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A PROPERTY FOR INVERSES IN A PARTIALLY ORDERED LINEAR ALGEBRA

BY

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ABSTRACT. We consider a Dedekind σ-complete partially ordered linear algebra A which has the following property: if $x \in A$ and $1 \leqslant x$, then $-u \leqslant x^{-1}$, where $u = u^2$. This property is used to show that A must be commutative. We also show that A is the direct sum of two algebras, each of which behaves like an algebra of real-valued functions.

1. **Introduction and definitions.** In his memoir [7] R. V. Kadison discusses various characterizations of an algebra of continuous real-valued functions on a compact Hausdorff space. In particular, §3 is devoted to characterizing an ordered algebra as an algebra of real-valued functions (the algebra theorem of M. H. Stone). In [2] and [3] the authors discuss various ways of characterizing a partially ordered linear algebra (pola) as an algebra of real-valued functions. The purpose of this paper is to characterize a pola having a special property for inverses. The motivation for this property comes from the following example. Let $\widetilde{B}$ be the pola of all continuous real-valued functions defined on $[0, 1]$, where the algebraic operations and the partial order are defined pointwise. It is well known that $\widetilde{B}$ has the Archimedean property but is not Dedekind σ-complete (definitions below). We may embed $\widetilde{B}$ in a Dedekind σ-complete pola A as follows. Define $A = \{(\widetilde{x}, \alpha) : \widetilde{x} \in \widetilde{B}$ and α is real$\}$. If the algebraic operations are defined componentwise, then A is a real linear algebra. The partial order in A is defined as follows. Take $x = (\widetilde{x}, \alpha) \in A$ and $y = (\widetilde{y}, \beta) \in A$. We write $x \leqslant y$ if and only if $0 \leqslant \widetilde{y}(\tau) - \widetilde{x}(\tau) \leqslant \beta - \alpha$ for all $\tau \in [0, 1]$. It is easy to verify that A is Dedekind σ-complete. See Example 1 for the details. Examples are given in §5.

Note that $B = \{(\widetilde{x}, 0) : \widetilde{x} \in \widetilde{B}\}$ is a subalgebra of A which is algebraically (but not order) isomorphic to $\widetilde{B}$. However, we may introduce a new partial order in B so that B is also order isomorphic to $\widetilde{B}$ (see §4). This can be done abstractly by noting that A has the following property. Take $u = (\widetilde{0}, 1) \in A$ and note that $0 \leqslant u = u^2$. Now if $x \in A$ and $1 \leqslant x$, then x has an inverse and $-u \leqslant x^{-1}$.

We now give the basic definitions needed in this paper. A pola (denoted by

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A) is a real linear associative algebra which is partially ordered so that it is a directed partially ordered linear space and $0 \leqslant xy$ whenever $x, y \in A$, $0 \leqslant x$, $0 \leqslant y$. We also assume that A has a multiplicative identity $1 \geqslant 0$. A Dedekind σ-complete pola (dsc-pola) A is one having the property: if $x_n \in A$, $0 \leqslant \cdots \leqslant x_2 \leqslant x_1$, then $\inf \{x_n\}$ exists. Order convergence is defined as usual. A dsc-pola A has the Archimedean property: if $x, y \in A$ and $nx \leqslant y$ for every positive integer n, then $x \leqslant 0$. For more details and examples see the references.

2. **Basic lemmas and theorems.** In this paper we assume that A is a dsc-pola (not necessarily commutative) which has the following property: if $x \in A$ and $1 \leqslant x$, then x has an inverse and $-u \leqslant x^{-1}$, where $u \in A$ is a fixed element such that $0 \leqslant u = u^2$.

We put $e = 1 - u$ and note that $e = e^2$ and $eu = ue = 0$. The first step is to show that u commutes with every element of A.

LEMMA 2.1. *If* $w \in A$, $0 \leqslant w$ *and* $w^2 = 0$, *then* $w = 0$.

PROOF. Since $1 \leqslant 1 + nw$ for every positive integer n, we have $-u \leqslant (1 + nw)^{-1} = 1 - nw$, so that $nw \leqslant 1 + u$ for all n. From the Archimedean property we get $w \leqslant 0$. Hence, $w = 0$.

