TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 213, 1975

ON π_3 OF A FINITE H-SPACE

BY

J. R. HUBBUCK AND R. KANE

ABSTRACT. The third homotopy group of a finite H-space is shown to have no torsion.

1. Our primary objective is to establish the truth of the conjecture that the third homotopy group of a finite H-space is torsion free. A (finite) H-space is a pointed topological space X which has the homotopy type of a connected (finite) CW-complex, together with a base point preserving map $\mu: X \times X \longrightarrow X$ with two sided homotopy unit. The universal covering space of a finite H-space is a finite H-space and so there is no loss of generality in assuming that X is simply connected. We investigate the $Z_2 = Z/2Z$ cohomology rings of such spaces. Recall that the Cartan formula for the Steenrod squares implies that Sq^i: $H^n(\, , Z_2) \longrightarrow H^{n+i}(\, , Z_2)$ induces a homomorphism of the indecomposable quotient modules Sq^i: $Q^n\{H^*(\, , Z_2)\} \longrightarrow Q^{n+i}\{H^*(\, , Z_2)\}$.

THEOREM 1.1. *Let X be simply connected and a finite H-space. Then* Sq^2: $Q^{4n-2}\{H^*(X, Z_2)\} \longrightarrow Q^{4n}\{H^*(X, Z_2)\}$ *is surjective for $n > 0$.*

For such an H-space, $H_1(X, Z) \cong \pi_1(X) = 0, H_2(X, Z) \cong \pi_2(X) = 0$ [1] and so $H_2(X, Z_2) = 0$. Thus $H^2(X, Z_2) = 0$ and Theorem 1.1 implies that $H^4(X, Z_2) = Q^4\{H^*(X, Z_2)\} = 0$. It follows that $H^4(X, Z)$ has no 2-torsion and so neither has $\pi_3(X) \cong H_3(X, Z)$.

The fact that $\pi_3(X)$ has no p-torsion for odd p is known. A result analogous to Theorem 1.2 for odd primes p from which this can be deduced is as follows.

(1.2) Let X be a finite H-space. Then
$$P^1: Q^{2n-2(p-1)}\{H^*(X, Z_p)\} \longrightarrow Q^{2n}\{H^*(X, Z_p)\}$$
is surjective if $n \neq 1 \bmod p$.

This can be proved from Theorem 0.2 of [11] and the Adem relations in the mod p Steenrod algebra and motivates Theorem 1.1. Indeed our proof of Theo-

Received by the editors October 21, 1974.

AMS (MOS) subject classifications (1970). Primary 55D45; Secondary 55G20, 57F20.

Key words and phrases. Finite H-space, homotopy groups, Steenrod algebra, secondary cohomology operations.

rem 1.1 uses the methods of [11] although (1.2) itself can be proved by rather more elementary arguments. Thus we have

COROLLARY 1.3. *Let X be a finite H-space. Then $\pi_3(X)$ is a finitely generated free abelian group.*

It is known that the rank of $\pi_3(X)$ is not zero if X is a (noncontractible) strictly associative finite H-space [4].(1) In fact the only known examples of a simply connected finite H-space with $\pi_3(X) = 0$ occur when X is a product of seven spheres. Much of the interest in Corollary 1.3 (and less directly Theorem 1.1) lies in its relationship to the loop space conjecture, that is, the conjecture that the homology of the loop space of a simply connected finite H-space is torsion free. We have shown that $H_2(\Omega X, Z) \cong \pi_2(\Omega X) \cong \pi_3(X)$ is torsion free.

2. Let X be an H-space with multiplication μ. The ring $H^*(X, Z_2)$ has a natural structure as a Hopf algebra over the mod 2 Steenrod algebra. We denote the comultiplication by μ^* and the reduced comultiplication ν: $\bar{H}^*(X, Z_2) \rightarrow \bar{H}^*(X, Z_2) \otimes \bar{H}^*(X, Z_2)$ is the linear map defined by $\nu x = \mu^* x - x \otimes 1 - 1 \otimes x$. All Hopf algebras considered in this paper will be graded, connected, of finite type and either associative and commutative like $H^*(X, Z_2)$ or coassociative and cocommutative like $H_*(X, Z_2)$. If $B \subset H^*(X, Z_2)$ is a sub-Hopf algebra over $A(2)$, then the quotient $H^*(X, Z_2)/\!/B$ is also a Hopf algebra over $A(2)$. However higher order cohomology operations do not pass to quotient modules in as simple a manner. Zabrodsky has developed in [11] a technique which sometimes enables one to overcome this difficulty.

