TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 215, 1976

COMPOSITION SERIES AND INTERTWINING OPERATORS FOR THE SPHERICAL PRINCIPAL SERIES. II

BY

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ABSTRACT. In this paper, we consider the connected split rank one Lie group of real type F_4 which we denote by F_4^1. We first exhibit F_4^1 as a group of operators on the complexification of A. A. Albert's exceptional simple Jordan algebra. This enables us to explicitly realize the symmetric space $F_4^1/\mathrm{Spin}(9)$ as the unit ball in $\mathbf{R}^{16}$ with boundary S^{15}. After decomposing the space of spherical harmonics under the action of Spin(9), we obtain the matrix of a transvection operator of $F_4^1/\mathrm{Spin}(9)$ acting on a spherical principal series representation. We are then able to completely determine the Jordan Holder series of any spherical principal series representation of F_4^1.

1. **Introduction and notation.** This is the second in a proposed series of papers dealing with intertwining operators and the composition series of the principal series. In [7], we considered the classical split rank one groups. In this paper, we consider the spherical principal series representations of the split rank one version of real type F_4 which we will denote in the following by F_4^1.

Our main results for F_4^1 will duplicate our results for the classical split rank one groups. That is, our results on spherical harmonics, irreducibility, existence of complementary series, composition series, and intertwining operators will mimic those of [7]. Furthermore, our techniques in obtaining these results will closely follow [7]. Our chief difference with [7] lies in the algebraic preliminaries we use to study F_4^1.

In this paper, we will first develop F_4^1 by considering A. A. Albert's exceptional simple Jordan algebra over the reals. We will then proceed as in [7] by determining the Poisson kernel, spherical harmonics and the action of the transvection on the M-fixed harmonics. This will allow us to obtain our results on composition series and intertwining operators. We will assume the notation of [7].

Our results on the irreducibility of the spherical principal series were first obtained by Kostant [9] and later by Helgason [4]. Our results on the existence of the complementary series may be found in Kostant [9].

Received by the editors November 30, 1974.

AMS (MOS) subject classifications (1970). Primary 43A65, 22E15; Secondary 43A80, 33A75.

Key words and phrases. F_4^1, exceptional simple Jordan algebra, spherical harmonics, Poisson kernel, transvection operator, composition series, intertwining operators.

Finally, I would like to thank Earl Taft for several useful discussions on Jordan algebras.

2. F_4^1 and the exceptional simple Jordan algebra. Let **F** be the Cayley numbers. Let $x \longrightarrow \bar{x}$ be the standard involution of **F** over **R**, and let $|x|^2 = \bar{x}x = x\bar{x}$. Let $A(3, \mathbf{F})$ be the set of 3×3 Hermitian symmetric matrices with entries in **F**.

We can convert $A(3, \mathbf{F})$ into a Jordan algebra by defining a multiplication on $A(3, \mathbf{F})$ such that

$$\text{if } a, b \in A(3, \mathbf{F}),\ a \cdot b = (ab + ba)/2 \text{ where}$$
$$ab \text{ denotes the standard matrix multiplication.}$$

As is well known $(A(3, \mathbf{F}), \cdot\,)$ is A. A. Albert's exceptional simple Jordan algebra.

If $a \in A(3, \mathbf{F})$ we denote by L_a the linear map $L_a\colon A(3, \mathbf{F}) \longrightarrow A(3, \mathbf{F})$ where for $x \in A(3, \mathbf{F})$ $L_a x = a \cdot x$. Consider the three idempotents $L_{e_1}, L_{e_2}, L_{e_3}$ where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If $\widetilde{A}_i$ is the kernel of L_{e_i} and $A_i = \{a \in \widetilde{A}_i | a_{jj} = 0, 1 \leqslant j \leqslant 3\}$ we have that

$$A(3, \mathbf{F}) = \mathbf{R}e_1 \oplus \mathbf{R}e_2 \oplus \mathbf{R}e_3 \oplus A_1 \oplus A_2 \oplus A_3.$$

A derivation of $A(3, \mathbf{F})$ is a linear map $D\colon A(3, \mathbf{F}) \longrightarrow A(3, \mathbf{F})$ such that for a, b in $A(3, \mathbf{F})$ $D(a \cdot b) = (Da) \cdot b + a \cdot (Db)$. Let $\mathcal{D}_0$ denote the derivations of $A(3, \mathbf{F})$ which send e_1, e_2 and e_3 to 0. If $\mathcal{D}_1 = \{[L_{e_2 - e_3}, L_a] | a \in A_1\}$, $\mathcal{D}_2 = \{[L_{e_1 - e_3}, L_a] | a \in A_2\}$, and $\mathcal{D}_3 = \{[L_{e_1 - e_2}, L_a] | a \in A_3\}$ we have by a theorem of Chevalley and Schafer [11, p. 112], that $\mathcal{D} = \mathcal{D}_0 \oplus \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \mathcal{D}_3$ is the Lie algebra of derivations of $A(3, \mathbf{F})$. Now $\mathcal{D}$ is the Lie algebra of Aut $A(3, \mathbf{F})$ which is a compact group of real type F_4 (see [10]).

The following properties of $\mathcal{D}$ are obvious:

$$[\mathcal{D}_0, \mathcal{D}_i] \subset \mathcal{D}_i \qquad \text{for all } i;$$

$$[\mathcal{D}_i, \mathcal{D}_j] \subset \mathcal{D}_k \qquad \text{if } 1 \leqslant i, j, k \leqslant 3 \text{ are all different; and}$$

$$[\mathcal{D}_i, \mathcal{D}_i] \subset \mathcal{D}_0 \qquad \text{for all } i.$$

Thus, if we let $k = \mathcal{D}_0 + \mathcal{D}_1$ and $p_0 = \mathcal{D}_2 + \mathcal{D}_3$ we have that $[k, k] \subset k$, $[k, p_0] \subset p_0$ and $[p_0, p_0] \subset k$. So if we set $p = ip_0$ we see that $k + p$ is the Lie algebra of a noncompact connected group of real type F_4. By classification (see Helgason [3, p. 354]), this group is F_4^1, and so in the future we will denote $k + p$ by f_4^1.

