where K is a finite discrete group and c is a nonnegative integer; K_2 is therefore contained in a compact subset K_3 of G of the form $[-M_1, M_1]^a \oplus T^b \oplus K \oplus [-M_1, M_1]^c$. Let j be a positive integer so large that

$$(2.9) \qquad\qquad ((j+1)/j)^{(a+c)} < 1 + \delta.$$

Define M to be the positive integer $(j + 1)M_1$, and let K_0 be the set $[-M, M]^a \oplus T^b \oplus K \oplus [-M, M]^c$. Let u and q be as in Lemma 1.

We first prove the inequality (2.4). For γ in Γ_1, by the inversion theorem,

$$\sum_1^\infty b_n(\phi f) * k(\gamma + \gamma_n) = \int_{K_0} dx (\phi f)\hat{\ }(x)\hat{k}(x)\sum_1^\infty b_n(\gamma + \gamma_n)(x)$$

$$= \int_{K_0} dx (\phi f)\hat{\ }(x)\hat{k}(x)\gamma(x)u(x).$$

So, for any sufficiently large positive integer N and any γ in Γ_1,

$$(2.10) \qquad \left| \sum_1^N b_n(\phi f) * k(\gamma + \gamma_n) - \int_{G_1} dx (\phi f)\hat{\ }.\hat{k}(x)\gamma(x)u(x) \right| < \delta.$$

By (2.7), $\|(\phi f)\hat{\ }.\hat{k} - (\phi f)\hat{\ }\|_\infty < \delta$, whence

$$\left| \int_{G_1} dx [(\phi f)\hat{\ }(x)\hat{k}(x) - (\phi f)\hat{\ }(x)]\gamma(x)u(x) \right| < \delta.$$

Thus, from (2.6) and (2.10), for any γ in Λ_0,

$$\left| \sum_1^N b_n(\phi f) * k(\gamma + \gamma_n) - (\phi f)\hat{\ }(0) \right| < 3\delta,$$

and therefore

$$\int_{\Lambda_0} d\gamma \left| \sum_1^N b_n(\phi f) * k(\gamma + \gamma_n) - (\phi f)\hat{\ }(0) \right| < 3\delta\mu(\Lambda_0).$$

From (2.7), we conclude that

$$\int_{\Lambda_0} d\gamma \left| \sum_1^N b_n\phi(\gamma + \gamma_n)f(\gamma + \gamma_n) - (\phi f)\hat{\ }(0) \right|$$

$$\leqslant \int_{\Lambda_0} d\gamma \left| \sum_1^N b_n[(\phi f) * k - \phi f](\gamma + \gamma_n) \right| + 3\delta \mu(\Lambda_0)$$

$$\leqslant \sum_1^N |b_n|\delta\, m(V_0) + 3\delta \mu(\Lambda_0) \leqslant \delta + 3\delta \mu(\Lambda_0),$$

and so if δ is chosen to be less than $\epsilon(1 + 3\mu(\Lambda_0))^{-1}$, inequality (2.4) is established.

We now estimate $\|\Sigma_1^N b_n f(\gamma + \gamma_n)(\gamma + \gamma_n)\|_{B_p(G)}$. Our first step, the most difficult, is to estimate

$$\left\| \sum_1^\infty b_n(f * k)(\gamma + \gamma_n) \cdot (\gamma + \gamma_n) \right\|_{B_p(G)}.$$

For any γ in Γ_1, $\Sigma_1^\infty b_n(f * k)(\gamma + \gamma_n) \cdot (\gamma + \gamma_n)$ is constant on cosets of H_1 in G, and so

$$\left\| \sum_1^\infty b_n (f * k)(\gamma + \gamma_n) \cdot (\gamma + \gamma_n) \right\|_{B_p(G)}$$

$$= \left\| \sum_1^\infty b_n (f * k)(\gamma + \gamma_n) \cdot (\gamma + \gamma_n) \right\|_{B_p(G_1)}$$

$$= \left\| \sum_1^\infty b_n (f * k)(\gamma + \gamma_n) \cdot \gamma_n \right\|_{B_p(G_1)}.$$