Let F_{q-1} be the $A(2)$ subalgebra of $H^* = H^*(X, Z_2)$ generated by $\Sigma_{i<q} H^i$. It is a sub-Hopf algebra of H^* and so $G_{q-1} = H^*/\!/F_{q-1}$ is also a Hopf algebra over $A(2)$. We write μ^* and ν for the induced comultiplication and reduced comultiplication and similarly we shall use the same symbol to denote an element of a module and its image in a quotient module, where this does not cause confusion. The symbol Q for the indecomposable quotient module of an algebra will be used in a functorial manner.

THEOREM 2.1. *Let*

$$(*) \qquad \mathrm{Sq}^{2n+1} = \sum a_i b_i$$

be a relation in $A(2)$. Suppose that we are given $x \in H^{2n}$ representing $x \in Q^{2n}H^*$ where $x \in QF_q$ but $x \notin QF_{q-1}$ such that

$$(2.2) \qquad b_i x \in \bar{F}_{q-1} \cdot \bar{F}_{q-1}.$$

(1)The proof of Corollary 1.3 announced in [4] when X is a strictly associative H-space was later withdrawn.

Then there exists $y \in H^{4n}$ given by a "secondary operation" associated with $(*)$ *defined on x such that in $QG_{q-1} \otimes QG_{q-1} \nu y = x \otimes x + c$ where $c \in \Sigma$* Image a_1.

This theorem is a 2-primary analogue of Theorem 0.1 of [11] and we refer to this for further details. To be more precise, we follow the proof of Theorem 0.1 given in [11]. §2 needs no alteration. Hypothesis (2.2) enables us to omit Lemma 3.1 (temporarily) and Lemma 3.2 is true for $p = 2$ by an argument similar to that given there for odd primes. The proof of Theorem 2.1 is completed as in §4 of [11]. (The authors are grateful to J. Stasheff for showing to them, while work was in progress on this paper, a preliminary version of a Princeton doctoral thesis by J. Lin. This includes a detailed exposition of results similar to those of [11] at the prime 2. The main theorem is closely related to Theorem 2.1 and from it Lin deduces that $\pi_3(X)$ has no higher 2-torsion when X is a finite H-space.) We use Theorem 2.1 to prove Proposition 2.3. Theorem 1.1 is a corollary of this.

PROPOSITION 2.3. *Let X be a finite H-space. Then* $\mathrm{Sq}^2\colon Q^{4n-2}H^* \longrightarrow Q^{4n}H^*$ *is surjective for $n > 1$.*

The proof proceeds by induction as n decreases. Assume that the result is true for $n > m$ where $m > 1$. We apply Theorem 2.1 when $(*)$ is the relation $\mathrm{Sq}^{4m+1} = (\mathrm{Sq}^{4m})(\mathrm{Sq}^1) + (\mathrm{Sq}^2\mathrm{Sq}^1)(\mathrm{Sq}^2\mathrm{Sq}^{4m-4})$.

Let $x \in Q^{4m}H^*$ and define q by $x \in QF_q$, $x \notin QF_{q-1}$. We assume now and establish later that we can find $x \in H^{4m}$ representing $x \in Q^{4m}H^*$ such that

(2.4) $$\mathrm{Sq}^1 x \in \overline{F}_{q-1} \cdot \overline{F}_{q-1},$$

(2.5) $$\mathrm{Sq}^2\mathrm{Sq}^{4m-4}x \in \overline{F}_{q-1} \cdot \overline{F}_{q-1}.$$

Theorem 2.1 implies the existence of $y \in H^{8m}$ such that

(2.6) $$\nu y = x \otimes x + \mathrm{Sq}^2\mathrm{Sq}^1 \textstyle\sum u \otimes v + 0 \quad \text{in } QG_{q-1} \otimes QG_{q-1}.$$

We now state some results derived from the Bockstein spectral sequence which enable us to show that (2.6) implies the inductive step in the proof. The first is well known.