Although f_4^1 no longer acts on $A(3, \mathbf{F})$, we do obtain a representation of f_4^1 on $A(3, \mathbf{F})_{\mathbf{C}} = A(3, \mathbf{F}) \otimes_{\mathbf{R}} \mathbf{C}$ and a representation on $\bar{A}_{\mathbf{C}}$ where $\bar{A} = \{a \in A(3, \mathbf{F}) \mid \operatorname{Tr} a = 0\}$ which is of course irreducible.

By classification and the Weyl dimension formula, it is easy to see that the connected subgroup of Aut $A(3, \mathbf{F})$ with Lie algebra k is Spin(9) $= K$.

If we let

$$a_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and $H = i[L_{e_1 - e_3}, L_{a_2}] \in \mathcal{D}_2$ we see that $\alpha = \mathbf{R}H$. Let $\mathfrak{n}$ be the subalgebra of f_4^1 which is spanned by the positive eigenvectors of ad H, and let N be its Lie algebra. We will not compute $\mathfrak{n}$ explicitly but rather compute the eigenspaces of H acting on $A(3, \mathbf{F})_{\mathbf{C}}$. They are as follows:

$$0\text{-eigenspace} = \mathbf{C}e_1 + \mathbf{C}(e_2 + e_3)$$
$$+ \mathbf{C}\left\{a \in A_2 \mid a = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ \bar{x} & 0 & 0 \end{pmatrix} \text{ and } x + \bar{x} = 0\right\},$$

$$\tfrac{1}{2}\text{-eigenspace} = \mathbf{C}\left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & \bar{x} & 0 \end{pmatrix} + i\begin{pmatrix} 0 & \bar{x} & 0 \\ x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| x \in \mathbf{F}\right\},$$

$$-\tfrac{1}{2}\text{-eigenspace} = \mathbf{C}\left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & \bar{x} & 0 \end{pmatrix} - i\begin{pmatrix} 0 & \bar{x} & 0 \\ x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| x \in \mathbf{F}\right\},$$

$$1\text{-eigenspace} = \mathbf{C}(a_2 + i(e_1 - e_3)),$$

$$-1\text{-eigenspace} = \mathbf{C}(a_2 - i(e_1 - e_3)).$$

Observe from Helgason [4, p. 79], that, since $K(2e_1 - (e_2 + e_3)) = 2e_1 - (e_2 + e_3)$, $MN(a_2 + i(e_1 - e_3)) = a_2 + i(e_1 - e_3)$ where M is the centralizer of α in K. Now $M = \{k \in K \mid ka_2 = a_2 \text{ and } ke_3 = e_3\}$.

The following theorem of E. Artin will be useful in computing exponentials.

THEOREM 2.1 (E. ARTIN, SEE [11, p. 29]). *Any two elements of* $\mathbf{F}$ *generate an associative involutive algebra.*

LEMMA 2.1. $M = \{k \in K \mid ka_2 = a_2\}$.

PROOF. Let $M_1 = \{k \in K \mid ka_2 = a_2\}$. Then clearly $M \subset M_1$. Let K_0 be the subgroup of K generated by $\mathcal{D}_0$. Then $K = K_0 \exp \mathcal{D}_1 = \exp \mathcal{D}_1 K_0$.

Suppose $k \in M_1$. Then $k = k_0 \exp D$ where $k_0 \in K_0$ and $D = [L_{e_2 - e_3}, L_a]$ where

$$a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & \bar{c} & 0 \end{pmatrix}.$$

By Theorem 2.1 we obtain

$$k(a_2 + i(e_1 - e_3)) = k_0 \exp D(a_2 + i(e_1 - e_3))$$

$$= k_0 \left[\left(\cos \frac{|c|}{6} \right) a_2 + \left(\sin \frac{|c|}{2} \right) \left(\frac{2a_2 \cdot a}{|c|} \right) + i(e_1 - ½(e_2 + e_3)) \right.$$

$$\left. + \left(\frac{\cos |c|}{2} \right) (e_2 - e_3) - (\sin |c|) \left(\frac{a}{2|c|} \right) \right].$$

Since $k \in M$, we must have $\sin(|c|/2) = 0$, and thus $k(a_2 + i(e_1 - e_3)) = k_0(\pm a_2 + i(e_1 - e_3)) = \pm k_0(a_2) + i(e_1 - e_3) = a_2 + i(e_1 - e_3)$. Therefore, $k \in M$ and we are done.

REMARK. We know that $K/M = S^{15}$ and K is connected and simply connected. Thus, from the exact sequence

$$\pi_2(K/M) \longrightarrow \pi_1(M) \longrightarrow \pi_1(K) \longrightarrow \pi_1(K/M) \longrightarrow \pi_0(M) \longrightarrow \pi_0(K)$$

we conclude that M is connected and simply connected. Since the Lie algebra of M is the same as the Lie algebra of SO(7) we conclude that $M = \text{Spin}(7)$.

It is well known that $F_4^1/\text{Spin}(9)$ is analytically isomorphic to the open unit ball. We now construct this isomorphism.