LEMMA 2.2. *If* $0 \leqslant z \leqslant u$, *then* $uz = zu$.

PROOF. Put $w = ezu$. Note that $0 \leqslant u - w = (u - w)^2$. Hence, $1 \leqslant 1 + n(u - w)$ for every positive integer n so that $-u \leqslant [1 + n(u - w)]^{-1} = 1 - n(n + 1)^{-1}(u - w)$, which gives $-(u - w) \leqslant (n + 1)(1 + w)$ for all n. Thus, $0 \leqslant 1 + w$ (from the Archimedean property). Since $w^2 = 0$, we get $0 \leqslant (1 + w)^n = 1 + nw$ for every positive integer n so that $-nw \leqslant 1$ for all n. Using the Archimedean property again, we get $0 \leqslant w$. From Lemma 2.1 we get $w = ezu = 0$.

We may repeat the above argument for the element uze to show that $uze = 0$. Hence, $uz = uz(u + e) = uzu = (u + e)zu = zu$.

LEMMA 2.3. *If* $-\beta u \leqslant x \leqslant \beta u$ *for some real number* $\beta \geqslant 0$, *then* $xu = ux$.

PROOF. This lemma follows directly from Lemma 2.2.

LEMMA 2.4. *If* $0 \leqslant y \leqslant z$ *and* $y^n \leqslant nz$ *for every positive integer* n, *then* $y \leqslant 1 + u$.

PROOF. If $0 \leqslant \lambda < 1$, then $\Sigma_{k=1}^{\infty} (\lambda y)^k \leqslant \Sigma_{k=1}^{\infty} \lambda^k (kz) < \infty$. Hence, $1 \leqslant (1 - \lambda y)^{-1}$; see Theorem I.6.1 of [2]. Therefore, $-u \leqslant 1 - \lambda y$ or $\lambda y \leqslant 1 + u$ for all $\lambda < 1$. From the Archimedean property we get $y \leqslant 1 + u$.

LEMMA 2.5. *If* $1 \leqslant x$, *then* $-u \leqslant x^{-1} \leqslant 1 + u(x - 1)$ *and* $-u \leqslant x^{-1} \leqslant 1 + (x - 1)u$.

PROOF. The basic property of this paper asserts that $-u \leqslant x^{-1}$. The other inequalities follow directly from $0 \leqslant (x^{-1} + u)(x - 1)$ and $0 \leqslant (x - 1)(x^{-1} + u)$.

LEMMA 2.6. *If* $1 \leqslant y$ *and* $yu = uyu$ (*or* $uy = uyu$), *then* $yu = uy$.

PROOF. From Lemma 2.5 we get $-u \leqslant y^{-1} \leqslant 1 + (y - 1)u$ or $0 \leqslant u + y^{-1} \leqslant 1 + yu$. We now show by mathematical induction that $0 \leqslant (u + y^{-1})^n \leqslant 1 + 3^n(u + yu)$ for every positive integer n. This is easy to verify for $n = 1$. If the above inequalities are true for $n = k$, then

$$0 \leqslant (u + y^{-1})^{k+1} \leqslant (u + y^{-1})\,[1 + 3^k(u + yu)]$$
$$\leqslant 1 + yu + 3^k(u + yu + yu + u) \leqslant 1 + 3^{k+1}(u + yu),$$

where we have used the fact that $uyu = yu$, $y^{-1}u + u \leqslant u + yu$ and other elementary inequalities. This completes the induction.

It follows that $[3^{-1}(u + y^{-1})]^n \leqslant 1 + 2yu$ for every positive integer n. Using Lemma 2.4, we get $0 \leqslant u + y^{-1} \leqslant 3(1 + u)$. Therefore, $-5u \leqslant uy^{-1} \leqslant 5u$ and $-5u \leqslant y^{-1}u \leqslant 5u$. From Lemma 2.3 we get $uy^{-1} = uy^{-1}u$ and $uy^{-1}u = y^{-1}u$. Hence, $uy^{-1} = y^{-1}u$ so that $yu = uy$.

