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MÜNTZ-SZÁSZ THEOREM WITH INTEGRAL COEFFICIENTS. II

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ABSTRACT. The classical Müntz-Szász theorem concerns uniform approximation on $[0, 1]$ by polynomials whose exponents are taken from a sequence of real numbers. Under mild restrictions on the exponents or the interval, the theorem remains valid when the coefficients of the polynomials are taken from the integers.

Let $C[a, b]$ be the continuous real valued functions defined on a closed bounded interval $[a, b]$ and $\|\cdot\|$ the supremum norm on $C[a, b]$ ($\|f\| = \sup\{|f(x)|: a \leq x \leq b\}$). Let $\Lambda = \{\lambda_i\}$ be a sequence of real numbers satisfying $0 < \lambda_1 < \lambda_2 < \ldots$. A Λ-polynomial is a function of the form

$$p(x) = a_0 + \sum_{i=1}^{n} a_i x^{\lambda_i} \tag{1}$$

where the a_i's are any real numbers. One version of the classical Müntz-Szász theorem reads as follows (cf. Müntz [7]).

THEOREM 1. *The Λ-polynomials are dense in* $C[0, 1]$ *if and only if* $\sum_{i=1}^{\infty} \lambda_i^{-1} = \infty$.

It is also well known that the ordinary polynomials with integer coefficients, i.e. integral polynomials, are dense in the subspace.

$$C_0[0, 1] = \{f \in C[0, 1]: f(0) \text{ and } f(1) \text{ are integers}\}$$

of $C[0, 1]$. This seems to be due originally to Kakeya [6]. For generalizations see Ferguson [2], [3], and Cantor [1].

Thus it is interesting to ask if Theorem 1 remains true for integral Λ-polynomials, i.e. functions of the form (1) where the a_i's are restricted to the ring of rational integers $\{0, \pm 1, \pm 2, \ldots\}$. The answer is yes under certain restrictions on the functions to be approximated, the interval $[0, 1]$, or the sequence of exponents Λ.

For $\alpha > 0$ the map $x \longrightarrow \alpha x$ induces an isometry between $C[0, 1]$ and

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$C[0, \alpha]$ under which Λ-polynomials correspond to Λ-polynomials. Thus for a given $\alpha > 0$ and sequence Λ the Λ-polynomials are dense in $C[0, 1]$ iff they are dense in $C[0, \alpha]$.

From Theorem 1 we have that $\Sigma_{i=1}^{\infty} \lambda_i^{-1} = \infty$ is a necessary condition for the density of the Λ-polynomials, and since the integral Λ-polynomials are a subset of these, the condition is also necessary for the density of the integral Λ-polynomials. This leads to obvious converses for the following theorems.

Clearly, every integral Λ-polynomial takes on integral values at $x = 0$ and $x = 1$. Since the integers form a closed subset of the reals, it is not possible to approximate functions outside of the set $C_0[0, 1]$ by integral Λ-polynomials.

THEOREM 2. *Let $\Lambda = \{\lambda_i\}$ be a sequence of integers satisfying* $0 < \lambda_1 < \lambda_2 < \dots$. *If $\Sigma_{i=1}^{\infty} \lambda_i^{-1} = \infty$, then the integral Λ-polynomials are dense in* $C_0[0, 1]$.

The proof will follow from a series of lemmas.

LEMMA 1. *For any two positive integers q and s, $q < s$, there exists a polynomial Q_{qs} of the form*

$$Q_{qs}(x) = \sum_{i=q+1}^{s} c_{iqs} x^{\lambda_i}$$

such that

$$A_{qs} = \|x^{\lambda_q} - Q_{qs}(x)\| \leq 2 \exp\left(-2\lambda_q \sum_{i=q+1}^{s} \lambda_i^{-1}\right) \tag{2}$$

and $Q_{qs}(1) = 1$, *where the first equality in* (2) *serves to define* A_{qs}.