If $a_1 + ib_1, a_2 + ib_2 \in A(3, \mathbf{F})_{\mathbf{C}}$ set

$$\langle a_1 + ib_1, a_2 + ib_2 \rangle = (\text{Tr } a_1 \cdot a_2) + (\text{Tr } b_1 \cdot b_2).$$

This is a symmetric positive definite bilinear form on $A(3, \mathbf{F})_{\mathbf{C}}$ and it is well known that Aut $A(3, \mathbf{F})$ preserves this bilinear form. Let P: $A(3, \mathbf{F})_{\mathbf{C}} \longrightarrow A_2 + A_3$ be the orthogonal projection onto $A_2 + A_3$ with respect to this bilinear form. Let $\|a + ib\|^2 = \|a\|^2 + \|b\|^2$.

THEOREM 2.2. *The map* T: $F_4^1 \longrightarrow A_2 + A_3$ *given by* $T(g) = \sqrt{2}P(ige_1)/\|ge_1\|$ *defines an analytic isomorphism of* $F_4^1/\text{Spin}(9)$ *with the open unit ball in* $A_2 + A_3$.

PROOF. Observe that $T(gk) = T(g)$ for k in Spin(9). Since the map $g \longrightarrow ige_1$ is analytic and P is linear, the map $g \longrightarrow \sqrt{2}\, P(ige_1)$ is analytic. Since $g \longrightarrow \|ge_1\|$ is never 0 it is analytic. Thus T is an analytic map.

Observe that $T(kg) = kT(g)$ for k in Spin(9). Since the representation of K on $A_2 + A_3$ is the spin-representation we have that K acts transitively on the unit sphere. Thus we see that the image of T is a ball.

Let a_2 and H be as above and set $\mathfrak{a}^+ = \{tH | t > 0\}$ and $A^+ = \exp \mathfrak{a}^+$.

Then as usual $F_4^1 = K\overline{A^+}K$. Observing that $iHe_1 = a_2/2$, $iHe_3 = -a_2/2$ and $iH^2e_1 = i((e_1 - e_3)/2)$ we obtain

$$i(\exp tH)e_1 = (\cosh t)i\left(\frac{e_1 - e_3}{2}\right) + \frac{i(e_1 + e_3)}{2} + (\sinh t)\,\frac{a_2}{2}.$$

Thus,

$$T(\exp tH) = \frac{\sqrt{2}\,(\sinh t)\,a_2}{\sqrt{2 + 2(\cosh t)^2 + 2(\sinh t)^2}}.$$

Observe that $\|T(\exp t_1H)\| < \|T(\exp t_2H)\|$ if $0 \leqslant t_1 < t_2$ and that $\lim_{t\to\infty} \|T(\exp tH)\| = 1$. Therefore, T is an analytic map of F_4^1 onto the unit ball in $A_2 + A_3$. We will now show that $T(g_1) = T(g_2)$ only if $g_1^{-1}g_2$ is in K.

Suppose $T(g_1) = T(g_2)$. Now $g_1 = k_1\bar{a}_1\bar{k}_1$ and $g_2 = k_2\bar{a}_2k_2$ where k_1, k_2, $\bar{k}_1$, $\bar{k}_2$ are in K and $\bar{a}_1$, $\bar{a}_2$ are in A^+. Since $\|T(g_1)\| = \|T(g_2)\|$ we must have $\bar{a}_1 = \bar{a}_2$. Hence $k_1^{-1}k_2T(\bar{a}_1) = T(\bar{a}_1)$ and thus $k_1^{-1}k_2$ are in M by Lemma 2.1, and therefore $g_1^{-1}g_2 \in K$.

It remains only to show that the map T has maximal rank everywhere. For $X \in \mathcal{V}_2 + \mathcal{V}_3$ we obtain a vector field $\widetilde{X}$ on $F_4^1/\mathrm{Spin}(9)$ defined by

$$(\widetilde{X}f)(x) = \frac{d}{dt}f((\exp tX)x)|_{t=0}$$

for f a C^∞-function on $F_4^1/\mathrm{Spin}(9)$. A simple calculation shows that

$$T_*(\widetilde{X})(T(g)) = \sqrt{2}\,\frac{P(iXge_1)}{\|ge_1\|} - \sqrt{2}\,\frac{P(ige_1)}{\|ge_1\|^3}\,\langle Xge_1, ge_1\rangle.$$

Writing $g = k(\exp tH)k^1$ and $y = k^{-1}Xk$ we have

$$T_*(\widetilde{X})(T(g)) = k\left[\sqrt{2}\,\frac{P(iY(\exp tH)e_1)}{\|(\exp tH)e_1\|} - \sqrt{2}\,\frac{P(i(\exp tH)e_1)\,\langle Y(\exp tH)e_1, (\exp tH)e_1\rangle}{\|(\exp tH)e_1\|^3}\right].$$

It suffices to show that $T_*(\widetilde{X})(T(g))$ is nonzero for a nonzero X. If $X \neq 0$, $Y \neq 0$, and $Y = i[L_{e_1-e_3}, L_{c_2}] + i[L_{e_1-e_2}, L_{c_3}]$ where $c_2 \in A_2$, and $c_3 \in A_3$. Now we have the following formulas

$$[L_{e_1-e_3}, L_{c_2}](e_1) = -c_2/2, \qquad [L_{e_1-e_3}, L_{c_2}](e_3) = c_2/2,$$

$$[L_{e_1-e_2}, L_{c_3}](e_1) = -c_3/2, \qquad [L_{e_1-e_2}, L_{c_3}](e_3) = 0.$$

From these formulas and the above formula for $i(\exp tH)e_1$ we have

$$P(iY(\exp tH)e_1) = (\cosh t)c_2/2 + [(\cosh t) + 1]c_3/4$$

and

$$P(i(\exp tH)e_1) = (\sinh t)a_2/2.$$

Thus if Y is not a scalar multiple of H we have $T_*(\widetilde{X})(T(g)) \neq 0$. If $Y = H$ we see that

$$\|(\exp tH)e_1\|^2 P(iH(\exp tH)e_1) - P(i(\exp tH)e_1)\langle H(\exp tH)e_1, (\exp tH)e_1\rangle$$
$$= \left\{[2(\cosh t)^2 + 2(\sinh t)^2]\frac{\cosh t}{8} - \sinh^2 t\frac{\cosh t}{2}\right\} a_2 \neq 0.$$

Thus T_* has maximal rank everywhere and this completes the proof of Theorem 2.2.