To prove this lemma when we assume that $uy = uyu$ one starts with the inequality $0 \leqslant u + y^{-1} \leqslant 1 + uy$ (from Lemma 2.5) and then shows (as above) that $0 \leqslant (u + y^{-1})^n \leqslant 1 + 3^n(u + uy)$ for all n. Hence, $u + y^{-1} \leqslant 3(1 + u)$ and the rest of the proof follows as above.

LEMMA 2.7. *If* $1 \leqslant x$, *then* $ux = xu$.

PROOF. Define $y_1 = 1 + u(x - 1)$ and $y_2 = 1 + (x - 1)u$. Note that $1 \leqslant y_1$, $1 \leqslant y_2$ and $y_1u = uy_1u = uxu = uy_2u = uy_2$. From Lemma 2.6 we get $uy_1 = y_1u = uy_2 = y_2u$. But $ux = uy_1 = y_2u = xu$.

THEOREM 2.8. *If* $z \in A$, *then* $uz = zu$.

PROOF. Since A is directed, we may write $z = x_1 - x_2$, where $1 \leqslant x_1$ and $1 \leqslant x_2$. The theorem follows from Lemma 2.7.

We may now define $B = \{x : ex = x\}$ and $N = \{x : ux = x\}$. It is easy to see that B and N are real linear algebras and that A is the direct sum of B and N. The remaining lemmas will be used later to describe the various properties of B and N.

LEMMA 2.9. *If* $1 \leqslant x$, *then* $-5u \leqslant ux^{-1} \leqslant 5u$.

PROOF. Since $ux = xu$, we have $xu = uxu$. We now refer to the proof of Lemma 2.6 to get $-5u \leqslant ux^{-1} \leqslant 5u$.

LEMMA 2.10. *If* $x \in A$, $t \in N$ *and both* x *and* $x + t$ *have inverses, then* $ex^{-1} = e(x + t)^{-1}$.

PROOF. Since $t \in N$, we have $et = 0$. Hence, $ex = e(x + t)$ and the result follows easily, but one must use the fact that $ex = xe$.

LEMMA 2.11. *If* $1 \leqslant x$, *then* $-u \leqslant ex^{-1}$.

PROOF. Since $1 \leqslant x(1 + nu)$ for every positive integer n, we have $-u \leqslant (1 + nu)^{-1}x^{-1} = [1 - n(n + 1)^{-1}u]x^{-1} = [e + (n+1)^{-1}u]x^{-1}$ for all n. From the Archimedean property it follows that $-u \leqslant ex^{-1}$.

LEMMA 2.12. *If* $1 \leqslant x$, *then* $ex^{-1} \leqslant 1$.

PROOF. Using Lemma 2.11, we get $0 \leqslant (u + ex^{-1})(x - 1)$, from which it follows that $0 \leqslant u + ex^{-1} \leqslant e + ux$. We now show by mathematical induction that $0 \leqslant (u + ex^{-1})^n \leqslant e + n(ux)$ for all n. The inequalities are clearly true for $n = 1$. If they are true for $n = k$, then

$$0 \leqslant (u + ex^{-1})^{k+1} \leqslant (e + kux)(u + ex^{-1}) = ex^{-1} + kux$$
$$\leqslant u + ex^{-1} + kux \leqslant e + ux + kux = e + (k + 1)ux.$$

This completes the induction.

Since $e \leqslant 1$, we get $0 \leqslant (u + ex^{-1})^n \leqslant n(1 + ux)$ for all n. From Lemma 2.4 we get $u + ex^{-1} \leqslant 1 + u$ so that $ex^{-1} \leqslant 1$.

LEMMA 2.13. *If* $1 \leqslant x \leqslant y$, *then* $-u \leqslant exy^{-1} \leqslant 1$.