Denote by E the almost periodic function on G_1 defined by the formula

$$E(y) = \sum_1^\infty b_n (f * k)(\gamma + \gamma_n)\gamma_n(y).$$

Since $\Sigma_1^\infty |b_n| < \infty$, we deduce from the inversion theorem that

$$E(y) = \sum_1^\infty b_n T_{-\gamma}(f * k)(\gamma_n) \cdot \gamma_n(y) = \int_{G_1} dx\, \gamma \widehat{f\widehat{k}}(x) \sum_1^\infty b_n \gamma_n(x + y)$$

$$= \int_{G_1} dx\, \gamma \widehat{f\widehat{k}}(x)\, q(x + y).$$

It follows from results of Lohoué [8] that

$$(2.11) \qquad \left\| \sum_1^\infty b_n (f * k)(\gamma + \gamma_n) \cdot (\gamma + \gamma_n) \right\|_{B_p(G)} \leqslant \|\overline{E}\|_{A_p(\overline{G}_1)},$$

$\overline{E}$ being the unique continuous extension of E to $\overline{G}_1$. We shall estimate $\|\overline{E}\|_{A_p(\overline{G}_1)}$ in terms of $\|\widehat{f}\|_{A_p(G)}$, and hence obtain the inequality (2.5) from (2.11) and (2.8).

The estimation of $\|\overline{E}\|_{A_p(\overline{G}_1)}$ perhaps merits some explanation. It is illuminating to consider the cases $G_1 = R$, T, and D separately. Suppose first that $G_1 = T$. Then

$$E(y) = \int_{G_1} dx\, \gamma \widehat{f\widehat{k}}(x)\, q(x + y) = \int_T dx\, \gamma \widehat{f\widehat{k}}(x)\, u(x + y).$$

From Lemma 1, u is in $L^1(T)$ and $\|u\|_1 \leqslant 1$. For any LCA group, $L^1 * A_p \subseteq A_p$ and $\|h * k\|_{A_p} \leqslant \|h\|_1 \|k\|_{A_p}$. Since also $T = \overline{T}$, it follows that

$$\|\overline{E}\|_{A_p(T)} \leqslant \|\gamma \widehat{f\widehat{k}}\|_{A_p(T)} = \|\widehat{f\widehat{k}}\|_{A_p(T)} \leqslant \|\widehat{f}\|_{A_p(T)} + \delta,$$

the last inequality being a consequence of (2.8).

Suppose now that $G_1 = R$. We can express $\gamma \widehat{f\widehat{k}}$ as a limit of sums of convolutions of $C_{K_0}(R)$-functions. Let us consider one such convolution, $g * h$ say. The function $\overline{q}$ on $\overline{R}$ is a convolution:

$$\overline{q} = m(V_0)^{-1} \overline{m}(U)^{-1} \chi_U \,\overline{*}\, \chi_U.$$

Informally, $(g * h) * (\chi_U \,\overline{*}\, \chi_U) = (g * \chi_U) \,\overline{*}\, (h * \chi_U)$ whence

$$\|(g * h) * (\chi_U \bar{*} \chi_U)\|_{A_p(\bar{R})} \leqslant \|f * \chi_U\|_{L^p(\bar{R})} \|h * \chi_U\|_{L^{p'}(\bar{R})}.$$

As a function on R, χ_U is a sum of translates of V_0:

$$\chi_U = \sum_{x \in IZ} T_x \chi_{V_0}.$$

Now g say is supported in $[-I/2, I/2]$, from which we deduce that

$$\|g * \chi_U\|_{L^p(\bar{R})} \leqslant \|g\|_{L^p(R)} \|\chi_U\|_{L^1(\bar{R})},$$

which leads us to the required inequality.