(2.7) $$\mathrm{Sq}^1\colon Q^{\mathrm{even}}H^* \longrightarrow Q^{\mathrm{odd}}H^* \quad \text{is zero.}$$

This is part of Lemma 4.5 of [2]. A second result which is also closely connected with this lemma is

LEMMA 2.8. *Let $u \in Q^{2k}H^*$. Then we can represent u by $v \in H^{2k}$ such that in $QH^* \otimes QH^*$,* (2.9) $$\nu v \in Q^{\mathrm{even}}H^* \otimes Q^{\mathrm{even}}H^*.$$

PROOF. Let w be any representative in H^{2k} for u. Suppose that we have chosen a basis for QH^* and thus a basis for $QH^* \otimes QH^*$. It is clearly sufficient to show that if x and y are odd dimensional members of the basis and we project νw into $QH^* \otimes QH^*$, then $x \otimes y$ occurs only if $y \otimes x$ occurs and $x \otimes x$ never occurs, for we can replace w by $w - x'\,y'$ where x' and y' are any two representatives in H^* for x and y. Choose a basis for $PH_*(X, Z_2)$, the submodule of primitive elements of $H_*(X, Z_2)$, dual to that chosen for QH^*. If ξ and η are dual to x and y we know that $\xi\eta = \eta\xi$ and $\xi^2 = 0$ [8, Lemma 2.3]. This is precisely what is required.

We return to the proof of Proposition 2.3 and (2.6). We can find $z \in H^{8m-2}$ satisfying (2.9) such that $\mathrm{Sq}^2(z) = y + w$ where w is decomposable. If y is indecomposable, this follows from the induction hypothesis and Lemma 2.8. If y is decomposable, simply let $w = y$ and $z = 0$. Then (2.6) becomes

(2.10) $$\nu w + \mathrm{Sq}^2 \nu z = x \otimes x + \sum \mathrm{Sq}^2 a \otimes b + \sum c \otimes \mathrm{Sq}^2 d$$

in $QG_{q-1} \otimes QG_{q-1}$, using the Cartan formula and the Adem relation $\mathrm{Sq}^1\mathrm{Sq}^1 = 0$. If $x \in \mathrm{Sq}^2 Q^{4m-2}H^*$, we have our result. Otherwise $x \notin \mathrm{Sq}^2 Q^{4m-2}G_{q-1}$ and we can write $Q^{4m-2}G_{q-1} = V \oplus W$ where $V = \mathrm{Sq}^2 Q^{4m-2}G_{q-1}$ and $x \in W$. Taking components of (2.10) in $W \otimes W$ and using (2.7) leads to the relation $\Sigma(u \otimes v + v \otimes u) = x \otimes x$, where the left-hand side arises from νw. This is not possible and so $x \in \mathrm{Sq}^2 Q^{4m-2}H^*$ as required. It remains to establish (2.4) and (2.5).

3. We shall use some elementary facts about Hopf algebras which we state in two lemmas. A is an associative, commutative Hopf algebra, B a sub-Hopf algebra and C a sub-Hopf algebra of B. We write θ for the reduced comultiplication of A.

LEMMA 3.1. (i) *Let* $x \in \bar{A}$ *be such that* $\theta x \in B \otimes B$. *Then* $x \in B$ *if and only if* $x \in \bar{B} \cdot A$.

(ii) $B/\!/C$ *is a sub-Hopf algebra of* $A/\!/C$ (*that is, the induced map* $B/\!/C \longrightarrow A/\!/C$ *is a monomorphism*).

Let $P(A)$ be the submodule of primitive elements of A and $I_{2k}(A)$ be the Hopf ideal generated by elements of positive dimension less than $2k + 1$.

LEMMA 3.2. (i) $P^{2k+1}(A) \cap I_{2k}(A) = 0$. (ii) $P^{4k+2}(A) \cap I_{2k}(A) = 0$.

The proof of the first lemma depends upon considering the function $\check{f}$: $A \longrightarrow B \otimes A/\!/B$ of [6, Theorem 4.4]. This is an isomorphism of right $A/\!/B$

comodules for part (i) and of left B-modules and therefore of left C-modules for part (ii). The second lemma is a simple exercise given a Borel decomposition of A. In fact the first part is true since odd dimensional primitive elements are indecomposable.

4. We now choose a representative $x \in H^{4m}$ for $x \in Q^{4m}H^*$ in Proposition 2.3. Choose $u \in Q^qH^*$ such that $bu = x$ in QH^* where $b \in A(2)$. Let $v \in H^q$ represent u and satisfy (2.9) if q is even. Let $x = bv$ in H^{4m}. Then this x also satisfies (2.9) by the Cartan formula, the Adem relation $\mathrm{Sq}^1\mathrm{Sq}^{2r} = \mathrm{Sq}^{2r+1}$ and (2.7). Also $\nu x \in F_{q-1} \otimes F_{q-1}$.

