

THE SETS THAT ARE SCISSOR CONGRUENT TO AN UNBOUNDED CONVEX SUBSET OF THE PLANE

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ABSTRACT. It is shown that an unbounded convex plane body is scissor congruent to the union of a congruent body with a finite number of arbitrary topological discs. It is proved that 'is scissor congruent to' is an equivalence relation. Thus two unbounded convex plane bodies are scissor congruent if and only if the union of one with a finite number of topological discs is scissor congruent to the other.

1. Introduction. The decomposition of a set into a finite number of parts and the rearrangement of these parts to form another set has been studied in the past. Bolyai and Gerwin (see [4] for example) have shown that if two polygons have equal area one can be decomposed into a finite number of triangles which can be rearranged to form the other. Dubins, Hirsch and Karush showed [5] that if one cuts a strictly convex, compact, planar body, E , along arbitrary Jordan curves, the pieces can be rearranged by rigid motions to form another strictly convex planar body (i.e. the bodies are scissor congruent) if and only if the bodies have the same area and the boundary of one can be cut into a finite number of pieces which can be rearranged to form the boundary of the other (i.e. the boundaries are scissor congruent).

The notion of scissor congruence is herein expanded to unbounded bodies. An easily stated preliminary result is:

THEOREM 1.1. *An unbounded convex body is scissor congruent to itself with a finite number of arbitrary topological discs adjoined.*

This theorem implies, for example, that if a continuous function on the real line agrees with a convex function (e.g. a quadratic) except possibly on a compact interval, then the graph of the continuous function, together with its interior, is scissor congruent to the graph of the convex function together with its interior.

It will also be demonstrated that 'is scissor congruent to' is an equivalence relation and thus one can conclude:

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THEOREM 1.2. *Two unbounded convex bodies, E_+ and E_- , are scissor congruent if and only if E_+ together with a finite number of arbitrary topological discs is scissor congruent to E_- .*

2. Definition of scissor congruence.

DEFINITIONS.

2.1. \emptyset will denote the empty set.

2.2. A *bounded arc* A is the image of a connected subset A' of the circumference C of a circle under a homeomorphism (of the plane onto itself) except that \emptyset , C and one point sets are not considered as arcs. A is open if A' is open in C .

2.3. $X + Y$ is the set with the same elements as $X \cup Y$, and is defined for sets whose interiors do not intersect. The set is called the *sum* of X and Y .

2.4. Let R be a rectangle of width $t > 0$, and A an open bounded arc contained in the interior of R with the endpoints of A contained in the opposite sides of length t of R , with no endpoint of A equal to a corner point of R . The two closed subsets, of sum R , defined by A are each called a *W-rectangle* of width t . Two *W-rectangles* whose sum is a rectangle, R , are called a *pair of W-rectangles* of R .

2.5. A *half-infinite strip*, of width $t > 0$, is a closed, unbounded, convex subset of the plane whose boundary is two parallel rays, a distance t apart, and a line segment of length t joining their endpoints.

LEMMA 2.6. *A half-infinite strip is scissor congruent to the sum of a congruent strip and a W-rectangle of the same width.*

Before proceeding with the proof, the relevant definitions will be given.

DEFINITIONS.

2.7. \cong will denote congruence under plane rigid motion.

2.8. For any line segment s , $|s|$ will denote its length.

2.9. $d(\cdot, \cdot)$ will denote the Euclidean distance function. The appropriate domain will be clear from the context. (We will be working in Euclidean space throughout.)

2.10. For any set S , $\text{int } S$, $\text{ext } S$ and ∂S will denote the interior, exterior and boundary of S , respectively.

2.11. A *topological disc* (*topological half-plane*) is the image of the closed unit disc (closed half-plane) under a homeomorphism of the plane onto itself.

Some notations, borrowed from [5], that will be used to define scissor congruence, are:

2.12. $D_1, \dots, D_m, D_{-1}, \dots, D_{-m}$ are $2m$ topological half-planes;
 $D_{m+1}, \dots, D_n, D_{-(m+1)}, \dots, D_{-n}$ are $2(n - m)$ topological discs, $m \leq n < \infty$.

2.13. T_1, \dots, T_n are n rigid motions of the plane.

$$2.14. E_+ = \bigcup_{i>0} D_i; E_- = \bigcup_{i<0} D_i.$$

$$2.15. K_+ = \partial E_+; K_- = \partial E_-.$$

$$2.16. K = K_+ \cup K_-.$$

$$2.17. J_i = \partial D_i, \text{ for all } i.$$

2.18. E_+ is scissor congruent to E_- if there is a collection of D_i and T_i which satisfy 2.12–2.14 and if:

$$(i) T_i(D_i) = D_{-i}, i > 0,$$

$$(ii) \text{int } D_i \cap \text{int } D_j = \emptyset \text{ for } i > j > 0 \text{ and for } i < j < 0.$$

The next assumption is automatic for unbounded figures:

(iii) $J_i \cap J_j \cap K_+$ is a bounded set for $i > j > 0$; $J_i \cap J_j \cap K_-$ is a bounded set for $i < j < 0$.