PROOF. Since $-u \leqslant x^{-1}$, we get $0 \leqslant (y - x)(x^{-1} + u)$, from which $1 \leqslant [y + u(y - x)x]x^{-1}$. From Lemmas 2.11 and 2.12 we get $-u \leqslant ex[y + u(y - x)x]^{-1} \leqslant 1$. Since $u(y - x)x \in N$, we may use Lemma 2.10 to get $-u \leqslant exy^{-1} \leqslant 1$.

LEMMA 2.14. *If* $0 \leqslant z \leqslant 1 + t$, *where* $t \in N$ *and* $0 \leqslant t$, *then* $-u \leqslant ez \leqslant 1$.

PROOF. Since $1 \leqslant 1 + nz \leqslant 1 + n(1 + t)$ for every positive integer n, we can use Lemmas 2.10 and 2.13 to get $-u \leqslant e(1 + nz)[(n + 1)1 + nt]^{-1} = (n + 1)^{-1}e(1 + nz) \leqslant 1$. From the Archimedean property we get $-u \leqslant ez \leqslant 1$.

LEMMA 2.15. *If* $a \in B$, $s \in N$ *and* $-s \leqslant a \leqslant s$, *then* $a = 0$. (*Note that* $ea = a$ *and* $es = 0$.)

PROOF. It is clear that $0 \leqslant s + a \leqslant 2s$. Therefore, $0 \leqslant n(s + a) \leqslant 1 + 2ns$ for every positive integer n. Hence, from Lemma 2.14 we get $-u \leqslant ne(s + a) = na \leqslant 1$ for all n. From the Archimedean property we get $a = 0$.

LEMMA 2.16. *If* $0 \leqslant x \leqslant y$ *and* $1 \leqslant y$, *then* $-u \leqslant exy^{-1} \leqslant 1$.

PROOF. Since $0 \leqslant y^{-1} + u$, we get $0 \leqslant x(y^{-1} + u) \leqslant y(y^{-1} + u) = 1 + yu$. Since $yu \in N$ and $0 \leqslant yu$, we may use Lemma 2.14 to get $-u \leqslant ex(y^{-1} + u) = exy^{-1} \leqslant 1$.

LEMMA 2.17. *If* $0 \leqslant z$ *and* $ez \leqslant x$, *where* $1 \leqslant x$, *then* $0 \leqslant ez + ux$.

PROOF. Since $0 \leq z = ez + uz \leq x + uz$ and $1 \leq x + uz$, we may use Lemma 2.16 to get $-u \leq ez(x + uz)^{-1} \leq 1$. Since $uz \in N$, it follows from Lemma 2.10 that $-u \leq ezx^{-1}$. Hence, $-ux \leq ez$ or $0 \leq ez + ux$.

3. **The structure of** N. Since $0 \leq u$, it is clear that N is a directed pola and that u is the identity for N. We will see that the structure of N is actually characterized in [2] but first we need two lemmas.

LEMMA 3.1. *N is order-convex.*

PROOF. We need only show that if $-t \leq x \leq t$ and $t \in N$, then $x \in N$ (recall that N is directed). Since $x = ex + ux$ and $-t \leq ux \leq t$, we get $-2t \leq ex \leq 2t$. From Lemma 2.15 we obtain $ex = 0$ so that $x = ux \in N$.

LEMMA 3.2. *N is closed with respect to order convergence.*

PROOF. Let $\{x_n\}$ be a sequence of elements from N such that $o\text{-}\lim x_n = x$. Thus, for some element $z \in A$ we have $-z \leq x_n \leq z$ for all n. Hence, $-uz \leq ux_n = x_n \leq uz$ for all n, which means $-uz \leq x \leq uz$. Since $uz \in N$ and N is order-convex, we have $x \in N$.

These lemmas enable us to assert that N is Dedekind σ-complete. Hence, N is a dsc-pola with identity u. We may now apply the results of [2]. Note that in [2] the term "polac" is used instead of "dsc-pola". Let N_1 denote the functional part of N as described in [2, p. 658].

THEOREM 3.3. $N = N_1$.