THE PROOF OF (2.4). We show first that $\mathrm{Sq}^1 x \in F_{q-1}$. We know that $\nu \mathrm{Sq}^1 x \in \bar{F}_{q-1} \otimes F_{q-1}$ and so by Lemma 3.1(i) it is sufficient to show that $\mathrm{Sq}^1 x \in \bar{F}_{q-1} \cdot H^*$, or that $\mathrm{Sq}^1 x = 0$ in G_{q-1}. But $\mathrm{Sq}^1 x$ is primitive in G_{q-1} and lies in $I_{4m}(G_{q-1})$ since it is decomposable by (2.7). Lemma 3.2(i) implies that it is zero.

It remains to show that $\mathrm{Sq}^1 x$ is decomposable in F_{q-1}. Let K be the sub-Hopf algebra of F_{q-1} generated by elements of dimension less than $4m + 1$. Lemma 3.1(ii) implies that $F_{q-1}/\!/K \longrightarrow H^*/\!/K$ is mono. $\mathrm{Sq}^1 x$ is primitive in these Hopf algebras and lies in the ideal $I_{4m}(H^*/\!/K)$. Thus $\mathrm{Sq}^1 x$ is zero in $H^*/\!/K$ by Lemma 3.2(i) and therefore in $F_{q-1}/\!/K$ as required.

THE PROOF OF (2.5). Let $Q_2H^* = H^*/\bar{H}^* \cdot \bar{H}^* \cdot \bar{H}^*$, so that μ^* induces ν: $Q_2H^* \longrightarrow QH^* \otimes QH^*$. Notice that if $v \in Q_2^{2k}H^*$ satisfies (2.9), then $\mathrm{Sq}^1 v = 0$ in Q_2H^*, again using (2.7).

Let $z = \mathrm{Sq}^2\mathrm{Sq}^{4m-4}x$. We choose $w \in H^{8m-4}$ such that (a) w is decomposable, (b) $\mathrm{Sq}^2 w = z$ in Q_2H^* and (c) $\nu w \in Q^{\mathrm{even}}H^* \otimes Q^{\mathrm{even}}H^*$. If $\mathrm{Sq}^{4m-4}x$ is decomposable, we can take this element to be w, and so assume that $t = \mathrm{Sq}^{4m-4}x$ is indecomposable. Our induction hypothesis implies that there exists an indecomposable v with $\mathrm{Sq}^2 v = t + w$, where w is decomposable. If we choose v satisfying (2.9), then w satisfies (c). It remains to check (b). But $\mathrm{Sq}^2 w = \mathrm{Sq}^2 t + \mathrm{Sq}^2\mathrm{Sq}^2 v = z + \mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^1 v$ and $\mathrm{Sq}^1 v = 0$ in Q_2H^*.

In Q_2H^*, w equals a product of even dimensional indecomposables by (c) and dimensional reasoning. Therefore $z = \mathrm{Sq}^2 w$ in $Q_2^{8m-2}H^*$ is also a product of even dimensional indecomposables, again using (2.7) and the Cartan formula. It follows for dimensional reasons that $z \in I_{4m-2}(H^*)$, the ideal in H^* generated by elements of dimension less than $4m - 1$.

The proof that $z \in \bar{F}_{q-1} \cdot \bar{F}_{q-1}$ is now similar to that which proved that $\mathrm{Sq}^1 x \in \bar{F}_{q-1} \cdot \bar{F}_{q-1}$ except that we use Lemma 3.2(ii) in place of Lemma 3.2(i).

5. In this section we complete the proof of Theorem 1.1 by showing that $H^4(X, Z_2) = 0$. Let PX be the projective plane of the simply connected finite

H-space. A detailed study of the cohomology of PX has been made in [3]. In particular it is shown that, for any coefficient group G, there exists an exact sequence

$$(5.1)\qquad \begin{aligned} \cdots \longrightarrow \widetilde{H}^{i-2}(X, G) &\xrightarrow{\nu} \widetilde{H}^{i-2}(X \wedge X, G) \xrightarrow{\lambda} \widetilde{H}^{i}(PX, G) \\ &\xrightarrow{j} \widetilde{H}^{i-1}(X, G) \longrightarrow \cdots . \end{aligned}$$

We will use (5.1) with $G = Z_2$ and $G = Z$. In the case $G = Z_2$, ν is essentially the reduced comultiplication defined previously and therefore Image j is the submodule of primitive elements of $H^*(X, Z_2)$. We have noted in §1 that $H^1(X, Z_2) \cong H^2(X, Z_2) = 0$ and so $H^i(X \wedge X, Z_2) = 0$ for $i < 6$. Thus j: $H^i(PX, Z_2) \longrightarrow H^{i-1}(X, Z_2)$ is an isomorphism for $i < 7$ and a monomorphism for $i = 7$.

