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A FORMULA FOR THE TANGENT BUNDLE OF FLAG MANIFOLDS AND RELATED MANIFOLDS

BY

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ABSTRACT. A formula is given for the tangent bundle of a flag manifold G in terms of canonically defined vector bundles over G. The formula leads to a unified proof of some parallelizability theorems of Stiefel manifolds. It can also be used to deduce some immersion theorems for flag manifolds.

1. **Introduction.** In this note **F** will denote either the field **R** of real numbers, the field **C** of complex numbers, or the skew-field **H** of quaternions. $Z(\mathbf{F})$ will denote the centre of **F**, and $d = \dim_{\mathbf{R}} \mathbf{F}$. The space $\mathbf{F}^n$ of all n-tuples of elements in **F** will be equipped with a "hermitian" product $\langle\ ,\ \rangle$ with values in **F**, and a dot product with values in **R**, as follows: if $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ are in $\mathbf{F}^n$, then

$$\langle x, y\rangle = x_1\bar{y}_1 + x_2\bar{y}_2 + \cdots + x_n\bar{y}_n \in \mathbf{F},$$

$$x \cdot y = \tfrac{1}{2}(\langle x, y\rangle + \langle y, x\rangle) \in \mathbf{R}.$$

Finally, all vector spaces over **F** are left vector spaces, unless otherwise specified.

Suppose now $n_1, n_2, \ldots, n_s$ are fixed positive integers such that $n_1 + n_2 + \cdots + n_s = n$. By a "flag", or more precisely a "$(n_1, \ldots, n_s)$-flag over **F**", we mean a collection σ of mutually orthogonal subspaces $(\sigma_1, \ldots, \sigma_s)$ in $\mathbf{F}^n$ such that $\dim_F \sigma_i = n_i$. The space of all such flags forms a compact smooth manifold (and a complex manifold in case $\mathbf{F} = \mathbf{C}$). We shall denote this manifold by $G_{\mathbf{F}}(n_1, \ldots, n_s)$, or simply by G if $n_1, \ldots, n_s$ and **F** are clearly understood. As an example, note that $G_{\mathbf{R}}(n_1, n_2)$ is just the Grassmannian manifold of n_1 planes in Euclidean $n_1 + n_2$ space.

One way to describe G as a smooth manifold is the following. Let $U_{\mathbf{F}}(n)$ be the Lie group of all $n \times n$ matrices A with entries in **F** satisfying $A\bar{A}^t = I_n$. Then $U_{\mathbf{F}}(n)$ acts on $\mathbf{F}^n$ from the right, the action being $(x, A) \mapsto xA$, where $x \in \mathbf{F}^n$ is regarded as a row vector. In a well-known manner, one can identify

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the space G with the homogeneous space $U_{\mathbf{F}}(n)/U_{\mathbf{F}}(n_1) \times \cdots \times U_F(n_s)$. In particular, G is a compact smooth manifold, whose dimension can be computed to be $\frac{1}{2}d(n^2 - \Sigma n_i^2)$.

The purpose of this note is to furnish an explicit formula for the tangent bundle of G, and to derive from it some applications to questions such as immersing G into Euclidean spaces, parallelizability of Stiefel manifolds, etc.

We begin by observing that there are naturally defined $\mathbf{F}$-vector bundles $\xi_1, \xi_2, \ldots, \xi_s$ over G. In fact we just define ξ_i to be the $\mathbf{F}$-vector bundle whose fibre at the point σ is the vector space σ_i. Note that since σ_i inherits the hermitian product of $\mathbf{F}^n$, ξ_i is automatically a $\mathbf{F}$-vector bundle with "hermitian" product $\langle\, , \rangle$. Next it should be noted that there is an $\mathbf{F}$-isomorphism

$$(1.0) \qquad \xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_s \approx_{\mathbf{F}} n\epsilon_{\mathbf{F}} \quad (= \text{trivial bundle of } \mathbf{F}\text{-dim } n),$$

because $\xi_1 \oplus \cdots \oplus \xi_s$ is obtained by simply "erecting $\mathbf{F}^n$ as fibre over each and every point $\sigma \in G$." Furthermore, this isomorphism preserves hermitian product.

THEOREM (1.1). *The tangent bundle $T(G)$ of G is isomorphic to*

$$\bigoplus_{1 \leq i < j \leq s} \mathrm{Hom}_{\mathbf{F}}(\xi_i, \xi_j)$$

as $Z(\mathbf{F})$-vector bundles.