PROOF. From von Golitschek [5, Lemma 2] there exist real numbers c_i, $q + 1 \leq i \leq s$, such that

$$\left\| x^{\lambda_q} - \sum_{i=q+1}^{s} c_i x^{\lambda_i} \right\| \leq \prod_{i=q+1}^{s} \frac{\lambda_i - \lambda_q}{\lambda_i + \lambda_q} \leq \prod_{i=q+1}^{s} \exp\left(\frac{-2\lambda_q}{\lambda_i}\right)$$

where the latter inequality follows from the inequality (applied factorwise) $(1 - x)/(1 + x) \leq e^{-2x}$, $x \geq 0$, which is proved by elementary methods. Now set

$$Q_{qs}(x) = \sum_{i=q+1}^{s} c_i x^{\lambda_i} + \left(1 - \sum_{i=q+1}^{s} c_i\right) x^{\lambda_s}. \quad \square$$

LEMMA 2. *Let r and s be positive integers, $r < s$. Suppose that* $|\Sigma_{i=j}^{s} d_i| < 1$, $r + 1 \leq j \leq s$, *and* $\Sigma_{i=r+1}^{s} d_i = 0$. *Then setting* $p_{rs}(x) = \Sigma_{j=r+1}^{s} d_j x^{\lambda_j}$ *we have* $\|p_{rs}\| \leq (\lambda_s - \lambda_r)/\lambda_r$.

PROOF. Since $p_{rs}(1) = \Sigma_{j=r+1}^{s} d_j = 0$ by hypothesis, we have

$$p_{rs}(x) = \sum_{\kappa=r+1}^{s} (x^{\lambda_\kappa} - x^{\lambda_{\kappa-1}})\left(\sum_{i=\kappa}^{s} d_i\right)$$

and for all x, $0 \leqslant x \leqslant 1$,

$$\begin{aligned} |p_{rs}(x)| &\leqslant \sum_{\kappa=r+1}^{s} (x^{\lambda_{\kappa-1}} - x^{\lambda_\kappa}) = x^{\lambda_r} - x^{\lambda_s} \\ &\leqslant \frac{\lambda_s - \lambda_r}{\lambda_r}. \end{aligned} \tag{3}$$

The second inequality in (3) can be established by elementary means. □

Suppose that $\Sigma_{i=1}^{\infty} \lambda_i = \infty$. In the following we will use implicitly the fact that there are infinitely many q such that $\lambda_q < q^{5/4}$. Indeed, if not, then $\lambda_q^{-1} \leqslant q^{-5/4}$ for all but finitely many q which contradicts the assumption $\Sigma_{i=1}^{\infty} \lambda_i^{-1} = \infty$.

LEMMA 3. *Let $\Lambda = \{\lambda_i\}$ satisfy the hypotheses of Theorem* 2, $0 < \epsilon < 1/25$, $K > 0$ *and let N be an integer with $N \geqslant 1 + 1/\epsilon^2$ and $\lambda_{N+1} \leqslant (N+1)^{5/4}$. There exist integers r and s such that $N < r < s$ and*

$$\lambda_s \leqslant s^{5/4}, \quad \lambda_s \leqslant (1 + 4\epsilon)\lambda_r, \quad \sum_{i=N}^{s} \lambda_i^{-1} \geqslant K \tag{4}$$

and

$$\lambda_q \sum_{i=q+1}^{s} \lambda_i^{-1} > \frac{\epsilon}{12}\sqrt{q} \quad \textit{whenever } N \leqslant q \leqslant r. \tag{5}$$

PROOF. Choose an integer M such that

$$M > N, \quad \lambda_M \leqslant M^{5/4}, \quad \text{and} \quad \sum_{i=N}^{M} \lambda_i^{-1} \geqslant K + 2. \tag{6}$$

Claim 1. There exists an integer s_0, $N < s_0 \leqslant M$, satisfying the following three conditions:

$$\lambda_{s_0} \leqslant s_0^{5/4}, \tag{7}$$

$$\sum_{i=N}^{s_0} \lambda_i^{-1} \geqslant K + 1, \tag{8}$$

$$\lambda_q \sum_{i=q+1}^{s_0} \lambda_i^{-1} + 5\lambda_q/s_0^{1/4} > \sqrt{q} \quad \text{whenever } N \leqslant q < s_0. \tag{9}$$

PROOF OF CLAIM 1. Set

$$Q = \left\{ q \,|\, N \leqslant q < M, \lambda_q \sum_{i=q+1}^{M} \lambda_i^{-1} \leqslant \sqrt{q} \right\}. \tag{10}$$