Observe that

$$v_0 = \lim_{t\to\infty} \frac{i \exp tHe_1}{\|\exp tHe_1\|} = \frac{1}{4}(a_2 + i(e_1 - e_3)).$$

It is now clear how to define an action of F_4^1 on S^{15}.

Let $\overline{T}$: $F_4^1 \longrightarrow S^{15}$ be the map defined by $\overline{T}(g) = \sqrt{2}\, P(ig(v_0)/\|gv_0\|$. Since $F_4^1 = KAN$ and M is the subgroup of K such that $\overline{T}(k) = \overline{T}(1)$ and $\overline{T}(AN) = \overline{T}(1)$ we have that MAN is the subgroup of F_4^1 such that $\overline{T}(MAN) = \overline{T}(1)$. Since $\overline{T}(K) = K/M = S^{15}$, $\overline{T}(F_4^1) = F_4^1/MAN = S^{15}$.

We have thus realized F_4^1/K as the open unit ball in $\mathbf{R}^{16}$ with F_4^1/MAN as its boundary.

It is an easy exercise to show that $T(g) = P(ig(v_0)/\|P(ig(v_0))\|$.

REMARK. In §§3 and 4, we shall have occasion to use the following inner product, also. If $a_1 + ib_1, a_2 + ib_2 \in A(3, \mathrm{F})_{\mathrm{C}}$ set

$$(a_1 + ib_1, a_2 + ib_2) = \mathrm{Tr}(a_1 \cdot a_2 + b_1 \cdot b_2) + i\mathrm{Tr}(b_1 \cdot a_2 - a_1 b_2).$$

3. **Decomposition of the spherical harmonics.** As in [7], we find it necessary to decompose the spherical harmonics into their Spin(9) irreducible subspaces. Since Spin(9) preserves a positive definite bilinear form we have Spin(9) $\subset$ SO(16). Hence H^m, the spherical harmonics of degree m, are invariant under the action of K.

We have seen that the subgroup of Spin(9) which leaves a point of S^{15} fixed is Spin (7). When we restrict the spin-representation of Spin(9) to Spin(7) we have $\mathbf{R}^{16} = V_0 \oplus V_1 \oplus V_2$ where Spin(7) acts trivially on V_0, Spin(7) acts as SO(7) on V_1, and the representation of Spin(7) on V_2 is the spin-representation.

If we consider S_1^6, the unit sphere in V_1, and S_2^7, the unit sphere in V_2, we obtain an action of Spin(7) on $S_1^6 \times S_2^7$ which we now show is doubly transitive. That is, given (p_1, p_2) and (q_1, q_2) on $S_1^6 \times S_2^7$ there is an A in Spin(7) such that $A(p_1, p_2) = (q_1, q_2)$. Since Spin(7) is transitive on S_1^6, we may assume $p_1 = q_1$. The subgroup of Spin(7) which leaves p fixed is clearly Spin(6) (SU(4)). When the spin-representation of Spin(7) is restricted to Spin(6) we obtain the irreducible representation of SU(4) on $\mathbf{C}^4 = \mathbf{R}^8$. As SU(4) is transitive on S^7 we obtain our result. (This result is contained in Kostant [9].)

Suppose now that f is a function on $\mathbf{R}^{16}$ which is invariant under the action of Spin(7). If we write $f(x, \vec{y}, \vec{z})$ with $x \in V_0$, $\vec{y} \in V_1$, $\vec{z} \in V_2$, we see that f depends only on x, $\|\vec{y}\|$, and $\|\vec{z}\|$. Furthermore, if we set $r^2 = x^2 + \|\vec{y}\|^2 + \|\vec{z}\|^2$, $r \cos \xi = \sqrt{x^2 + \|y\|^2}$ with $0 \leqslant \xi \leqslant \pi/2$, and $r \cos \xi \cos \varphi = x$ with $0 \leqslant \varphi \leqslant \pi$, we see that f depends only on r, ξ, and ϕ.

We know already that E, the Euler operator of $\mathbf{R}^{16}$, and Δ, the Laplacian of $\mathbf{R}^{16}$ commute with the action of Spin(9). We wish to construct another operator which commutes with Spin(9) and Δ. To do so we need the following:

LEMMA 3.1. *Let $U_1, \ldots, U_d$ be a basis for the skew symmetric $n \times n$ real matrices such that* $-\mathrm{Tr}\ U_iU_j = g_{ij}$ $(d = n(n-1)/2)$. *Let $A \in O(n)$ and e a vector of $\mathbf{R}^n$. Then*

(1) $\Sigma_{i,j}\, g^{ij}U_iU_j = -(n-1)I/2$ *where* $(g^{ij}) = (g_{ij})^{-1}$.

(2) *If* $g_{ij} = \delta_{ij}$ *then* $2\,\Sigma_i\, \langle AU_ie, e\rangle^2 = \langle e, e\rangle^2 - \langle Ae, e\rangle^2$.

PROOF. (1) $\Sigma_{i,j}g^{ij}U_iU_j$ is a scalar multiple of the Casimir operator of the Lie algebra of SO(n). Hence as a matrix $\Sigma_{i,j}g^{ij}U_iU_j$ is a scalar multiple of I. By a convenient choice of the U_i we obtain (1).