From elementary linear algebra one has

$$\mathrm{Hom}_{\mathbf{F}}(\xi_i, \xi_j) \approx_{Z(\mathbf{F})} \mathrm{Hom}_{\mathbf{F}}(\xi_i, \epsilon_{\mathbf{F}}) \otimes_{\mathbf{F}} \xi_j,$$

where $\epsilon_{\mathbf{F}}$ is the trivial bundle $G \times \mathbf{F}$ and $\mathrm{Hom}_{\mathbf{F}}(\xi_i, \epsilon_{\mathbf{F}})$ is the dual bundle of right $\mathbf{F}$-vector spaces. Furthermore, the hermitian product in ξ_i permits us to identity $\mathrm{Hom}_{\mathbf{F}}(\xi_i, \epsilon_{\mathbf{F}})$ with the conjugate bundle $\bar{\xi}_i$, whose typical fibre is the right $\mathbf{F}$-vector space obtained by forming the "conjugate" of the corresponding fibre of ξ_i (see §4). Thus theorem (1.1) can be stated as

COROLLARY (1.2). *The tangent bundle $T(G)$ of G is isomorphic to*

$$\bigoplus_{1 \leq i < j \leq s} \bar{\xi}_i \otimes_{\mathbf{F}} \xi_j$$

as $Z(\mathbf{F})$-vector bundles.

2. **Proof of Theorem (1.1).** To prove Theorem (1.1) we first show that for each $i < j$, $\mathrm{Hom}_{\mathbf{F}}(\xi_i, \xi_j)$ can be embedded into $T(G)$ as a $Z(\mathbf{F})$-subbundle in a natural way. For notational convenience we shall describe the embedding of $\mathrm{Hom}_{\mathbf{F}}(\xi_1, \xi_2)$ into $T(G)$, the description for the other embeddings being entirely similar.

Let $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_s)$ be a flag, or a point in G. The fibre of

$\mathrm{Hom}_{\mathbf{F}}(\xi_1, \xi_2)$ over σ is the vector space $\mathrm{Hom}_{\mathbf{F}}(\xi_1, \xi_2)_\sigma = \mathrm{Hom}_{\mathbf{F}}(\sigma_1, \sigma_2)$. Let $f\colon \sigma_1 \longrightarrow \sigma_2$ be an element of this fibre. For any $t \in \mathbf{R}$ let $\gamma(tf)$ be the graph of tf in the direct sum $\sigma_1 \oplus \sigma_2$, and let $\gamma^\perp(tf)$ be the orthogonal complement of $\gamma(tf)$ in this direct sum. Clearly, $\gamma(tf)$ is an n_1-plane in $\mathbf{F}^n$, and coincides with σ_1 when $t = 0$. As t varies, the flag

$$\sigma_t = (\gamma(tf), \gamma^\perp(tf), \sigma_3, \ldots, \sigma_s)$$

describes a smooth curve in G passing through σ, and we can take its tangent vector $\iota(f)$ at σ.

LEMMA (2.1). *The correspondence $f \longrightarrow \iota(f)$ induces an embedding ι of $\mathrm{Hom}_{\mathbf{F}}(\xi_1, \xi_2)$ into $T(G)$.*

To prove Lemma (2.1) we shall introduce coordinates in a neighbourhood of the point σ, as follows. By recoordinating $\mathbf{F}^n$ if necessary, one can suppose that the first n_1 rows of the identity matrix I_n form a basis of σ_1, the next n_2 rows of I_n form a basis of σ_2, and so on. Consider now a matrix

$$A = \begin{bmatrix} I_{n_1} & * & \cdots & * \\ 0 & I_{n_2} & \cdots & * \\ & & \ddots & \vdots \\ 0 & 0 & \cdots & I_{n_s} \end{bmatrix}$$

in upper diagonal block form with entries in $\mathbf{F}$. Let $V_i \subset \mathbf{F}^n$ be the subspace spanned by the first $n_1 + n_2 + \cdots + n_i$ rows of A. Let σ_i^A be the orthogonal complement of V_{i-1} in V_i. Then $\sigma(A) = (\sigma_1^A, \sigma_2^A, \ldots, \sigma_s^A)$ is a flag, and hence a point in G. Furthermore if B is another upper diagonal block matrix, then $\sigma(A) = \sigma(B)$ implies $A = B$. Thus we can use the upper diagonal entries of A as a coordinate chart (a complex coordinate chart in case $\mathbf{F} = \mathbf{C}$) for an open neighbourhood U of the point $\sigma \in G$, with σ being the origin.