Finally, if $G_1 = D$, we need, as above, to estimate $\|g * \chi_U\|_{L^p(\bar{D})}$ in terms of $\|g\|_{L^p(D)}$. As a function on $\bar{D}$, $g * \chi_U$ is a sum of translates of χ_U whose supports are disjoint, i.e.

$$g * \chi_U = \sum g(x_n) T_{x_n} \chi_U,$$

whence

$$\|g * \chi_U\|_{L^p(\bar{D})} = \left[\sum \|g(x_n) T_{x_n} \chi_U\|_{L^p(\bar{D})}^p\right]^{p^{-1}} = \|g\|_{L^p(D)} \|\chi_U\|_{L^p(\bar{D})},$$

which again gives the required estimate.

We now formalise the above ideas. The function $\gamma \hat{\hat{fk}}$ is in $A_c(G_1)$, a subspace of $A_p(G_1)$, so there exist sequences $(g_n)_1^\infty$ and $(h_n)_1^\infty$ of $C_c(G_1)$-functions such that

$$\sum_1^\infty \|g_n\|_p \|h_n\|_{p'} \leqslant \|\gamma \hat{\hat{fk}}\|_{A_p(G_1)} + \delta$$

and

$$\sum_1^\infty g_n * h_n = \gamma \hat{\hat{fk}}.$$

Note that, since $\|\gamma\|_{B(G_1)} = \|\gamma\|_{C(G_1)} = 1$, $\|\gamma\|_{B_p(G_1)} = 1$ for any γ in Γ_1. The same equality holds for the inverse of γ, whence

$$\|\gamma \hat{\hat{fk}}\|_{A_p(G_1)} = \|\hat{\hat{fk}}\|_{A_p(G_1)} \leqslant \|\hat{f}\|_{A_p(G_1)} + \delta = \|f\|_{A_p(G)} + \delta,$$

the last inequality being a consequence of (2.8). We conclude that

(2.12) $$\sum_1^\infty \|g_n\|_p \|h_n\|_{p'} \leqslant \|\hat{f}\|_{A_p(G)} + 2\delta.$$

Recall that k is supported inside a compact subset K_2 of G_1 of the form $[-M_1, M_1]^a \oplus T^b \oplus (K \oplus [-M_1, M_1]^c)$, and that K_0 was chosen to be the subset $[-M, M]^a \oplus T^b \oplus (K \oplus [-M, M]^c)$ of G_1 where $M = (j + 1)M_1$. Denote by ψ_0 and ψ_3 the characteristic functions of K_0 and K_3 respectively, where

$$K_3 = [M_1 - M, M - M_1]^a \oplus T^b \oplus (K \oplus [M_1 - M, M - M_1]^c).$$

Then $\|\psi_3\|_1^{-1} \psi_0 * \psi_3$ takes the value one on $\mathrm{supp}(\hat{k})$, so

$$\gamma \widehat{f\widehat{k}}(x) = \gamma \cdot \hat{f} \cdot \hat{k} \cdot \|\psi_3\|_1^{-1} \psi_0 * \psi_3(x) = \|\psi_3\|_1^{-1} \left[\sum_1^\infty g_n * h_n\right] \cdot [\psi_0 * \psi_3](x)$$

$$= \|\psi_3\|_1^{-1} \sum_1^\infty \int_{G_1} dw (T_w g_n \cdot \psi_0) * (T_{-w} h_n \cdot \psi_3)(x)$$

by a standard argument.

Next, we recall the definition of E:

$$E(y) = \int_{G_1} dx \gamma \cdot \hat{f} \cdot \hat{k}(x) q(x + y).$$

We see at once that

$$E(y) = \int_{G_1} dx \|\psi_3\|_1^{-1} \left[\sum_1^\infty \int_{G_1} dw (T_w g_n \cdot \psi_0) * (T_{-w} h_n \cdot \psi_3)(x)\right] q(x + y)$$

(2.13)

$$= \|\psi_3\|_1^{-1} \sum_1^\infty \int_{G_1} dw \int_{G_1} dx [(T_w g_n \cdot \psi_0) * (T_{-w} h_n \cdot \psi_3)](x) q(x + y)$$

from Fubini's theorem.