If Q is empty take $s_0 = M$. Suppose Q is not empty. Define $M^* = \min Q$. Then from (10)

$$\sum_{i=M^*+1}^{M} \lambda_i^{-1} \leqslant \frac{\sqrt{M^*}}{\lambda_{M^*}} \leqslant \frac{1}{\sqrt{M^*}}. \tag{11}$$

Define $s_0 = \max\{q \mid N \leqslant q \leqslant M^*, \lambda_q \leqslant q^{5/4}\}$. By hypothesis this set is not empty and $N < s_0 \leqslant M^* < M$. Since $\lambda_q > q^{5/4}$ whenever $s_0 + 1 \leqslant q \leqslant M^*$ we have

$$\sum_{i=s_0+1}^{M^*} \lambda_i^{-1} < \int_{s_0}^{\infty} \frac{dx}{x^{5/4}} = 4\, s_0^{-1/4}. \tag{12}$$

From (11) and (12) we have

$$\sum_{i=s_0+1}^{M} \lambda_i^{-1} < \frac{4}{s_0^{1/4}} + \frac{1}{\sqrt{M^*}} < \frac{5}{s_0^{1/4}} \leqslant 1. \tag{13}$$

This, together with (6), establishes (8). Inequality (7) follows from the definition of s_0. From the definition of M^* and $s_0 \leqslant M^*$ it follows that $\lambda_q \Sigma_{i=q+1}^{M} \lambda_i^{-1} > \sqrt{q}$ whenever $N \leqslant q < s_0$. Inequality (9) follows from this and (13) which completes the proof of Claim 1.

We next define, by induction, a finite sequence $s_1, s_2, \ldots, s_{\kappa+1}$ satisfying

$$s_{j+1} + [\epsilon s_{j+1}] = s_j \quad \text{or} \quad s_j - 1, \quad 0 \leqslant j \leqslant \kappa, \tag{14}$$

and $s_{\kappa+1} \leqslant N < s_\kappa$.

Since $s_j > N > 1 + 1/\epsilon^2$ and $\epsilon < 1/25$ by hypothesis, the sequence $\{s_j\}_{j=0}^{\kappa+1}$ is strictly decreasing. It is also well defined since the left-hand side of (14), as a function of s_{j+1}, decreases by at most 2 when s_{j+1} is decreased by 1.

Claim 2. Let $1 \leqslant k \leqslant \kappa$. If

$$\lambda_{s_j} > (1 + 4\epsilon)\lambda_{s_{j+1}}, \quad 0 \leqslant j \leqslant k - 1, \tag{15}$$

then

$$\lambda_{s_j} \leqslant s_j^{5/4}, \quad 0 \leqslant j \leqslant k, \tag{16}$$

and

$$\sum_{i=s_k+1}^{s_0} \lambda_i^{-1} \leqslant \frac{1}{2} \frac{s_k}{\lambda_{s_k}}. \tag{17}$$

PROOF OF CLAIM 2. Inequality (16) holds for $j = 0$ by Claim 1. We proceed by induction. By the induction hypothesis and (15)

$$\lambda_{s_{j+1}} < \frac{\lambda_{s_j}}{1+4\epsilon} < \frac{s_j^{5/4}}{1+4\epsilon}.$$

Hence by (14)

$$\lambda_{s_{j+1}} \leqslant \frac{(1+\epsilon+1/s_{j+1})^{5/4}}{1+4\epsilon} s_{j+1}^{5/4} \leqslant s_{j+1}^{5/4}$$

where the second inequality can be verified by taking logarithms and noting that $(x-1) \geqslant \ln x \geqslant (x-1)/2$ for $1 \leqslant x \leqslant 2$.