Using the bases of σ_1 and σ_2 given above, we can write down the $n_1 \times n_2$ matrix $M(f)$ of $f\colon \sigma_1 \longrightarrow \sigma_2$. If now

$$A_t = \begin{bmatrix} I_{n_1} & tM(f) & \cdots & 0 \\ 0 & I_{n_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & I_{n_s} \end{bmatrix},$$

then the curve σ_t is just $\sigma(A_t)$, so that in terms of coordinates the components of the tangent vector $\iota(f) = (d\sigma_t/dt)_{t=0}$, when arranged in blocks in upper diagonal positions, are simply

$$\begin{matrix} 0 & M(f) \cdots 0 \\ & 0 \cdots 0 \\ & \cdots 0 \end{matrix}$$

This clearly shows that ι is a $Z(\mathbf{F})$-linear embedding of $\mathrm{Hom}_{\mathbf{F}}(\xi_1, \xi_2)$ into $T(G)$.

To finish the proof of Theorem (1.1), let $\iota_{i,j}$ $(i < j)$ be the natural embedding of $\mathrm{Hom}_{\mathbf{F}}(\xi_i, \xi_j)$ into $T(G)$. If $f_{i,j}: \sigma_i \longrightarrow \sigma_j$ is an arbitrary vector in the fibre of $\mathrm{Hom}_{\mathbf{F}}(\xi_i, \xi_j)$ over σ, then in the coordinate neighbourhood U, the components for the tangent vector $\Sigma_{1 \leqslant i < j \leqslant s} \iota_{i,j}(f_{i,j})$, when arranged in blocks in upper diagonal positions, are just

$$\begin{matrix} M(f_{1,2}) M(f_{1,3}) & \cdots & M(f_{1,s}) \\ M(f_{2,3}) & \cdots & M(f_{2,s}) \\ & \cdots & M(f_{s-1,s}). \end{matrix}$$

This makes it clear that

$$\bigoplus_{1 \leqslant i < j \leqslant s} \mathrm{Hom}_{\mathbf{F}}(\xi_i, \xi_j) \approx_{Z(\mathbf{F})} T(G),$$

and Theorem (1.1) is proved.

REMARK. In the case of Grassmannian manifolds, the formula for $T(G)$ has been established in [9, p. 411] for $\mathbf{F} = \mathbf{R}$, and also in [5, p. 217] using Lie group representations. (See also [8].) Our formula for the tangent bundle of flag manifolds can be regarded both as an extension to that of [9], and as a totally different approach in the Grassmannian case that is considerably simpler than the treatment of [5].

3. **Tangent bundle of some related manifolds.** It is easy to see that there is an analogous formula for the tangent bundle of manifolds formed by the set of all mutually orthogonal subspaces $(\hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_k, \sigma_{k+1}, \ldots, \sigma_s)$ in $\mathbf{R}^n$ with $\mathbf{R}$-dimensions equal to $n_1, n_2, \ldots, n_s$ respectively, and with $\hat{\sigma}_1, \ldots, \hat{\sigma}_k$ oriented. In particular, real Stiefel manifolds can be looked upon as manifolds of this type. However, we shall give a unified treatment for the tangent bundle of real, complex and quaternionic Stiefel manifolds in the next paragraph.

By the Stiefel manifold $V = V_{\mathbf{F}}(n, k)$ we mean the manifold of all k-frames θ in $\mathbf{F}^n$ whose k vectors $\theta_1, \theta_2, \ldots, \theta_k$ satisfy $\langle \theta_i, \theta_j \rangle = \delta_{ij}$ for $1 \leqslant i, j \leqslant k$. Alternatively we can define V to be the homogeneous space $U_{\mathbf{F}}(n)/U_{\mathbf{F}}(n-k)$. If $\theta \in V$ is a k-frame in $\mathbf{F}^n$, let $\mathbf{F}\theta_i$ be the $\mathbf{F}$-subspace spanned by the ith vector θ_i of the frame θ, and let $\theta^{\perp}$ be the orthogonal complement of $\mathbf{F}\theta_1 \oplus \cdots \oplus \mathbf{F}\theta_k$ in $\mathbf{F}^n$. Then $\pi(\theta) = (\mathbf{F}\theta_1, \ldots, \mathbf{F}\theta_n, \theta^{\perp})$ is a $(1, \ldots, 1, n-k)$-flag in $\mathbf{F}^n$, and in this way we obtain a principal fibration