From Lemma 1, for any g and h in $C_c(G_1)$, and any y in G_1,

$$\int_{G_1} dx\, g * h(x) q(x + y)$$

$$= m(V_0)^{-1} \overline{m}(U)^{-1} \int_{G_1} dx \left[\int_{G_1} dz\, g(x - z) h(z)\right]$$

$$\cdot \left[\int_{\overline{G}_1} d\overline{v} \chi_U(x + y - \overline{v}) \chi_U(\overline{v})\right]$$

$$= m(V_0)^{-1} \overline{m}(U)^{-1} \int_{\overline{G}_1} d\overline{v}\, G(y - \overline{v}) H(\overline{v}),$$

where

$$G(\overline{v}) = \int_{G_1} dx\, g(x) \chi_U(x + \overline{v}) \quad \forall \overline{v} \in G_1$$

and

$$H(\overline{v}) = \int_{G_1} dx\, h(x) \chi_U(x + \overline{v}) \quad \forall \overline{v} \in G_1$$

by Fubini's theorem. Defining the functions $G_{n,w}$ and $H_{n,w}$ by the formulae:

$$G_{n,w}(\overline{v}) = \int_{G_1} dx (T_w g_n \cdot \psi_0)(x) \chi_U(x + \overline{v}) \quad \forall \overline{v} \in \overline{G}_1,$$

$$H_{n,w}(\overline{v}) = \int_{G_1} dx (T_{-w} h_n \cdot \psi_3)(x) \chi_U(x + \overline{v}) \quad \forall \overline{v} \in \overline{G}_1$$

it follows from (2.13) that

$$E(y) = \|\psi_3\|_1^{-1} m(V_0)^{-1} \overline{m}(U)^{-1} \sum_1^\infty \int_{G_1} dw \, G_{n,w} \mp H_{n,w}(y)$$

for every y in the dense subgroup G_1 of $\overline{G}_1$. We deduce by a standard argument that

$$\overline{E} = \|\psi_3\|_1^{-1} m(V_0)^{-1} \overline{m}(U)^{-1} \sum_1^\infty \int_{G_1} dw \, G_{n,w} \mp H_{n,w}$$

in $A_p(\overline{G}_1)$ and that

$$\|\overline{E}\|_{A_p(\overline{G}_1)} \leqslant \|\psi_3\|_1^{-1} m(V_0)^{-1} \overline{m}(U)^{-1} \sum_1^\infty \int_{G_1} dw \|G_{n,w}\|_{L^p(\overline{G}_1)} \|H_{n,w}\|_{L^{p'}(\overline{G}_1)}$$

provided, of course, that this last expression is finite. From Hölder's inequality, we conclude that

$$\|\overline{E}\|_{A_p(\overline{G}_1)} \leqslant \|\psi_3\|_1^{-1} m(V_0)^{-1} \overline{m}(U)^{-1} \sum_1^\infty \left[\int_{G_1} dw \|G_{n,w}\|_{L^p(\overline{G}_1)}^p \right]^{p-1}$$

(2.14)
$$\cdot \left[\int_{G_1} dw \|H_{n,w}\|_{L^{p'}(\overline{G}_1)}^{p'} \right]^{p'-1}.$$

If g is any bounded measurable function on G_1 which vanishes off K_0, and G is the function on $\overline{G}_1$ defined by the formula

$$G(\overline{v}) = \int_{G_1} dx \, g(x) \chi_U(x + \overline{v}) \qquad \forall \overline{v} \in \overline{G}_1,$$

then $\|G\|_1 \leqslant \|g\|_1 \|\chi_U\|_1$ from Fubini's theorem. Moreover,

$$\|G\|_\infty = \sup \left\{ \left| \int_{G_1} dx \, g(x) \chi_U(x + \overline{v}) \right| : \overline{v} \in G_1 \right\}$$

$$\leqslant \sup \left\{ \int_{K_0} dx \|g\|_\infty \chi_U(x + \overline{v}) | : \overline{v} \in G_1 \right\} \leqslant \|g\|_\infty m(V_0)$$

from Lemma 1. By the Riesz-Thorin interpolation theorem, for any p in $[1, \infty]$,

$$\|G\|_p' \leqslant \|g\|_p \|\chi_U\|_1^{p-1} m(V_0)^{p'-1} = \|g\|_p \overline{m}(U)^{p-1} m(V_0)^{p'-1}.$$