(2) Suppose $(U_1^0, \ldots, U_d^0)$ is another set of skew symmetric matrices with the property $-\mathrm{Tr}\ U_i^0 U_j^0 = \delta_{ij}$. We show first that

$$\sum_i \langle AU_ie, e\rangle^2 = \sum_i \langle AU_i^0e, e\rangle^2.$$

Now $U_i = \Sigma_j\, c_{ji}U_j^0$ and $C = (c_{ij})$ is a $d \times d$ orthogonal matrix. Hence, if we set $\xi_i = \langle AU_i^0e, e\rangle$ and $\vec{\xi} = (\xi_1, \ldots, \xi_{n(n-1)/2})$ we have that

$$\sum_i \langle AU_ie, e\rangle^2 = \langle C\vec{\xi}, C\vec{\xi}\rangle = \langle \vec{\xi}, \vec{\xi}\rangle = \sum_i \langle AU_i^0e, e\rangle^2.$$

Assume $e = (r, 0 \cdots 0)$. We now select a convenient basis $U_1^0, \ldots, U_d^0$. If $i \leqslant n-1$ let $\sqrt{2}U_i = (\varphi_{j,k}^i)$ where $\varphi_{1,i+1}^i = 1$, $\varphi_{i+1,1}^i = -1$ and $\varphi_{j,k}^i = 0$ otherwise. If $i > n-1$ let U_i be such that $U_ie = 0$. If $A = (a_{jk})$ a simple calculation yields

$$2\sum_i \langle AU_ie, e\rangle^2\ (a_{12}^2 + \cdots + a_{1n}^2)r^4 = (1 - a_{11}^2)r^4 = \langle e, e\rangle^2 - \langle Ae, e\rangle^2.$$

If v is another vector in $\mathbf{R}^n$ such that $\|v\| = r$ we have that $v = Be$ for some B in $O(n)$. Thus,

$$2\sum_i \langle AU_iBe, Be\rangle^2 = 2\sum_i \langle (B^tAB)(B^tU_iB)e, e\rangle^2$$
$$= \langle e, e\rangle^2 - \langle B^tABe, e\rangle^2 = \langle e, e\rangle^2 - \langle ABe, Be\rangle^2 = \langle v, v\rangle^2 - \langle Av, v\rangle^2.$$

Hence, we have our result.

We now consider the action of Spin(9) on $A_2 \oplus A_3 = \mathbf{R}^{16} = V_0 \oplus V_1 \oplus V_2$. Here

$$V_0 = \mathbf{R}a_2, \quad V_1 = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ \bar{x} & 0 & 0 \end{pmatrix}: x + \bar{x} = 0, \quad V_2 = A_3.$$

Then, if $c_2 + c_3 \in A_2 \oplus A_3$ with $c_2 \in A_2$ and $c_3 \in A_3$, we have $r^2 = (c_2 + c_3, c_2 + c_3)$, $(c_2, c_2) = r^2 \cos^2 \xi$ and $r \cos \xi \cos \phi = (a_2/\sqrt{2}, c_2)$ in the angular coordinates used above.

LEMMA 3.2. *Let* $U_1, \ldots, U_d$ *be a basis defined in Lemma* 4.1(2) *where* $n = 8$ $(d = 27)$. *Then the differential operator* D *on* S^{15} *defined as follows*

$$Df(kM) = \left(\frac{\partial^2}{\partial t_i^2} + \cdots + \frac{\partial^2}{\partial t_d^2}\right)$$
$$\cdot f(k \exp(t_1U_1 + \cdots + t_dU_d)M)|_{(t_1 \cdots t_d)=(0,\ldots,0)}$$

is a left K-invariant differential operator which commutes with Δ. *Moreover, if* f *is an M-invariant function on* S^{15}

$$2Df(\xi, \varphi) = \frac{1}{(\sin^6\varphi)} \frac{\partial}{\partial\varphi} (\sin^6\varphi) \frac{\partial}{\partial\varphi} f(\xi, \varphi).$$

PROOF. As $U_1 \cdot U_1 + \cdots + U_d \cdot U_d$ is in the center of the enveloping algebra of $\mathfrak{D}_0$ ($X \cdot Y$ denotes multiplication in the enveloping algebra). The fact that D is a left K-invariant differential operator may be found in Helgason [3, p. 395]. This also proves that D and Δ commute.

To prove the second statement let $k = e^D k_0$ and let

$$k_0 \exp(t_1U_1 + \cdots + t_dU_d)a_2/\sqrt{2} = a(t_1, \ldots, t_d) \in A_2$$

($a_2/\sqrt{2}$ is the point on S^{15} left fixed by M).

Now

$$e^D a(t_1 \cdots t_d) = \left(\cos\frac{|c|}{2}\right) a(t_1, \ldots, t_d) + \left(\sin\frac{|c|}{2}\right) \frac{2a_1 \cdot a(t_1, \ldots, t_d)}{|c|}$$

where $D = [L_{e_2-e_3}, L_{a_1}]$ with

$$a_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & \bar{c} & 0 \end{pmatrix}.$$

As $\|a(t_1, \ldots, t_d)\| = \|a(0, \ldots, 0)\|$ we see that $\xi(t_1, \ldots, t_d)$ is constant and $\cos \varphi(t_1, \ldots, t_d) = (a_2/\sqrt{2}, a(t_1, \ldots, t_d))$. As $\partial f/\partial t_i = (\partial f/\partial\varphi)(\partial\varphi/\partial t_i)$ we have that

$$Df(kM) = \frac{\partial^2 f}{\partial\varphi^2} \sum_{i=1}^{d} \left(\frac{\partial\varphi}{\partial t_i}\right)^2(0) + \frac{\partial f}{\partial\varphi} \sum_{i=1}^{d} \frac{\partial^2\varphi}{\partial t_j^2}(0).$$