$$\pi\colon V_{\mathbf{F}}(n, k) \longrightarrow G_{\mathbf{F}}(\underbrace{1, \dots, 1}_{k}, n-k),$$

whose fibre is $U_{\mathbf{F}}(1) \times \cdots \times U_{\mathbf{F}}(1)$ (k factors). Furthermore, if $\xi_1, \dots, \xi_k$ and ξ_{k+1} are the canonical bundles over $G_{\mathbf{F}}(1, \dots, 1, n-k)$ defined in §1, then the pull-back bundles $\pi^*(\xi_i)$ have the following description:

(1) For $1 \leqslant i \leqslant k$, $\pi^*(\xi_i)$ is the trivial bundle $\epsilon_{\mathbf{F}}$. This is because the fibre $\pi^*(\xi_i)_\theta$ is none other than $\mathbf{F}\theta_i$, in which the vector θ_i serves as a canonical non-zero section.

(2) $\pi^*(\xi_{k+1})$ is obtained by "erecting $\theta^\perp$ as the fibre over $\theta \in V_{\mathbf{F}}(n, k)$". Henceforth, we shall refer to this bundle as $\eta_{\mathbf{F}}(n, k)$, or simply as η if n, k and $\mathbf{F}$ are clearly understood.

(3) Since $\xi_1 \oplus \cdots \oplus \xi_k \oplus \xi_{k+1} \approx_{\mathbf{F}} n\epsilon_{\mathbf{F}}$, applying π^* gives $k\epsilon_{\mathbf{F}} \oplus \eta \approx_{\mathbf{F}} n\epsilon_{\mathbf{F}}$, or $kd\epsilon_{\mathbf{R}} \oplus \eta \approx nd\epsilon_{\mathbf{R}}$ if the underlying real bundle of η is to be emphasised.

In particular, this shows that

$$(*) \qquad l\epsilon_{\mathbf{R}} \oplus \eta \approx \text{trivial bundle}, \quad \text{whenever } l \geqslant kd.$$

The above information now enables us to prove

THEOREM (3.1). *For $n \geqslant k \geqslant 2$, the Stiefel manifold $V_{\mathbf{F}}(n, k)$ is parallelizable, i.e. its tangent bundle is $\mathbf{R}$-trivial.*

PROOF. Since π is a principal fibration, the bundle along the fibre is the trivial bundle $k(d-1)\epsilon_{\mathbf{R}}$. Hence, with "$\approx$" meaning "isomorphisms over $\mathbf{R}$",

$$\begin{aligned} T(V_{\mathbf{F}}(n, k)) &\approx \pi^*(T(G_{\mathbf{F}}(1, \dots, 1, n-k))) \oplus \text{bundle along the fibre} \\ &\approx \bigoplus_{1 \leqslant i < j \leqslant k+1} \pi^*(\mathrm{Hom}_{\mathbf{F}}(\xi_i, \xi_j)) \oplus k(d-1)\epsilon_{\mathbf{R}} \\ &\approx \bigoplus_{1 \leqslant i < j \leqslant k} \mathrm{Hom}_{\mathbf{F}}(\epsilon_{\mathbf{F}}, \epsilon_{\mathbf{F}}) \oplus k\, \mathrm{Hom}_{\mathbf{F}}(\epsilon_{\mathbf{F}}, \eta) \oplus k(d-1)\epsilon_{\mathbf{R}} \\ &\approx \tfrac{1}{2}k(k-1)d\epsilon_{\mathbf{R}} \oplus k\eta \oplus k(d-1)\epsilon_{\mathbf{R}} \\ &\approx \tfrac{1}{2}k(kd+d-2)\epsilon_{\mathbf{R}} \oplus k\eta. \end{aligned}$$

Since $\frac{1}{2}k(kd+d-2) \geqslant kd$ unless $\mathbf{F} = \mathbf{R}$ and $k = 2$, we can repeatedly apply $(*)$ to conclude that, apart from $V_{\mathbf{R}}(n, 2)$, $T(V_{\mathbf{F}}(n, k))$ is trivial for all $k \geqslant 2$. As for $V_{\mathbf{R}}(n, 2)$, our argument only shows that $T(V_{\mathbf{R}}(n, 2)) \oplus \epsilon_{\mathbf{R}}$ is trivial. Although one can refine the argument to trivialise $T(V_{\mathbf{R}}(n, 2))$ in case $n \equiv 0 \pmod 4$, using the fact that S^{n-1} has three vector fields that lift to three sections in $\eta \oplus \epsilon_{\mathbf{R}}$, the parallelizability of $V_{\mathbf{R}}(n, 2)$ for general n seems to require a different type of argument, such as for example in [2] or [10].