PROOF. Take any $t \in N$ such that $u \leq t$. Put $x = e + t = 1 + (t - u)$ so that $1 \leq x$. From Lemma 2.9 we get $-5u \leq ux^{-1} \leq 5u$, which means that $ux^{-1} \in N_1$. Hence, $0 \leq (ux^{-1})^2$. Since $u \leq t \leq t^2$ and $t^2(ux^{-1})^2 = (ux^{-1})^2 t^2 = u$, we see that $t^2 \in N_1$. Since N_1 is order-convex, we can assert that $t \in N_1$. Since N is directed, we get $N = N_1$.

In particular this means that N is commutative. The reader is referred to [2] for a more detailed discussion.

4. **The structure of** B. In the special case that $u \leq 1$ we have $0 \leq e$ and one may show that $B = B_1$ = the functional part of B. In fact, one may show that $A = A_1$ = the functional part of A. We leave this as an exercise for the reader.

In general, we must introduce a new partial order in B. To do this we define $K = \{ez : 0 \leq z\} \subset B$. The reader may easily verify that K is a generating cone in B and that K is closed with respect to multiplication.

LEMMA 4.1. *If $a \in K$ and $-a \in K$, then $a = 0$.*

PROOF. Suppose $0 \leq y$ and $0 \leq z$ are such that $a = ey$ and $-a = ez$. Since

$ey \leqslant y$ and $ez \leqslant z$, we get $-(y+z) \leqslant a \leqslant y+z$. But $e(y+z) = a - a = 0$ so that $y + z = u(y+z) \in N$. From Lemma 2.15 we get $a = 0$.

We may now define a partial order $\leqslant_o$ in B as follows: for $b, c \in B$ we write $b \leqslant_o c$ if and only if $c - b \in K$. Lemma 4.1 is used to assert that $\leqslant_o$ is antisymmetric.

LEMMA 4.2. *The real linear algebra B with the partial order $\leqslant_o$ is a directed pola which has the Archimedean property.*

PROOF. It is easily verified that B is a directed pola. Let us now assume that $b, c \in B$ and $nb \leqslant_o c$ for every positive integer n. Thus, for each n we can find $z_n \in A$ such that $0 \leqslant z_n$ and $e(nz_n) = c - nb$. We may now take $y \in A$ so that $0 \leqslant y$ and $c \leqslant y$. Since $0 \leqslant y$ and $n \geqslant 1$, we get $ez_n \leqslant y - b$ for all n. Next select $x \in A$ so that $1 \leqslant x$ and $ez_n \leqslant y - b \leqslant x$. From Lemma 2.17 we get $0 \leqslant ez_n + ux$ for all n. Therefore, $0 \leqslant n(ez_n + ux) = c + n(ux - b)$ for all n. Since the partial order $\leqslant$ in A has the Archimedean property, we have $0 \leqslant ux - b$. This means that $e(ux - b) = -b \in K$. Hence, $b \leqslant_o 0$.

LEMMA 4.3. *If $f \in B$ and $e \leqslant_o f$, then there exists $g \in B$ such that $0 \leqslant_o g$ and $fg = gf = e$.*

PROOF. There exists $y \in A$ such that $0 \leqslant y$ and $ey = f - e$. Since $f = e(1+y)$ and $1 \leqslant 1 + y$, we have $f(1+y)^{-1} = (1+y)^{-1}f = e$ and $0 \leqslant u + e(1+y)^{-1}$ from Lemma 2.11. If we put $g = e(1+y)^{-1}$, then it is easy to show that g has the desired properties.

THEOREM 4.4. *B is commutative.*

PROOF. Define $F = \bigcup_{n=1}^{\infty} \{f \in B : -ne \leqslant_o f \leqslant_o ne\}$. It is clear that F is an order-convex subalgebra of B which has e as an order unit. Also, F has the Archimedean property. It is known that F is isomorphic to an algebra of bounded real-valued functions; see [8, p. 255, Exercise 24]. This means that F is commutative. If $a, b \in B$ and $e \leqslant_o a$ and $e \leqslant_o b$, then there exist elements $c, d \in B$ such that $0 \leqslant_o c$, $0 \leqslant_o d$ and $ac = ca = e = bd = db$. But $0 \leqslant_o ec = c \leqslant_o ac = e$ and $0 \leqslant_o ed = d \leqslant_o bd = e$, which means that $c \in F$ and $d \in F$. Thus, $cd = dc$ and it follows easily that $ab = ba$. Since B is directed, it follows that B is commutative.