It follows from the inequality (2.14) that

$$\|\overline{E}\|_{A_p(\overline{G}_1)} \leqslant \|\psi_3\|_1^{-1} \sum_1^\infty \left[\int_{G_1} dw \|T_w g_n \cdot \psi_0\|_p^p \right]^{p-1}$$

$$\cdot \left[\int_{G_1} dw \|T_{-w} h_n \cdot \psi_3\|_{p'}^{p'} \right]^{p'-1}.$$

Now

$$\left[\int_{G_1} dw \|T_w g_n \cdot \psi_0\|_p^p\right]^{p-1} = \left[\int_{G_1} dw \int_{G_1} dx \, |g_n(x-w)\psi_0(x)|^p\right]^{p-1}$$

$$= \|g_n\|_p \|\psi_0\|_p$$

by Fubini's theorem, and the translation and reflection invariance of Haar measure. A similar estimate holds for

$$\left[\int_{G_1} dw \|T_{-w} h_n \cdot \psi_3\|_{p'}^{p'}\right]^{p'-1}$$

so

$$\|\bar{E}\|_{A_p(\bar{G}_1)} \leqslant \|\psi_3\|_1^{-1} \|\psi_0\|_p \|\psi_3\|_{p'} \sum_1^\infty \|g_n\|_p \|h_n\|_{p'}$$

$$\leqslant m(K_0)^{p-1} m(K_3)^{-p-1} [\|\hat{f}\|_{A_p(G)} + 2\delta]$$

from the definitions of ψ_0 and ψ_3, and the inequality (2.12). Thus

$$\|\bar{E}\|_{A_p(\bar{G}_1)} \leqslant ((j+1)/j)^{(a+c)\cdot p-1} [\|\hat{f}\|_{A_p(G)} + 2\delta]$$

$$\leqslant (1+\delta)[\|\hat{f}\|_{A_p(G)} + 2\delta]$$

from the definitions of K_0 and K_3, and the inequality (2.9). We conclude, from (2.11), that

$$\left\|\sum_1^\infty b_n(f * k)(\gamma + \gamma_n) \cdot (\gamma + \gamma_n)\right\|_{B_p(G)} \leqslant (1+\delta)[\|\hat{f}\|_{A_p(G)} + 2\delta].$$

We can now complete the proof of Lemma 2 very easily. Because

$$\sum_1^\infty |b_n| |(f * k)(\gamma + \gamma_n)| \leqslant \sum_1^\infty |b_n| \|f * k\|_\infty < \infty,$$

(2.15) $$\left\|\sum_1^N b_n(f * k)(\gamma + \gamma_n) \cdot (\gamma + \gamma_n)\right\|_{B_p(G)} \leqslant (1+\delta)[\|\hat{f}\|_{A_p(G)} + 3\delta]$$

for all sufficiently large N. Consequently,

$$\left\|\sum_1^N b_n f(\gamma + \gamma_n) \cdot (\gamma + \gamma_n)\right\|_{B_p(G)}$$

$$\leqslant \left\|\sum_1^N b_n(f - f * k)(\gamma + \gamma_n) \cdot (\gamma + \gamma_n)\right\|_{B_p(G)} + (1+\delta)[\|\hat{f}\|_{A_p(G)} + 3\delta]$$

$$\leqslant \sum_1^N |b_n| |(f - f * k)(\gamma + \gamma_n)| + (1+\delta)[\|\hat{f}\|_{A_p(G)} + 3\delta],$$