REMARK. The parallelizability of $V_{\mathbf{F}}(n, k)$ for $k \geqslant 2$ is a known result, see Sutherland [10] and Handel [6]. Nevertheless, the proof given in Theorem (3.1) is more elementary and unifies the cases $k \geqslant 3$ for all possible $\mathbf{F}$. It also provides a rather explicit trivialisation of the tangent bundle of Stiefel manifolds, save for the exceptional case of $V_{\mathbf{R}}(n, 2)$.

We now confine ourselves to $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$, and turn our attention to the so called "projective Stiefel manifolds", which have found some applications in topology (see [1], [3] and [4]). The projective Stiefel manifold $PV_{\mathbf{F}}(n, k)$, by definition, is obtained from $V_{\mathbf{F}}(n, k)$ by identifying any frame $\theta = (\theta_1, \ldots, \theta_k)$ with the frame $\alpha\theta = (\alpha\theta_1, \ldots, \alpha\theta_k)$ for any $\alpha \in U_{\mathbf{F}}(1)$. For convenience, we can refer to a point $[\theta]$ in $PV_{\mathbf{F}}(n, k)$ as a "projective k-frame" in $\mathbf{F}^n$. Note that as a homogeneous space, $PV_{\mathbf{F}}(n, k)$ is just $U_{\mathbf{F}}(n)/\Delta_{\mathbf{F}}(k) \times U_{\mathbf{F}}(n-k)$, where $\Delta_{\mathbf{F}}(k)$ is the subgroup of $U_{\mathbf{F}}(1) \times \cdots \times U_{\mathbf{F}}(1)$ consisting of matrices αI_k with $\alpha \in U_{\mathbf{F}}(1)$. There is a projection map

$$p\colon PV_{\mathbf{F}}(n, k) \longrightarrow G_{\mathbf{F}}(1, \ldots, 1, n-k)$$

obtained by sending a projective frame $[\theta]$ to the flag $(\mathbf{F}\theta_1, \ldots, \mathbf{F}\theta_k, \theta^{\perp})$. Note that p is a principal fibration with structural group $U_{\mathbf{F}}(1) \times \cdots \times U_{\mathbf{F}}(1)/\Delta_{\mathbf{F}}(k)$. (Remember that $U_{\mathbf{F}}(1)$ is communative for $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$.) Under p, the pull-back bundles $p^*(\xi_i)$, $1 \leqslant i \leqslant k+1$, have the following description:

(1) If ζ is the $\mathbf{F}$-line bundle over $PV_{\mathbf{F}}(n, k)$ well defined by erecting $\mathbf{F}\theta_1$ as fibre over the projective frame $[\theta] = [\theta_1, \ldots, \theta_k]$, then for $1 \leqslant i \leqslant k$, $p^*(\xi_i) \approx_{\mathbf{F}} \zeta$ as $\mathbf{F}$-vector bundles. This is because the map

$$p^*(\xi_i)_{[\theta]} \ni \lambda\theta_i \longmapsto \lambda\theta_1 \in \zeta_{[\theta]} \qquad (\lambda \in \mathbf{F})$$

gives a well-defined $\mathbf{F}$-isomorphism from $p^*(\xi_i)$ to ζ.

(2) $p^*(\xi_{k+1})$ is the bundle which erects the vector space $\theta^{\perp}$ as fiber over the projective frame $[\theta]$. We shall also denote this bundle by η, for the reason that under the identification map $V_{\mathbf{F}}(n, k) \longrightarrow PV_{\mathbf{F}}(n, k)$, it pulls back to what we previously called η over the Stiefel manifold $V_{\mathbf{F}}(n, k)$.