Inequality (17) is established as follows. Using (14) we see that

$$\sum_{i=s_j+1}^{s_{j-1}} \lambda_i^{-1} \leqslant (s_{j-1} - s_j)\lambda_{s_j}^{-1} \leqslant \left(\epsilon + \frac{1}{s_j}\right)\frac{s_j}{\lambda_{s_j}}, \qquad 1 \leqslant j \leqslant \kappa + 1. \tag{18}$$

Also from (14)

$$s_j \leqslant \left(1 + \epsilon + \frac{1}{s_{j+1}}\right)s_{j+1}, \qquad 0 \leqslant j \leqslant \kappa,$$

hence

$$s_j \leqslant \left(1 + \epsilon + \frac{1}{s_k}\right)^{k-j} s_k, \qquad 0 \leqslant j \leqslant k. \tag{19}$$

Iterating on (15) gives

$$\lambda_{s_j} > (1+4\epsilon)^{k-j}\lambda_{s_k}, \qquad 0 \leqslant j \leqslant k.$$

This, together with (19) gives

$$\frac{s_j}{\lambda_{s_j}} \leqslant \left(\frac{1+\epsilon+1/s_k}{1+4\epsilon}\right)^{k-j} \frac{s_k}{\lambda_{s_k}}, \qquad 0 \leqslant j \leqslant k,$$

and by (18) we have

$$\sum_{i=s_j+1}^{s_{j-1}} \lambda_i^{-1} \leqslant \left(\epsilon + \frac{1}{s_j}\right)\left(\frac{1+\epsilon+1/s_k}{1+4\epsilon}\right)^{k-j} \frac{s_k}{\lambda_{s_k}}, \qquad 1 \leqslant j \leqslant k.$$

Hence

$$\begin{aligned}
\sum_{i=s_k+1}^{s_0} \lambda_i^{-1} &= \sum_{j=k}^{1} \sum_{i=s_j+1}^{s_{j-1}} \lambda_i^{-1} \\
&\leqslant \sum_{j=k}^{1} \left(\epsilon + \frac{1}{s_k}\right)\left(\frac{1+\epsilon+1/s_k}{1+4\epsilon}\right)^{k-j} \frac{s_k}{\lambda_{s_k}} \\
&\leqslant \frac{s_k}{\lambda_{s_k}}\left(\epsilon + \frac{1}{s_k}\right)\sum_{j=0}^{k-1}\left(\frac{1+\epsilon+1/s_k}{1+4\epsilon}\right)^{j}.
\end{aligned}$$

But $s_k > N > 1/\epsilon^2$ so

$$\begin{aligned}\sum_{i=s_k+1}^{s_0} \lambda_i^{-1} &\leqslant \frac{s_k}{\lambda_{s_k}}(\epsilon+\epsilon^2)\bigg/\left(1-\left(\frac{1+\epsilon+\epsilon^2}{1+4\epsilon}\right)\right)\\ &= \frac{s_k}{\lambda_{s_k}}\epsilon(1+\epsilon)\frac{(1+4\epsilon)}{\epsilon(3-\epsilon)}\\ &\leqslant \frac{s_k}{\lambda_{s_k}}(1+1/25)\frac{1+4/25}{3-1/25}\\ &< \frac{1}{2}\frac{s_k}{\lambda_{s_k}}\end{aligned}$$

which establishes (17), hence Claim 2.

We have, using (14),

$$\begin{aligned}\sum_{i=N}^{s_\kappa} \lambda_i^{-1} &\leqslant (1+s_\kappa - s_{\kappa+1})\lambda_N^{-1} \leqslant \frac{2+\epsilon s_{\kappa+1}}{N}\\ &\leqslant (2+\epsilon N)/N = 2/N + \epsilon < 2\epsilon.\end{aligned}$$

This, together with (8), shows that (17) does not hold with $k = \kappa$. Thus, by Claim 2, (15) does not hold for $k = \kappa$ and we can define l to be the smallest integer satisfying $0 \leqslant l < \kappa$ and

$$\lambda_{s_l} \leqslant (1+4\epsilon)\lambda_{s_{l+1}} \tag{20}$$

Setting $s = s_l$ and $r = s_{l+1}$, we see that (4) and (5) are satisfied as follows.