By Lemma 3.1 and the equation for $\cos \varphi(t_1, \ldots, t_d)$,

$$2 \sin^2\varphi \sum_{i=1}^{d} \left(\frac{\partial\varphi}{\partial t_i}\right)^2(0) = 2 \sum_{i=1}^{d} \left(\frac{a_2}{\sqrt{2}}, U_i\frac{a_2}{\sqrt{2}}\right)^2 = 1 - \cos^2\varphi = \sin^2\varphi,$$

and

$$-2 \cos \varphi \sum_{i=1}^{d} \left(\frac{\partial\varphi}{\partial t_i}\right)^2(0) - 2 \sin \sum_{i=1}^{d} \frac{\partial^2\varphi}{\partial t_i^2}(0)$$
$$= \sum_{i=1}^{d} \left(\frac{a_2}{\sqrt{2}}, k_0 U_i^2 \frac{a_2}{\sqrt{2}}\right) = -7 \cos \varphi.$$

Thus

$$\sum_{i=1}^{d} \frac{\partial^2\varphi}{\partial t_i^2}(0) = 6 \frac{\cos\varphi}{\sin\varphi}$$

and our lemma follows.

We can now state our main result of this section.

THEOREM 3.1 (1) (KOSTANT [9]). *As a representation of K,*

$$L^2(S^{15}) \sim \sum_{r\in\hat{K}_0} V_\gamma \qquad (\hat{K}_0 = \{\gamma \in \hat{K} \mid V_\gamma^M \neq (0)\}).$$

Hence each irreducible subrepresentation of K appears exactly once.

(2) *Let*

$$e_{m,l} = \cos^l\varphi\, F\left(-\frac{l}{2}, \frac{-l+1}{2}; \frac{7}{2}; -\tan^2\varphi\right)$$
$$\times \cos^m\xi\, F\left(\frac{l-m}{2}, \frac{-m-l-6}{2}; 4; -\tan\xi\right).$$

Then if m, l are nonnegative integers such that $m \geqslant l \geqslant 0$ and $m - l$ is even, $e_{m,l} \in H^m$. Moreover, under these assumptions, if $V^{m,l}$ is the K-cyclic space for $e_{m,l}$, $V^{m,l}$ is irreducible and $(V^{m,l})^M = \mathbf{C}e_{m,l}$. Finally, $L^2(S^{15}) \sim \sum V^{m,l}$ where the sum is taken over all such m and l.

PROOF. Let f be a function which is M-invariant. From Helgason [3, p. 387],

$$\Delta f = \frac{1}{r^{15}} \frac{\partial}{\partial r} r^{15} \frac{\partial}{\partial r} f$$

$$+ \frac{1}{r^2} \frac{1}{(\cos^7 \xi)(\sin^7 \xi)} \frac{\partial}{\partial \xi} (\cos^7 \xi)(\sin^7 \xi) \frac{\partial}{\partial \xi} f$$

$$+ \frac{1}{r^2 \cos^2 \xi} \frac{1}{\sin^6 \varphi} \frac{\partial}{\partial \varphi} \sin^6 \varphi \frac{\partial}{\partial \varphi} f.$$

Let $f \in \mathcal{H}^m$. Then $f(r, \xi, \varphi) = r^m h(\xi, \varphi)$ and we obtain

$$\frac{1}{(\cos^7 \xi)(\sin^7 \xi)} \frac{\partial}{\partial \xi} (\cos^7 \xi)(\sin^7 \xi) \frac{\partial h}{\partial \xi} + \frac{1}{(\cos^2 \xi)} \frac{1}{(\sin^6 \varphi)} \frac{\partial}{\partial \varphi} (\sin^6 \varphi) \frac{\partial}{\partial \varphi} h$$
$$= -m(m + 14)h.$$

As Δ and D commute on $\mathcal{H}^m$ we may diagonalize D on $\mathcal{H}^m$. Setting $h(\xi, \varphi) = \theta(\xi) \Xi(\varphi)$ we have by a lemma of Vilenkin [14, p. 495], (see also [7, Lemma 4.2]) that

$$\frac{1}{\sin^6 \varphi} \frac{\partial}{\partial \varphi} \sin^6 \varphi \frac{\partial}{\partial \varphi} \Xi(\varphi) = -l(l + 6) \Xi(\varphi)$$

where l is a nonnegative integer. Again by Vilenkin [14, p. 499],

$$\Xi(\varphi) = c_1 \cos^l \varphi \, F\left(\frac{-l}{2}, \frac{-l+1}{2}; \frac{7}{2} - \tan^2 \varphi\right),$$

$$\theta(\xi) = c_2 \cos^m \xi \, F\left(\frac{l-m}{2}, \frac{-m-l-6}{2}; 4; -\tan^2 \xi\right),$$

and $m - l \geqslant 0$ and even.

This proves Theorem 4.1.

We now express the representation of Spin(9) on $V^{p,q}$ in terms of highest weights. The Dynkin diagram for Spin(9) is

0——0——0⟹0

α_1 α_2 α_3 α_4

where $\alpha_1, \alpha_2, \alpha_3$, and α_4 are simple roots. Let λ_i be the fundamental weight such that

$$2\langle \lambda_i, a_j \rangle / \langle \alpha_j, \alpha_j \rangle = \delta_{ij}.$$

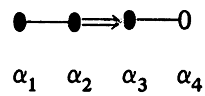
Then, from [7, Lemma 4.3] and the formula for $(V^{p,q})^M$, we see that the highest weight on $V^{p,q}$ is $((p - q)/2)\lambda_1 + q\lambda_4$.

4. Poisson kernel and the transvection operator. In order to study the spherical principal series representations of F_4^1, we need to compute the action of the transvection operator on X^k (see [7]). For this we need a convenient expression for $P(gK, kM)$, the Poisson kernel of $F_4^1/\mathrm{Spin}(9)$. Now $P(gK, kM) = e^{-2\rho H(g^{-1}k)}$.