Since p is a principal fibration, the bundle along the fibre for p is trivial, of $\mathbf{R}$-dimension $(k-1)(d-1)$. Consequently, the tangent bundle of a projective Stiefel manifold as a real bundle is given by

$$\begin{aligned} T(PV_{\mathbf{F}}(n, k)) &\approx p^*T(G(1, \ldots, 1, n-k)) \oplus (k-1)(d-1)\epsilon_{\mathbf{R}} \\ &\approx p^*\Bigg(\bigoplus_{1 \leqslant i < j \leqslant k+1} \mathrm{Hom}_{\mathbf{F}}(\xi_i, \xi_j)\Bigg) \oplus (k-1)(d-1)\epsilon_{\mathbf{R}} \\ &\approx \bigoplus_{1 \leqslant i < j \leqslant k} \mathrm{Hom}_{\mathbf{F}}(\zeta, \zeta) \oplus k\, \mathrm{Hom}_{\mathbf{F}}(\zeta, \eta) \oplus (k-1)(d-1)\epsilon_{\mathbf{R}}. \end{aligned}$$

Since $\mathrm{Hom}_{\mathbf{F}}(\zeta, \zeta)$ is trivial for $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$, we obtain

THEOREM (3.2). *The tangent bundle of a projective Stiefel manifold is given by*

$$(**)\qquad T(PV_{\mathbf{F}}(n, k)) \approx k\,\mathrm{Hom}_{\mathbf{F}}(\zeta, \eta) \oplus \tfrac{1}{2}(k-1)(kd + 2d - 2)\epsilon_{\mathbf{R}}.$$

COROLLARY (3.3). *This tangent bundle, abbreviated as T, satisfies the relationship* $T \oplus (\tfrac{1}{2}k(k-1)d + k + d - 1)\epsilon_{\mathbf{R}} \approx kn\zeta$.

PROOF. The relationship $\xi_1 \oplus \cdots \oplus \xi_k \oplus \xi_{k+1} \approx_{\mathbf{F}} n\epsilon_{\mathbf{F}}$ pulls back under p^* to give $k\zeta \oplus \eta \approx_{\mathbf{F}} n\epsilon_{\mathbf{F}}$. Multiplying by k, one obtains $k^2\zeta \oplus k\eta \approx_{\mathbf{F}} kn\epsilon_{\mathbf{F}}$. Taking $\mathrm{Hom}_{\mathbf{F}}(\zeta, \_)$, using $\mathrm{Hom}_{\mathbf{F}}(\zeta, \zeta) \approx d\epsilon_{\mathbf{R}}$ and $\mathrm{Hom}_{\mathbf{F}}(\zeta, \epsilon_{\mathbf{F}}) \approx_{\mathbf{R}} \zeta$, we get

$$k^2 d\epsilon_{\mathbf{R}} \oplus k\,\mathrm{Hom}_{\mathbf{F}}(\zeta, \eta) \approx kn\zeta,$$

from which the corollary follows by adding to both sides of (**) the trivial bundle $(\tfrac{1}{2}k(k-1)d + k + d - 1)\epsilon_{\mathbf{R}}$.

Note that when $k = 1$, $PV_{\mathbf{F}}(n, 1)$ is just a real or complex projective space, and ζ is the canonical Hopf line bundle over it. In this case Corollary (3.3) simply reduces to the well-known facts about the tangent bundle of such spaces.

We remark finally that although an analogous manifold $PV_{\mathbf{F}}(n, k)$ can be defined for $\mathbf{F}$ the quaternions, the description of its tangent bundle is not as simple as in Theorem (3.2), due to the noncommutativity of the quaternions.

4. **The functor μ^2.** In this section we discuss a functor μ^2 analogous to the "symmetric power functor" s^2 on vector spaces. The result will be used in §5 to obtain immersions of flag manifolds into Euclidean spaces.

If V is a left vector space over $\mathbf{F}$, the conjugate vector space $\overline{V}$ is the right $\mathbf{F}$-vector space having V as underlying abelian group, with scalar multiplication given by $v\alpha = \bar{\alpha}v$ for any $v \in V$ and α in $\mathbf{F}$. The tensor product $\overline{V} \otimes_{\mathbf{F}} V$ is then a vector space over $\mathbf{R}$, and the map $T\colon \overline{V} \otimes_{\mathbf{F}} V \longrightarrow \overline{V} \otimes_{\mathbf{F}} V$ given by $T(u \otimes v) = -v \otimes u$ is a well-defined, $\mathbf{R}$-linear involution. (Although it is true that in case $\mathbf{F} = \mathbf{C}$, $\overline{V} \otimes_{\mathbf{F}} V$ is a $\mathbf{C}$-vector space and T is actually conjugate linear, this additional structure will not be used in the sequel.) Let $\mathrm{Fix}(T)$ be the subspace of $\overline{V} \otimes_{\mathbf{F}} V$ consisting of all fixed points of T. It can be verified that $\mathrm{Fix}(T)$ is spanned by tensors of the form $u \otimes v - v \otimes u$, together with those of the form $u \otimes \beta u$, where β is any element in $\mathbf{F}$ such that $\bar{\beta} = -\beta$. Define