and so

$$\int_{\Lambda_0} d\gamma \left\| \sum_1^N b_n f(\gamma + \gamma_n) \cdot (\gamma + \gamma_n) \right\|_{B_p(G)}$$

$$\leqslant \sum_1^N |b_n| \, \|f - f * k\|_1 + (1 + \delta)\mu(\Lambda_0)[\|\hat{f}\|_{A_p(G)} + 3\delta]$$

$$\leqslant \delta + (1 + \delta)\mu(\Lambda_0)[\|\hat{f}\|_{A_p(G)} + 3\delta],$$

by (2.8). Therefore, if δ be chosen to satisfy the conditions

$$\delta[1 + 3\mu(\Lambda_0)] < \epsilon$$

and

$$\delta + (1 + \delta)\mu(\Lambda_0)[\|\hat{f}\|_{A_p(G)} + 3\delta] < \mu(\Lambda_0)\|\hat{f}\|_{A_p(G)} + \epsilon,$$

and N is then chosen so that both (2.10) and (2.15) are satisfied, the lemma is proved.

PROOF OF THEOREM 2 (continued). To prove Theorem 2, we show that the inequality (2.1) is satisfied.

Suppose that f is a bounded measurable function vanishing outside the compact subset Λ_0 of Λ, and that ϵ is any small positive quantity. By Lemma 2, there exists a trigonometric polynomial $\Sigma_1^N b_n \gamma_n$ such that

$$(2.16) \qquad \int_{\Lambda_0} d\gamma \left| \sum_1^N b_n \phi(\gamma + \gamma_n) f(\gamma + \gamma_n) - (\phi f)\hat{\,}(0) \right| < \epsilon^2 \mu(\Lambda_0)$$

and

$$(2.17) \quad \int_{\Lambda_0} d\gamma \left\| \sum_1^N b_n \cdot f(\gamma + \gamma_n) \cdot (\gamma + \gamma_n) \right\|_{B_p(G)} < \mu(\Lambda_0)[\|\hat{f}\|_{A_p(G)} + \epsilon].$$

Define the measurable subset Λ_1 of Λ_0 by the formula

$$\Lambda_1 = \left\{ \gamma \in \Lambda_0 : \left| \sum_1^N b_n \cdot \phi(\gamma + \gamma_n) f(\gamma + \gamma_n) - (\phi f)\hat{\,}(0) \right| > \epsilon \right\}.$$

Now

$$\epsilon \mu(\Lambda_1) \leqslant \int_{\Lambda_1} d\gamma \left| \sum_1^N b_n \cdot \phi(\gamma + \gamma_n) f(\gamma + \gamma_n) - (\phi f)\hat{\,}(0) \right|$$

so $\epsilon \mu(\Lambda_1) < \epsilon^2 \mu(\Lambda_0)$ from (2.16). We deduce that $\mu(\Lambda_1) < \epsilon \mu(\Lambda_0)$.

Similarly, defining Λ_2 by the formula

$$\Lambda_2 = \left\{ \gamma \in \Lambda_0 : \left\| \sum_1^N b_n \cdot f(\gamma + \gamma_n)(\gamma + \gamma_n) \right\|_{B_p(G)} > (1 - \epsilon)^{-1} [\|\hat{f}\|_{A_p(G)} + \epsilon] \right\}$$

we can deduce that $\mu(\Lambda_2) < (1 - \epsilon)\mu(\Lambda_0)$. Consequently, $\mu(\Lambda_1 \cup \Lambda_2) < \mu(\Lambda_0)$; take γ_0 in $\Lambda_0 \backslash (\Lambda_1 \cup \Lambda_2)$. Then

$$(2.18) \qquad \left| \sum_1^N b_n \phi(\gamma_0 + \gamma_n) f(\gamma_0 + \gamma_n) - (\phi f)\hat{\ }(0) \right| \leqslant \epsilon$$

and

$$(2.19) \quad \left\| \sum_1^N b_n f(\gamma_0 + \gamma_n) \cdot (\gamma_0 + \gamma_n) \right\|_{B_p(G)} \leqslant (1 - \epsilon)^{-1} [\|\hat{f}\|_{A_p(G)} + \epsilon].$$