Let $v = a_2 + i(e_1 - e_3)$.

LEMMA 4.1. $\exp -2\rho(H(g)) = |(gv, e_1)|^{-11}$.

PROOF. By Araki [1], F_4^1 has the Satake diagram



The smallest positive restricted root is $\alpha = (\alpha_4 + \nabla\alpha_4)/2$ where $\nabla\alpha_4 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4$. The highest weight of the representation of F_4^1 on $\bar{A}_C$ is easily seen to be $\Lambda = 2\alpha$. As $2\rho = 22\alpha = 11\Lambda$ and $|(gv, e_1)| = e^{\Lambda(H(g))}$ our result follows.

REMARKS. (1) If we let $H_1 = 2H$ where $H = i[L_{e_1-e_3}, L_{a_2}]$ we can see that the eigenvalues of ad H_1 are ± 2, ± 1 and 0.

(2) After normalizing as in [7] we identify the spherical principal series representation π_μ with the representation on $L^2(S^{15})$ such that for $g \in F_4^1$, $ka_2/\sqrt{2} \in S^{15}$, and $f \in L^2(S^{15})$,

$$(\pi_\mu(g)f)(ka_2/\sqrt{2}) = |(g^{-1}kv, e_1)|^{-\mu/2} f(\overline{T}(g^{-1}k)).$$

We now write out a more convenient expression for $|((\exp - tH_1)kv, e_I)|$ for $k \in K$. Now $k = (\exp D)k_0$ where $k_0 \in K_0$ and $D = [L_{e_2-e_3}, L_{a_1}]$ where

$$a_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & \bar{c} & 0 \end{pmatrix}.$$

Setting $k_0 a_2 = a \in A_2$ we have

$$(\exp D)k_0 v = \left(\cos\frac{|c|}{2}\right)a + \left(\sin\frac{|c|}{2}\right)\left(\frac{2a_1 \cdot a}{|c|}\right)$$
$$+ i\left\{e_1 - \tfrac{1}{2}(e_2 + e_3) + \left(\frac{\cos|c|}{2}\right)(e_2 - e_3) - (\sin|c|)\left(\frac{a_1}{2|c|}\right)\right\}.$$

Furthermore, as

$$(\exp - tH_1)e_1 = (\cosh 2t)\frac{e_1 - e_3}{2} + \frac{e_1 + e_3}{2} + i(\sinh 2t)\frac{a_2}{2},$$

and

$$|((\exp - tH_1)kv, e_1| = |(kv, (\exp - tH_1)e_1)|,$$

we have that

$$|((\exp - tH_1)kv, e_1)| = \cosh^2 t - 2(\cos\xi)(\cos\varphi)(\cosh t)(\sinh t) + (\sinh^2 t)(\cos^2\xi)$$

once we observe $|c|/2 = \xi$ and $\cos\varphi = (a, a_2/2)$.

LEMMA 4.2. *Let f be an M-invariant function on S^{15}. Then*

$$\pi_\nu(H_1)f(\xi, \varphi) = \mu \cos\xi \cos\varphi\, f(\xi, \varphi) + \sin\xi \cos\varphi\ \partial f(\xi, \varphi)/\partial\xi$$

$$+ \left(\frac{1 + \cos^2\xi}{\cos\xi}\right) \sin\varphi \frac{\partial f}{\partial\varphi}(\xi, \varphi).$$

PROOF. $d|((\exp - tH_1)kv, e_1)|^{-\mu/2}/dt = \mu \cos\xi \cos\varphi$. Let $k = e^D k_0$ as above. Now $P((\exp - tH)kv) = \alpha_2(t) + \alpha_3(t)$ where $\alpha_i(t) \in A_i$. An easy calculation gives

$$\alpha_2(t) = (\cosh tH_1)\left(\cos\frac{|c|}{2}\right)a + i[(\sinh - tH_1)e_1$$

$$- \tfrac{1}{2}(1 + \cos|c|)(\sinh - tH_1)e_3]$$

and

$$\alpha_3(t) = (\cosh tH_1)\left(\sin\frac{|c|}{2}\right)\frac{2a_1 \cdot a}{|c|}$$

$$+ i(\sinh - tH)(-\sin|c|)\left(\frac{a_1}{2|c|}\right).$$

Thus,

$$\cos\xi(t) = \|\alpha_2(t)\|/(\|\alpha_2(t)\|^2 + \|\alpha_3(t)\|^2)^{1/2}$$

and

$$\cos\varphi(t)\cos\xi(t) = \left(\frac{a_2}{\sqrt{2}}, \frac{\alpha_2(t)}{(\|\alpha_2(t)\|^2 + \|\alpha_3(t)\|^2)^{1/2}}\right).$$

As $\xi = \xi(0)$ and $\varphi = \varphi(0)$ we have

$$d\xi(0)/dt = \sin\xi\cos\varphi \quad \text{and } d\varphi(0)/dt = \left(\frac{1 + \cos^2\xi}{\cos\xi}\right)\sin\varphi.$$

We now obtain

THEOREM 4.1.

$$\pi_\nu(H_1)e_{m,l} = \frac{1}{(6+2l)(14+2m)}\{(6+l)(\nu+m+l)(14+m+l)e_{m+1,l+1} + l(\nu+m-l-6)(8+m-l)e_{m+1,l-1} + (6+l)(\nu-m+l-14)(m-l)e_{m-1,l+1} + l(\nu-20-m-l)(m+l+6)e_{m-1,l-1}\}.$$

PROOF. Let $\chi_l(\varphi) = \cos^l \varphi\, F(-l/2, (-l+1)/2; 7/2; -\tan^2\varphi)$ and $h_l^m(\xi) = \cos^m \xi\, F((l-m)/2, -m-l-6; 4; -\tan^2\varphi)$. From Lemma 5.3 of [7] we have

$$\cos\varphi\,\chi_l = \left(\frac{6+l}{6+2l}\right)\chi_{l+1} + \frac{l}{6+2l}\chi_{l-1},$$

$$\sin\varphi\frac{d\chi_l}{d\varphi} = \left(\frac{(6+l)l}{6+2l}\right)(\chi_{l+1} - \chi_{l-1}).$$

The rest of the proof now duplicates Theorem 5.1 of [7].