If $l = 0$ then $\lambda_s \leqslant s^{5/4}$ by (7), $\lambda_s \leqslant (1+4\epsilon)\lambda_r$ by (20), and by (8) we have (4). Otherwise $l \geqslant 1$ and (15) is valid for $0 \leqslant j \leqslant l-1$. From (16), it follows that $\lambda_{s_l} = \lambda_s \leqslant s_l^{5/4} = s^{5/4}$. Also, $\lambda_s \leqslant (1+4\epsilon)\lambda_r$ follows from (20). Finally, from (8) and (17) there follows

$$\sum_{i=N}^{s} \lambda_i^{-1} = \sum_{i=N}^{s_0} \lambda_i^{-1} - \sum_{i=s+1}^{s_0} \lambda_i^{-1} \geqslant K + 1 - \frac{1}{2}\frac{s_l}{\lambda_{s_l}} > K$$

which establishes (4).

To establish (5) we note first that from (14) and the fact that $r > N > \epsilon^{-2}$ we have $s - r \geqslant \epsilon s/2$. Hence, for $N \leqslant q \leqslant r$,

$$\sum_{i=q+1}^{s} \lambda_i^{-1} \geqslant \sum_{i=r+1}^{s} \lambda_i^{-1} \geqslant \frac{s-r}{\lambda_s} \geqslant \frac{\epsilon s}{2\lambda_s}.$$

From (9) and (17)

$$\begin{aligned}\sqrt{q} &< \lambda_q \sum_{i=q+1}^{s_0} \lambda_i^{-1} + 5\lambda_q s_0^{-1/4} \\ &\leqslant \lambda_q \left(\sum_{i=q+1}^{s} \lambda_i^{-1} + \sum_{i=s+1}^{s_0} \lambda_i^{-1} + 5s/\lambda_s \right) \\ &\leqslant \lambda_q \left(\sum_{i=q+1}^{s} \lambda_i^{-1} + \frac{1}{2}\frac{s}{\lambda_s} + 5\frac{s}{\lambda_s} \right) \\ &\leqslant \lambda_q \left(\sum_{i=q+1}^{s} \lambda_i^{-1} \right) \left(1 + \frac{11}{2} \cdot \frac{2}{\epsilon} \right).\end{aligned}$$

Thus, for $N \leqslant q \leqslant r$,

$$\lambda_q \sum_{i=q+1}^{s} \lambda_i^{-1} > \sqrt{q} \left(1 + \frac{11}{\epsilon} \right)^{-1} > \sqrt{q}\epsilon/12$$

which establishes (5). □

LEMMA 4. *Let r and s be positive integers, $r < s$, $f \in C_0[0, 1]$, and $f(0) = 0$. Define*

$$E_s(f) = \inf_{a_j \in \mathbf{R}} \left\| f(x) - \sum_{j=1}^{s} a_j x^{\lambda_j} \right\|.$$

Then there exist integers b_j, $1 \leqslant j \leqslant s$, such that

$$\left\| f(x) - \sum_{j=1}^{s} b_j x^{\lambda_j} \right\| \leqslant 2E_s(f) + \sum_{q=1}^{r} A_{qs} + \frac{\lambda_s - \lambda_r}{\lambda_r} \tag{21}$$

where A_{qs} is defined in (2).

PROOF. By a standard compactness argument there exists a polynomial $\widetilde{P}_s$ of degree s or less such that $\|f - \widetilde{P}_s\| = E_s(f)$. Setting $P_s = \widetilde{P}_s - \widetilde{P}_s(1)x^{\lambda_1} = \Sigma_{j=1}^{s} a_{j0} x^{\lambda_j}$ it is easy to see that

$$\|f - P_s\| \leqslant 2E_s(f) \quad \text{and} \quad P_s(1) = \sum_{j=1}^{s} a_{j0} = 0. \tag{22}$$

We define coefficients b_j and a_{jq} by induction on q. By (21) we have (23) and (24) below when $q = 0$:

$$\left\| f(x) - \sum_{j=1}^{q} b_j x^{\lambda_j} - \sum_{j=q+1}^{s} a_{jq} x^{\lambda_j} \right\| \leqslant 2E_s(f) + \sum_{j=1}^{q} \|x^{\lambda_j} - Q_{js}(x)\| = A_q \tag{23}$$

and

$$\sum_{j=1}^{q} b_j + \sum_{j=q+1}^{s} a_{jq} = 0, \tag{24}$$

where the equality in (23) serves to define A_q.