Let Λ_3 be the set $\Lambda_0 \cap \{\gamma_0 + \gamma_n : n = 1, 2, \ldots, N\}$. The hypothesis of the theorem implies that there is a trigonometric polynomial t on Γ such that $\|t\|_{M_p} \leqslant C$ and

$$|t(\gamma) - \phi(\gamma)| < \epsilon \left[\sum_1^N |b_n| \cdot \|f\|_\infty \right]^{-1} \qquad \forall \gamma \in \Lambda_3.$$

Since $f(\gamma_0 + \gamma_n)$ vanishes unless $\gamma_0 + \gamma_n$ is in Λ_0,

$$\left| \sum_1^N b_n t(\gamma_0 + \gamma_n) f(\gamma_0 + \gamma_n) - (\phi f)\hat{\ }(0) \right| \leqslant 2\epsilon$$

from (2.18). Therefore

$$\left| \int_\Lambda d\gamma \, \phi(\gamma) f(\gamma) \right| \leqslant \left| \sum_1^N b_n t(\gamma_0 + \gamma_n) f(\gamma_0 + \gamma_n) \right| + 2\epsilon$$

$$\leqslant \left| \int_G d\hat{t}(x) \sum_1^N b_n f(\gamma_0 + \gamma_n)(\gamma_0 + \gamma_n)(x) \right| + 2\epsilon.$$

However $\hat{t}$ is a discrete measure with finite support, and $A_p(G)$ contains a local unit, so

$$\left| \int_\Lambda d\gamma \, \phi(\gamma) f(\gamma) \right| \leqslant \|t\|_{M_p} \left\| \sum_1^N b_n f(\gamma_0 + \gamma_n)(\gamma_0 + \gamma_n) \right\|_{B_p(G)} + 2\epsilon.$$

Taking (2.19) into account, we deduce that

$$\left| \int_\Lambda d\gamma \, \phi(\gamma) f(\gamma) \right| \leqslant C(1 - \epsilon)^{-1} [\|\hat{f}\|_{A_p(G)} + \epsilon] + 2\epsilon,$$

but ϵ can be arbitrarily small, proving the inequality (2.1) and thereby the theorem.

The author wishes to thank Professor G. I. Gaudry of the Flinders Universi-

ty of South Australia, under whose auspices this paper was prepared as part of work for a Ph.D. degree, for his aid.

REFERENCES

1. M. G. Cowling, *Spaces A_p^q and $L^p - L^q$ Fourier multipliers*, Doctoral Dissertation, Flinders University of South Australia, Bedford Park, 1974.

2. K. de Leeuw and C. Herz, *An invariance property of spectral synthesis*, Illinois J. Math. 9 (1965), 220–229. MR 31 #3787.

3. R. Doss, *Approximations and representations for Fourier transforms*, Trans. Amer. Math. Soc. 153 (1971), 211–221. MR 42 #3494.

4. N. Dunford and J. T. Schwartz, *Linear operators.* I: *General theory*, Pure and Appl. Math., vol. 7, Interscience, New York, 1958. MR 22 #8302.

5. A. Figà-Talamanca, *Translation invariant operators in L^p*, Duke Math. J. 32 (1965), 495–501. MR 31 #6095.

6. C. Herz, *Remarques sur la note précédente de M. Varopoulos*, C. R. Acad. Sci. Paris 260 (1965), 6001–6004. MR 31 #6096.

7. C. Herz, *Séminaire Orsay harmonique*, juin 1970.

8. N. Lohoué, *La synthèse des convoluteurs sur un group abélien localement compact*, C. R. Acad. Sci. Paris Ser. A-B 272 (1971), A27–A29. MR 43 #820.

9. S. Saeki, *Translation invariant operators on groups*, Tôhoku Math. J. (2) 22 (1970), 409–419. MR 43 #815.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA, VANCOUVER 8, BRITISH COLUMBIA, CANADA