5. **The composition series and intertwining operators.** In this section we will explicitly compute the composition series, the intertwining and partial intertwining operators from (π_ν, χ^ν) to $(\pi_\lambda, \chi^\lambda)$. Since the proofs are the same as those for the corresponding results for the classical rank one groups we will omit the proofs.

As in [7], we need the following

PROPOSITION 5.1. (1) *There is an F_4^1-invariant nondegenerate sesquilinear pairing between (π_ν, H^ν) and $(\pi_{22-\bar\nu}, H^{22-\bar\nu})$.*

(2) *Let f be a function in $L^2(S^{15})$ such that the span of Kf is irreducible. Then, if f is a cyclic vector for (π_ν, H^ν), the cyclic space for f in $(\pi_{22-\bar\nu}, H^{22-\bar\nu})$ is irreducible.*

THEOREM 5.2. (1) (KOSTANT [9]) *1_ν is a cyclic vector for (π_ν, χ^ν) if and only if $\nu-6$ is not a nonpositive even integer. (π_ν, χ^ν) is irreducible if and only if neither $\nu-6$ nor $16-\bar\nu$ is a nonpositive even integer.*

(2) *Suppose $\nu-6=-2l$ where l is a nonnegative integer. Then $W_l = \Sigma m + k \leqslant 2l - 6 V^{m,l}$, $M_l = \Sigma_{m-k<l} V^{m,k}$, and X^{-2l+6} are invariant under π_{-2l+6}. $W_l = 0$ if $l<3$, and W_l, M_l/W_l and X^{-2l+6}/M_l are irreducible under the induced representation. By duality, $\widetilde{W}_l = \Sigma_{m+k>2l-6} V^{m,k}$, $\widetilde{M}_l = \Sigma_{m-k>2l} V^{m,k}$, and X^{16+2l} are invariant under π_{16+2l} with $\widetilde{M}_l$, $\widetilde{W}_l/\widetilde{M}_l$, and $X^{16+2l}/\widetilde{W}_l$ irreducible.*

For a proof of this theorem see the proof of theorem 7.1 [7].

THEOREM 5.2. *Suppose $A \neq 0$ intertwines (π_ν, χ^ν) and $(\pi_\lambda, \chi^\lambda)$. Then either $\lambda = \nu$, or $\lambda = 22 - \nu$. Moreover, if $\lambda = \nu$, A is a scalar multiple of the identity, and if $\lambda = 22 - \nu$*

$$A|_{Vm,k} = c \prod_{j=1}^{(m-k)/2} \left(\frac{\nu - 2j - 14}{8 - 2j - \nu}\right) \prod_{j=1}^{(m+k)/2} \left(\frac{20 + 2j - \nu}{\nu + 2j - 2}\right)$$

where c is some constant.

PROPOSITION 5.2. *Define $A_\nu: X^\nu \longrightarrow X^{22-\nu}$ by setting*

$$A_\nu|_{Vm,k} = c_{m,k}(\nu)$$

$$= \left(\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{\nu-6}{2}\right)\right)^{-1} \prod_{j=1}^{(m-k)/2} \left(\frac{\nu - 2j - 14}{8 - 2j - \nu}\right) \prod_{j=1}^{(m+k)/2} \left(\frac{20 - \nu + 2j}{\nu + 2j - 2}\right).$$

Then $c_{m,k}(\nu)$ is an analytic function on $\mathbf{C}$, and thus A_ν defines an intertwining operator from (π_ν, X^ν) to $(\pi_{22-\nu}, X^{22-\nu})$.

If ν is real and $\langle\ ,\ \rangle$ is the nondegenerate sesquilinear pairing of X^ν with $X^{22-\nu}$, we have that $(f, g)_\nu = \langle f, A_\nu g\rangle$ defines an F_4^1-invariant Hermitian form on X^ν.

THEOREM 5.3. (1) (KOSTANT [9]) *If ν is real and not an even integer $\leqslant 6$ or $\geqslant 16$, $(\ ,\ )_\nu$ defines a nondegenerate Hermitian form on X^ν. If $3 < \nu < 8$ $(\ ,\ )_\nu$ turns X^ν into a pre-Hilbert space and by Nelson's theorem* [10] *we thus obtain a unitary representation of F_4^1 on the completion of X^ν.*

(2) *If $\nu - 6 = -2l$ where l is a nonnegative integer, the Hermitian product $(\ ,\ )_{-2l+6}$ is positive semidefinite on X^{-2l+6}. Furthermore, $(\ ,\ )_{-2l+6}$ induces a positive definite invariant inner product on X^{-2l+6}/M_l.*

Let $B_\nu = \Gamma((\nu - 6)/2)A_\nu$. If $\nu - 6 = -2l$ where l is a nonnegative integer, $B_{-2l+6}|_{M_{-2l+6}}$ is well defined and if we set $\{f, g\}_{-2l+6} = \langle f, B_{-2l+6}g\rangle$ we obtain

THEOREM 5.4. *$\{\ ,\ \}_{-2l+6}$ defines a nondegenerate Hermitian bilinear form on M_l/W_l which is positive definite only if $l = 0$.*

REMARK. In [7], we encountered a good deal of difficulty in examining the reducible principal series representations of SU(2, 1) and Sp(2, 1) for square integrability. These same difficulties are encountered with F_4^1.

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