$$\mu^2_{\mathbf{F}}(V) = \overline{V} \otimes_{\mathbf{F}} V/\mathrm{Fix}(T).$$

Then $\mu^2_{\mathbf{F}}$ is a covariant functor from the category of left $\mathbf{F}$-vector spaces to the category of $\mathbf{R}$-vector spaces. Analogous to the property

$$s^2(W \oplus W') \approx s^2(W) \oplus W \otimes W' \oplus s^2(W')$$

of the symmetric power functor s^2 for vector spaces over any field, we have

PROPOSITION (4.1). *If V, W are* **F**-*vector spaces, then there is a natural isomorphism*

$$\mu_{\mathbf{F}}^2(V \oplus W) \approx \mu_{\mathbf{F}}^2(V) \oplus \overline{V} \otimes_{\mathbf{F}} W \oplus \mu_{\mathbf{F}}^2(W).$$

PROOF. Consider the diagram

$$\begin{array}{ccc} (\overline{V \oplus W}) \otimes_{\mathbf{F}} (V \oplus W) & \xrightarrow{\approx} & \overline{V} \otimes_{\mathbf{F}} V \oplus \overline{V} \otimes_{\mathbf{F}} W \oplus \overline{W} \otimes_{\mathbf{F}} V \oplus \overline{W} \otimes_{\mathbf{F}} W \\ \Big\downarrow \pi_{V \oplus W} & & \Big\downarrow \pi_V \oplus (1+\tau) \oplus \pi_W \\ \mu_{\mathbf{F}}^2(V \oplus W) & \overset{\phi}{\dashrightarrow} & \mu_{\mathbf{F}}^2(V) \oplus \overline{V} \otimes_{\mathbf{F}} W \oplus \mu_{\mathbf{F}}^2(W) \end{array}$$

where the top map is the natural identification, the right vertical map is the direct sum of

$$\pi_V \oplus \pi_W \colon \overline{V} \otimes_{\mathbf{F}} V \oplus \overline{W} \otimes_{\mathbf{F}} W \longrightarrow \mu_{\mathbf{F}}^2(V) \oplus \mu_{\mathbf{F}}^2(W)$$

with the map

$$1 + \tau \colon \overline{V} \otimes_{\mathbf{F}} W \oplus \overline{W} \otimes_{\mathbf{F}} V \longrightarrow \overline{V} \otimes_{\mathbf{F}} W$$

defined by $(1+\tau)(v \otimes w + w_1 \otimes v_1) = v \otimes w + v_1 \otimes w_1$. This right vertical map annihilates precisely $\mathrm{Fix}(T_{V \oplus W})$, and so induces an isomorphism ϕ as indicated by the dotted arrow of the diagram.

COROLLARY (4.2). *If* $\dim_{\mathbf{F}} V = m$, *then* $\dim_{\mathbf{R}} \mu^2(V) = \frac{1}{2}m(m-1)d + m$.

PROOF. Under the identification $\overline{\mathbf{F}} \otimes_{\mathbf{F}} \mathbf{F} \approx \overline{\mathbf{F}}$, T becomes the map $T(\alpha) = -\overline{\alpha}$ for any $\alpha \in \overline{\mathbf{F}}$. Thus $\mathrm{Fix}(T) = \{\alpha \in \overline{\mathbf{F}} \mid \alpha = -\overline{\alpha}\}$, and so $\mu_{\mathbf{F}}^2(\mathbf{F}) = \mathbf{R}$. The corollary then follows from Proposition (4.1) by induction on m.