To describe the induction step we assume (23) and (24) hold. Define $b_{q+1} = [a_{q+1,q}]$ and $a_{j,q+1} = a_{jq} + (a_{q+1,q} - b_{q+1})c_{j,q+1,s}$ $(q + 2 \leqslant j \leqslant s)$ where $c_{j,q+1,s}$ are the coefficients of the polynomial $Q_{q+1,s}$ in Lemma 1. Then

$$\left\| f(x) - \sum_{j=1}^{q+1} b_j x^{\lambda_j} - \sum_{j=q+2}^{s} a_{j,q+1} x^{\lambda_j} \right\|$$

$$= \left\| f(x) - \sum_{j=1}^{q} b_j x^{\lambda_j} - \sum_{j=q+1}^{s} a_{jq} x^{\lambda_j} - (a_{q+1,q} - b_{q+1})(Q_{q+1,s}(x) - x^{\lambda_{q+1}}) \right\|$$

$$\leqslant A_q + \|Q_{q+1,s}(x) - x^{\lambda_{q+1}}\|,$$

and

$$\sum_{j=1}^{q+1} b_j + \sum_{j=q+2}^{s} a_{j,q+2} = \sum_{j=1}^{q+1} b_j + \sum_{j=q+2}^{s} (a_{jq} + (a_{q+1,q} - b_{q+1})c_{j,q+1,s})$$

$$= b_{q+1} - a_{q+1,q} + (a_{q+1,q} - b_{q+1}) \sum_{j=q+2}^{s} c_{j,q+1,s} = 0$$

since $Q_{q+1,s}(1) = \Sigma_{j=q+2}^{s} c_{j,q+1,s} = 1$. Thus (23) and (24) hold for $q + 1$ in place of q for this definition of b_{q+1} and $a_{j,q+1}$ $(q + 2 \leqslant j \leqslant s)$.

We stop the above induction at $q = r$ and proceed differently to define $b_{r+1}, \ldots, b_s$. Thus we have

$$(25) \quad \left\| f(x) - \sum_{j=1}^{r} b_j x^{\lambda_j} - \sum_{j=r+1}^{s} a_{jr} x^{\lambda_j} \right\| \leqslant 2E_s(f) + \sum_{j=1}^{r} \|x^{\lambda_j} - Q_{js}(x)\| = A_r$$

and

$$(26) \qquad \sum_{j=1}^{r} b_j + \sum_{j=r+1}^{s} a_{jr} = 0.$$

Define, recursively, for $j = s, s - 1, \ldots, r + 1$, $d_s = a_{sr} - [a_{sr}]$ and

$$(27) \qquad d_j = \begin{cases} a_{jr} - [a_{jr}] & \text{if } \sum_{i=j+1}^{s} d_j \leqslant 0 \\ a_{jr} - [a_{jr}] - 1 & \text{if } \sum_{i=j+1}^{s} d_i > 0 \end{cases} \qquad (s - 1 \geqslant j \geqslant r + 1).$$

Then the d_j's satisfy the inequality in the hypotheses of Lemma 2. Also, from (27), $d_i \equiv a_{ir} \pmod 1$ $(r + 1 \leqslant i \leqslant s)$ so $\Sigma_{i=r+1}^{s} d_i \equiv \Sigma_{i=r+1}^{s} a_{ir} \pmod 1$. But, by (26), $\Sigma_{i=r+1}^{s} a_{ir} \equiv 0 \pmod 1$ and since $|\Sigma_{i=r+1}^{s} d_i| < 1$ we have $\Sigma_{i=r+1}^{s} d_i = 0$. Define $b_j = a_{jr} - d_j$ $(r+1 \leqslant j \leqslant s)$ and $p_{rs}(x) = \Sigma_{j=r+1}^{s} d_j x^{\lambda_j}$. The polynomial p_{rs} satisfies the hypotheses of Lemma 2. Hence $\|p_{rs}\| \leqslant (\lambda_s - \lambda_r)/\lambda_r$. Thus

$$\left\| f(x) - \sum_{j=1}^{s} b_j x^{\lambda_j} \right\| = \left\| f(x) - \sum_{j=1}^{r} b_j x^{\lambda_j} - \sum_{j=r+1}^{s} a_{jr} x^{\lambda_j} + p_{rs}(x) \right\|$$

$$\leqslant A_r + \|p_{rs}\| \leqslant A_r + (\lambda_s - \lambda_r)/\lambda_r. \quad \square$$