2.19. A collection $\{D_i, D_{-i}, T_i, i = 1, \dots, n, \text{ such that 2.18 (i)–(iii) is satisfied}\}$ is a *scissor congruence* of E_+ and E_- .

If two topological discs are contained in a union, all the residual domains of the union of their boundaries are topological discs (Kerékjártó's theorem; see [10], for example). Therefore, we can consider the sum, rather than the union, of a topological disc with a topological disc or topological half-plane in our proofs, without loss of generality.

For the remainder of this section we will assume that $t > 0$.

PROOF OF LEMMA 2.6. Let R^u and R^b be any pair of W -rectangles of a rectangle R . We can assume R is contained in the upper half-plane with its lower boundary parallel to the x -axis. Let T be the translation in the direction of the $+y$ -axis so that $R \cap T(R)$ is a side of R . Let $U = R^u + T(R^u) + T^2(R^u) + \dots$ and $B = R^b + T(R^b) + T^2(R^b) + \dots$. U and B are obviously topological half-planes and $E_+ = U + B$ is a half-infinite strip and $E_- = R^b + T(U + B)$ is a W -rectangle + a half-infinite strip congruent to E_+ .

Intuitively, E_+ is scissor congruent to E_- because one can slip U along the half-infinite strip and thus expose the W -rectangle, R^b . The slipped U , namely $T(U)$, is congruent to U , and B is congruent to itself, i.e. E_+ and E_- are scissor congruent as follows:

$$\begin{array}{ccc} U + B & & = E_+ \\ \downarrow T \quad \downarrow \text{identity map} & & \\ (T(U) + B) & = & T(U + B) + R^b = E_- \quad \square \end{array}$$

LEMMA 2.20. A half-infinite strip is scissor congruent to the union of a congruent strip and a topological disc.

For the proof of Lemma 2.20 we will need Lemma 2.22.

DEFINITION 2.21. If p_i and p_j are any two points of the plane, let $\overline{p_i p_j}$ denote the closed line segment with endpoints p_i and p_j .

LEMMA 2.22. For any $t > 0$ and any topological disc, D , there is a (possibly empty) polygon, P^* , and a W -rectangle, R^* , of width t , such that $D + P^*$ is scissor congruent to R^* .

Before proceeding with the proof of Lemma 2.22 for the case of a general topological disc, it would be instructive to consider a convex topological disc, D . For any $t > 0$, we can choose points $d_1, \dots, d_M = d_0 \in \partial D$ so that the sector D_{i+1} bounded by $\overline{d_i d_{i+1}}$ and the subarc of ∂D between d_i and d_{i+1} (call this subarc A_{i+1}) can be contained in the interior of a square of width t . Then, obviously, we can string the D_i 's out so they are all contained in a rectangle, R , of width t . We can fill out R with the polygon $D - \bigcup D_i$ and another polygon, P^* , if necessary, to obtain a W -rectangle, R^* .

The method for a general disc is more complicated, for we do not necessarily have sectors, but must define a finite number of small topological discs, D_i , with the property that $D - \bigcup D_i$ is a polygon.

PROOF OF LEMMA 2.22. Let S be the unit disc. By the Riemann Mapping Theorem there is a homeomorphism $f: S \rightarrow D$ which is a conformal map of $\text{int } S \rightarrow \text{int } D$. Since S is compact, f is uniformly continuous. Therefore, given $t/2 > 0$, there is a δ' so that the closed disc, S' , concentric with S and of radius $1 - \delta$, where $\delta = \min(\frac{1}{2}, \delta')$ is mapped into $\text{int } D$ and $\partial f(S')$ is within $t/2$ of ∂D . $\partial f(S')$ is analytic and can therefore be approximated, within $t/2$, by a polygonal line, $P \subset \text{int } D$. Choose $M < \infty$ points, $d_1, \dots, d_M = d_0 \in \partial D$ so that $d(f^{-1}(d_{i-1}), f^{-1}(d_i)) < \delta$, $1 \leq i \leq M$. For $i = 1, \dots, M$: Let p_i be a point of P with $f^{-1}(p_i)$ on the radius to $f^{-1}(d_i) \in \partial S$ so that

$$\overline{f^{-1}(p_i) f^{-1}(d_i)} \cap f^{-1}(P) = f^{-1}(p_i).$$

Let

$$U_i = \overline{f(f^{-1}(p_i) f^{-1}(d_i))},$$

A_i = the closed subarc of ∂D with endpoints d_{i-1} and d_i . (Without loss of generality we can assume $\text{int } A_i \cap \text{int } A_j = \emptyset$ for $i \neq j$.)

L_i = the closed polygonal line segment of P joining p_{i-1} and p_i .

Then the closed set, D_i , bounded by $U_{i-1} + A_i + U_i + L_i$ is a topological disc since it is the homeomorphic image of a topological disc contained in S . Choose T_i to be a rigid motion satisfying the following conditions:

(i) $T_i(\overline{d_{i-1} d_i})$ are consecutive segments along some oriented line, L , with nonintersecting interiors. $T_i(\overline{d_{i-1} d_i})$ is before $T_j(\overline{d_{j-1} d_j})$ for $i < j$.