Suppose now V is equipped with a nondegenerate hermitian product $\langle\ ,\ \rangle$ over **F**. Since $\langle u, v\rangle = \overline{\langle v, u\rangle}$ and

$$\langle u, \alpha v\rangle - \langle u\alpha, v\rangle = \langle u, v\rangle\overline{\alpha} - \overline{\alpha}\langle u, v\rangle$$

for any $u, v \in V$, $\alpha \in \mathbf{F}$, the **R**-linear map $\overline{V} \otimes_{\mathbf{F}} V \longrightarrow \mathbf{R}$ given by $u \otimes v \mapsto u \cdot v = \frac{1}{2}(\langle u, v\rangle + \langle v, u\rangle)$ is well defined, and induces an epimorphism $j \colon \mu_{\mathbf{F}}^2(V) \longrightarrow \mathbf{R}$. Such an epimorphism j will be useful in §5.

5. **Immersion of flag manifolds into Euclidean spaces.** The basic immersion theorem of M. Hirsch [7] states that if M is an m-dimensional differentiable manifold with tangent bundle $T(M)$, such that there exists a real vector bundle ν of dimension $q > 0$ over M satisfying $\nu \oplus T(M) = (m+q)\epsilon_{\mathbf{R}}$, then M immerses in $\mathbf{R}^{m+q}$. We shall now combine it with Theorem (1.1) to get immersion results for flag manifolds.

THEOREM (5.1). *The flag manifold $G = G_{\mathbf{F}}(n_1, \ldots, n_s)$ can be immersed in Euclidean space with codimension*

$$(n - s) + \tfrac{1}{2} \sum_{i=1}^{s} n_i(n_i - 1)d,$$

provided this codimension is nonzero.

PROOF. Take the isomorphism $\xi_1 \oplus \cdots \oplus \xi_s \approx_{\mathbf{F}} n\epsilon_{\mathbf{F}}$, and apply the functor $\mu_{\mathbf{F}}^2$ to both sides, using Proposition (4.1) along the way. The result is

$$(\#) \qquad \bigoplus_{i=1}^{s} \mu_{\mathbf{F}}^2(\xi_i) \oplus \left(\bigoplus_{1 \leqslant i < j \leqslant s} \bar{\xi}_i \otimes_{\mathbf{F}} \xi_j \right) \approx \text{a trivial bundle.}$$

By Corollary (1.2), the second summand on the left side is precisely the tangent bundle $T(G)$. Furthermore, since each ξ_i is a bundle with nondegenerate hermitian product over $\mathbf{F}$, the final remark in §4 shows that there is a bundle epimorphism

$$j\colon \bigoplus_{i=1}^{s} \mu_{\mathbf{F}}^2(\xi_i) \longrightarrow s\epsilon_{\mathbf{R}}.$$

From these facts, it follows by standard argument that

$$\ker(j) \oplus T(G) \approx \text{a trivial bundle.}$$

Since the fibre dimension of $\ker(j)$ is $(n - s) \oplus \tfrac{1}{2}\Sigma_{i=1}^{s} n_i(n_i - 1)d$, Hirsch's theorem can be applied to yield the desired immersions of G, proving Theorem (5.1).

In the case $\mathbf{F} = \mathbf{R}$, there is no need to distinguish a real vector bundle from its conjugate, so that $T(G)$ is simply $\bigoplus_{1 \leqslant i < j \leqslant s} \xi_i \otimes_{\mathbf{R}} \xi_j$. Using the second exterior power functor λ^2 instead of μ^2, and arguing exactly the same way as in Theorem (5.1), one can get

COROLLARY (5.2). *The real flag manifold $G_{\mathbf{R}}(n_1, \ldots, n_s)$ can be immersed in Euclidean space with codimension $\tfrac{1}{2}\Sigma n_i(n_i - 1)$, provided this codimension is nonzero.*

This, of course, is a better result than the real case of Theorem (5.1).

In both the theorem and the corollary, if the codimension number turns out to be zero, then (#) shows that the manifold in question (which is $G_{\mathbf{F}}(1, \ldots, 1)$) must be stably parallelizable, and thus immerses in codimension one by Hirsch's theorem.

The above immersions are not always interesting. For example, if $G = G_{\mathbf{F}}(1, n - 1)$ is the projective space, then the immersion codimension far exceeds dim G. However, there are cases in which the codimension is considerably less than dim G (such as when all n_i's are close to n/s). These then constitute the interesting cases of our theorem.

REMARK. The immersion of complex flag manifolds is obtained by Tornehave [11], using a quite different method. To the best of our knowledge, the immersion results for quaternionic flag manifolds have not been given before in the literature.

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