PROOF OF THEOREM 2. Let $f \in C_0[0, 1]$. Since it suffices to approximate $f - f(1)x^{\lambda_1} - f(0)(1 - x^{\lambda_1})$, we can assume that $f(0) = 0 = f(1)$. Let $0 < \epsilon < 1/25$. By the classical Müntz theorem $E_i(f) \longrightarrow 0$ as $i \longrightarrow \infty$. Also $\lambda_i < i^{5/4}$ for infinitely many i or else we would have $\Sigma_{i=1}^{\infty} \lambda_i^{-1} < \infty$. Thus there exists an integer N such that $E_N(f) \leqslant \epsilon$, $N \geqslant 4!(6/\epsilon)^5$, and $\lambda_N < N^{5/4}$. Choose $K > 0$ such that $\exp(-2K) \leqslant \epsilon$. By Lemma 3 there exist integers r and s, $N < r < s$, such that (4) and (5) hold. Applying Lemma 4 to these integers r and s we see that there exist integers b_j $(1 \leqslant j \leqslant s)$ such that (21) holds. We estimate the right-hand side of (21) as follows:

$$2E_s(f) \leqslant 2E_N(f) \leqslant 2\epsilon,$$

$$(\lambda_s - \lambda_r)/\lambda_r \leqslant 4\epsilon \quad \text{(using (4))},$$

$$\sum_{q=1}^{N-1} A_{qs} \leqslant 2 \sum_{q=1}^{N-1} \exp\left(-2\lambda_q \sum_{i=q+1}^{s} \lambda_i^{-1}\right) \quad \text{(using Lemma 1)}$$

$$\leqslant 2 \sum_{q=1}^{N-1} \exp(-2\lambda_q K) \quad \text{(using (4))}$$

$$\leqslant 2 \sum_{q=1}^{N-1} \epsilon^{\lambda_q} < 3\epsilon,$$

$$\sum_{q=N}^{r} A_{qs} \leqslant 2 \sum_{q=N}^{r} \exp\left(-2\lambda_q \sum_{i=q+1}^{s} \lambda_i^{-1}\right) \quad \text{(using Lemma 1)}$$

$$\leqslant 2 \sum_{q=N}^{r} e^{-\epsilon\sqrt{q}/6} \quad \text{(using (5))}$$

$$< 2 \sum_{q=N}^{r} 4! \left(\frac{6}{\epsilon}\right)^4 \frac{1}{q^2}$$

$$< 4 \cdot 4! \left(\frac{6}{\epsilon}\right)^4 \frac{1}{N} < \epsilon.$$

Thus (21) gives

$$\left\| f(x) - \sum_{j=1}^{s} b_j x^{\lambda_j} \right\| \leqslant 2\epsilon + (3\epsilon + \epsilon) + 4\epsilon = 10\epsilon. \quad \square$$

Results similar to the above can be established more simply under certain conditions as follows. A preliminary version of these results appeared in Ferguson [4].

Let Λ be any subset of the positive real numbers.

THEOREM 3. *If the set Λ has a limit point x_0 with $0 < x_0 < \infty$ then the integral Λ-polynomials are dense in $C_0[0, 1]$.*

PROOF. Let $f \in C_0[0, 1]$, $\epsilon > 0$, and $\lambda \in \Lambda$. Since $f(0)$ and $f(1)$ are integers, it suffices to approximate $f - f(0) - (f(1) - f(0))x^\lambda$, and we assume without loss of generality that $f(0) = f(1) = 0$. Since x_0 is a positive limit point of Λ, it is easy to see that we can extract from Λ a sequence $\{\lambda_i\}$ satisfying

(1) $$\lambda_i \longrightarrow x_0,$$

(2) $$\lambda_i \text{ is monotone,}$$

(3) $$\lambda_i > 1, \quad \text{all } i,$$

or

(4) $$\lambda_i < 1, \quad \text{all } i,$$

and

(5) $$|\lambda_1 - \lambda_j|/\lambda_k < \epsilon, \quad \text{all } j, k.$$

Since $x_0 > 0$ we have $\Sigma_{j=1}^{\infty} \lambda_j/(1 + \lambda_j^2) = \infty$; hence (cf. Paley-Weiner [8, Theorem XV]) there is a Λ-polynomial p_0 where $p_0(x) = a + \Sigma_{j=1}^{n} b_j x^{\lambda_j}$ with