From the above results it follows that A is a commutative algebra. In this connection the authors recommend the very interesting paper of Jamison [6].

5. **Examples.** The basic example was described in the introduction. The reader should note that Example 4 is an important counterexample.

EXAMPLE 1. This example was described in the introduction. Here we wish to show that A is Dedekind σ-complete. Let $x_n \in A$ be a sequence such that

$0 \leqslant \cdots \leqslant x_2 \leqslant x_1$. Now $x_n = (\widetilde{x}_n, \lambda_n)$ and we must have $0 \leqslant \cdots \leqslant \lambda_2 \leqslant \lambda_1$. Define $\mu = \inf \{\lambda_n\}$ and note that $0 \leqslant \widetilde{x}_n(\tau) - \widetilde{x}_k(\tau) \leqslant \lambda_n - \mu$ for all $\tau \in [0, 1]$ and all $n \leqslant k$. Thus, the sequence $\{\widetilde{x}_n\}$ of functions converges uniformly to a continuous function $\widetilde{y}$. If we put $y = (\widetilde{y}, \mu)$, then it is easy to show that $y = \inf \{x_n\}$.

EXAMPLE 2. Let $\widetilde{B}$ be the real linear algebra of all continuous real-valued functions defined on the real line. Let

$$A = \{(\widetilde{x}; \alpha_1, \alpha_2, \ldots): \widetilde{x} \in \widetilde{B} \text{ and } \alpha_n \text{ real for all } n\}.$$

If the algebraic operations are defined componentwise, then A is a real linear algebra. The partial order $\leqslant$ in A is defined as follows: if $x = (\widetilde{x}; \alpha_1, \alpha_2, \ldots)$ and $y = (\widetilde{y}; \beta_1, \beta_2, \ldots)$, then $x \leqslant y$ if and only if $0 \leqslant \widetilde{y}(\tau) - \widetilde{x}(\tau) \leqslant \beta_n - \alpha_n$ for all n and for all $\tau \in [-n, n]$. As in Example 1 we can show that A is a dsc-pola. The reader should note that order convergence of sequence of elements from A implies that the corresponding functions converge uniformly on every finite interval. The above idea can be generalized to any algebra $\widetilde{B}$ of functions on a locally compact space.

We may define $u = (\widetilde{0}; 1, 1, \ldots)$ and then show that if $1 \leqslant x$, then $-u \leqslant x^{-1}$. Note that in this case the elements of B and N need not be bounded functions.

EXAMPLE 3. Let A be the set of all real-valued functions defined on the real line. If $x, y \in A$, we define $x = y$ and $x \leqslant y$ to mean that $x(\tau) = y(\tau)$ and $x(\tau) \leqslant y(\tau)$ for almost all τ (Lebesgue measure). Thus, A is a dsc-pola, where the algebraic operations are defined pointwise (almost everywhere). Let $u \in A$ be the characteristic function of the set of positive real numbers. Thus, $u \leqslant 1$ so that $0 \leqslant e$. Note that the elements of B and N behave like real-valued functions but cannot be represented by real-valued functions which are defined everywhere on some set.

EXAMPLE 4. Let A be the real linear algebra of all matrices of the form $x = \left[\begin{smallmatrix} \alpha & \beta \\ 0 & \nu \end{smallmatrix}\right]$. If we define $0 \leqslant x$ to mean that $0 \leqslant \alpha$, $0 \leqslant \beta$ and $\alpha + \beta \leqslant \nu$, then A is a dsc-pola. If we put $u = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 2 \end{smallmatrix}\right]$, then it is a routine computation to show that if $1 \leqslant x$, then $-u \leqslant x^{-1}$ and that $-u$ is the best possible lower bound for inverses of elements x such that $1 \leqslant x$. Note that $0 \leqslant 2u = u^2$ and that A is not commutative.

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