Assume for the moment that we have established that both of the following homomorphisms are trivial:

$$(5.2)\qquad \mathrm{Sq}^1 : H^5(PX, Z_2) \longrightarrow H^6(PX, Z_2),$$

$$(5.3)\qquad \mathrm{Sq}^1 : H^7(PX, Z_2) \longrightarrow H^8(PX, Z_2).$$

Let $x \in H^4(X, Z_2)$ and $ju = x$. Then

$$u^2 = \mathrm{Sq}^5 u = \mathrm{Sq}^4\mathrm{Sq}^1 u + \mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^2 u = 0,$$

using (5.2) and (5.3) But $\lambda(x \otimes x) = u^2$ (see [3, 3.3]) and so there exists $y \in H^8(X, Z_2)$ with $\nu y = x \otimes x$. If x is nonzero, then y is indecomposable and so, by Proposition 2.3, there exists $z \in H^6(X, Z_2)$ with $\mathrm{Sq}^2 z = y + w$, where w is decomposable. But Lemma 2.8 implies that we can choose z to be primitive and we obtain a contradiction by applying the comultiplication, unless $x = 0$.

Therefore to complete the proof of Theorem 1.1, it remains to establish that the homomorphisms (5.2) and (5.3) are zero. We prove that (5.3) is trivial; a similar but simpler argument implies the same for (5.2). It is sufficient to show that c: $H^7(PX, Z) \longrightarrow H^7(PX, Z_2)$ is surjective, where c is induced from c: $Z \longrightarrow Z_2$, reduction mod 2. Let $x \in H^7(PX, Z_2)$. Then $jx \in H^6(X, Z_2)$ is nonzero and primitive and so by [1, Theorem 6.13] is an integral class, say $cy = jx$. We may assume that y is a torsion element. For $H^*(X, Z)$/Torsion is an exterior algebra on odd dimensional generators and so $y = t + \Sigma\, uv$, where t is a torsion class and u and v are distinct (nontorsion) classes of $H^3(X, Z)$. Applying the comultiplication, we see that cy is primitive only if $c(\Sigma\, uv) = 0$. Now $H^6(X \wedge X, Z)$ is torsion free and so $\nu y = 0$. Thus by the exactness of (5.1) with $G = Z$, there exists $z \in H^7(PX, Z)$ with $jz = y$. The homomorphism j is induced from the inclusion $SX \longrightarrow PX$ and so commutes with c and j is a monomorphism in dimension 7. Thus $cz = x$, as required.

REFERENCES

1. W. Browder, *Torsion in H-spaces*, Ann. of Math. (2) **74** (1961), 24–51. MR **23** #A2201.

2. ———, *On differential Hopf algebras*, Trans. Amer. Math. Soc. **107** (1963), 153–176. MR **26** #3061.

3. W. Browder and E. Thomas, *On the projective plane of an H-space*, Illinois J. Math. **7** (1963), 492–502. MR **27** #1955.

4. A. Clark, *On π_3 of finite dimensional H-spaces*, Ann. of Math. (2) **78** (1963), 193–196. MR **27** #1956.

5. J. R. Hubbuck, *Simply connected H-spaces of rank 2 with 2-torsion*, Quart. J. Math. Oxford Ser. (3) **26** (1975), 169–177.

6. J. W. Milnor and J. C. Moore, *On the structure of Hopf algebras*, Ann. of Math. (2) **81** (1965), 211–264. MR **30** #4259.

7. R. Kane, *On loop spaces without p-torsion*, Pacific J. Math. (to appear).

8. ———, *Primitivity and finite H-spaces*, Quart. J. Math. (to appear).

9. J. Lin, *H-spaces with finitely generated cohomology algebras*, Thesis, Princeton University, Princeton, N. J., 1974.

10. N. Steenrod and D. Epstein, *Cohomology operations*, Ann. of Math. Studies, no. 50, Princeton Univ. Press, Princeton, N. J., 1962. MR **26** #3056.

11. A. Zabrodsky, *Secondary cohomology operations in the module of indecomposables*, Algebraic Topology Conference, Aarhus, 1970.

12. ———, *On sphere extensions of classical Lie groups*, Proc. Sympos. Pure Math., vol. 22, Amer. Math. Soc., Providence, R. I., 1971, pp. 279–283. MR **47** #5869.

13. ———, *Secondary operations in the cohomology of H-spaces*, Illinois J. Math. **15** (1971), 648–655. MR **48** #1234.

MATHEMATICAL INSTITUTE, OXFORD, ENGLAND

MAGDALEN COLLEGE, OXFORD, ENGLAND

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASSACHUSETTS 02139