(6) $$\|f - p_0\| < \epsilon$$

and a constant. By (1), since $f(0) = 0$, $|a| < \epsilon$, hence

(7) $$\|p_0 - p_1\| < \epsilon$$

where $p_1 = p_0 - a$. It is easy to see that we can write p_1 in the form $p_1(x) = cx^{\lambda_1} + \Sigma_{j=2}^{n} a_j(x^{\lambda_j} - x^{\lambda_{j-1}})$. From (6), (7) and $f(1) = 0$, $|c| < 2\epsilon$, and we have

(8) $$\|p_1 - p_2\| < 2\epsilon$$

where $p_2 = p_1 - cx^{\lambda_1}$. Define an integral Λ-polynomial $[p_2]$ by $[p_2](x) = \Sigma_{j=2}^{n} [a_j](x^{\lambda_j} - x^{\lambda_{j-1}})$ where $[a_j]$ denotes the greatest integer less than or equal to a_j. Then

$$
\begin{aligned}
|p_2(x) - [p_2](x)| &= \left|\sum_{j=2}^{n} (a_j)(x^{\lambda_j} - x^{\lambda_{j-1}})\right| \\
&\leq \sum_{j=2}^{n} (a_j)|x^{\lambda_j} - x^{\lambda_{j-1}}| \\
&\leq \sum_{j=2}^{n} |x^{\lambda_j} - x^{\lambda_{j-1}}| = \left|\sum_{j=2}^{n} (x^{\lambda_j} - x^{\lambda_{j-1}})\right| \\
&= |x^{\lambda_n} - x^{\lambda_1}|
\end{aligned} \tag{9}
$$

where the second equality follows from the monotonicity of the numbers x^{λ_i} as i increases. This monotonicity in turn follows from the properties (2), (3) and (4) of the sequence $\{\lambda_i\}$ and well-known results concerning exponentiation.

An elementary analysis shows that $|x^{\lambda_n} - x^{\lambda_1}| \leq |\lambda_1 - \lambda_n|/\min\{\lambda_1, \lambda_n\}$; hence by (9) and (5)

$$\|p_2 - [p_2]\| \leq \epsilon. \tag{10}$$

From (6), (7), (8) and (10), $\|f - [p_2]\| < 5\epsilon$. □

Another direction in which the above results can be extended is the following. Let $C_0[0, \alpha]$, $\alpha < 1$, denote the real valued continuous functions on the interval $[0, \alpha]$ which take on integer values at 0, and $\|\cdot\|$ the supremum norm on $C_0[0, \alpha]$.

THEOREM 4. *Let Λ be a subset of the positive real numbers with no finite limit point and $\Sigma_{\lambda \in \Lambda} \lambda^{-1} = \infty$. Then the integral Λ-polynomials are dense in $C_0[0, \alpha]$ for any $\alpha < 1$.*

PROOF. Let $f \in C_0[0, \alpha]$ and $\epsilon > 0$. Since Λ has no finite limit points, there are only finitely many λ's in any bounded interval and we can assume without loss of generality that $\alpha^\lambda < \epsilon$, all $\lambda \in \Lambda$. Next extract from Λ a sequence $\{\lambda_i\}$ which is monotone increasing and satisfies $\Sigma_i \lambda_i^{-1} = \infty$, hence $\Sigma_i \lambda_i/(1 + \lambda_i^2) = \infty$. Proceeding as in the proof of Theorem 3 above we construct a Λ-polynomial p_1 satisfying

$$\|f - p_1\| < 2\epsilon. \tag{11}$$

Then

$$
\begin{aligned}
\|p_1 - [p_1]\| &\leq \|x^{\lambda_1}\| + \|x^{\lambda_n} - x^{\lambda_1}\| \\
&\leq 2\alpha^{\lambda_1} + \alpha^{\lambda_n} \\
&< 3\epsilon.
\end{aligned}
$$

This and (11) gives $\|f - [p_1]\| < 5\epsilon$ by the triangle inequality. □

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