- (ii) $\text{int } D_i \cap \text{int } D_j = \emptyset$, for $i < j < 0$, where $T_i(D_i) = D_{-i}$.
- (iii) The polygonal arcs, $T_i(L_i)$ are all on the same side of L .
- (iv) $d(T_1(d_0), T_M(d_M)) < \infty$.

Thus the D_{-i} 's will lie within a rectangle, R , of width t . Join $T_1(p_0)$ and $T_M(p_M)$ to the nearest side (but not a corner) of R by a line segment. Join $T_j(p_j)$ to $T_{j+1}(p_{j+1})$ for $j = 1, \dots, M-1$, by a polygonal line, L' , which intersects none of the D_{-i} 's so that $A = \bigcup_i T_i(U_{i-1} \cup A_i \cup U_i) \cup L'$ is an arc. We have thus defined a W -rectangle, $R^b \subset R$, whose nonlinear boundary segment is A . Obviously $\bigcup_i D_{-i} \subset R^b$. Let $P^b = R^b - \bigcup_{i < 0} D_i$. P^b is a finite union of polygons, and $\bar{P} = D - \bigcup_{i > 0} D_i$ is a finite union of polygons.

If $\text{area } \bar{P} \leq \text{area } P^b$ then let P^* be any polygon disjoint from D with $\text{area } P^* = \text{area } P^b - \text{area } \bar{P}$. Then $\bar{P} + P^*$ is scissor congruent to P^b by Bolyai's theorem [4] and hence $D + P^*$ is scissor congruent to $R^b (= R^*)$, a W -rectangle.

If $\text{area } P > \text{area } P^b$, then \bar{P} is scissor congruent to $P^b +$ a rectangle of width t adjoined to R^b and sharing a side with R^b , i.e. another W -rectangle, R^* , of width t and P^* is empty. \square

PROOF OF LEMMA 2.20. Without loss of generality, we can assume the strip and the topological disc have disjoint interior. Then this lemma follows from 2.22 and 2.6 since we do not decompose the W -rectangle, R^b , in Lemma 2.6 to obtain scissor congruence. That is: Let I and I' be half-infinite strips of width t . Let D , P^* and R^* be as in Lemma 2.22. We can choose P^* to be a rectangle so that $I + P^* \cong I$. Then $I + P^* + D$ is scissor congruent to $I' + R^*$. But we can slip R^* into I' as in Lemma 2.6 and thus have $I + P^* + D$ scissor congruent to I' . \square

LEMMA 2.23. *A half-infinite strip of width t is scissor congruent to the union of a congruent strip with a finite number of topological discs.*

PROOF. Follows from Lemma 2.20 if we note that we can sum the W -rectangles associated with each topological disc to form another W -rectangle. \square

DEFINITION 2.24. A *strip* of width t is the union of a half-infinite strip of width t with a finite number of topological discs.

LEMMA 2.25. *Two strips of the same width are scissor congruent.*

PROOF. Follows immediately from Lemma 2.23, noting that if the strips are of width t , the topological discs of each can be made scissor congruent to a W -rectangle of width $t/2$. \square

3. Definition of S -equivalence.

DEFINITIONS.

3.1. A *bounded figure* is the union of a finite number of topological discs. An *unbounded figure* is the union of a topological half-plane and a bounded figure.

3.2. An unbounded figure is *eventually convex* if it is a union of an

unbounded convex figure and a finite number of topological discs.

3.3. An *axis* of a set E is a ray which is a subset of E . Grünbaum, on p. 23 of [6], proves two lemmas which can be stated as follows:

LEMMA 3.4. *Every unbounded convex figure has at least one axis.*

LEMMA 3.5. *If X is an axis of a convex figure, E , and if T is a translation with T (endpoint of X) $\in E$, then $T(X)$ is an axis of E .*

If E is an unbounded convex figure in the plane, we will assume, without loss of generality, that:

- (i) E is contained in the closed upper half-plane,
- (ii) the x -axis is a line of support of E , and
- (iii) E has an axis perpendicular to the x -axis, say $x = X$.

Let g be a convex function so that the boundary of E is the graph of g . Then, if x_1 and x_2 are any two real numbers, let

$$G(x_1, x_2) = x_2 / (g(x_1 + x_2) - g(x_1)).$$

$G(x_1, x_2)$ is a monotone function in x_2 for a fixed x_1 since g is convex. Therefore, the following limits exist:

$$\inf_{x_1 \geq X} \sup_{x_2 \downarrow 0} G(x_1, x_2) = \inf_{x_1 \geq X} (g'_R(x_1))^{-1} = \tan B_R(E)$$

and

$$\inf_{x_1 \leq X} \sup_{x_2 \uparrow 0} G(x_1, x_2) = \inf_{x_1 \leq X} (g'_L(x_1))^{-1} = \tan B_L(E)$$

where $g'_R(x_1)$ and $g'_L(x_1)$ are the right-hand and left-hand derivatives of g at x_1 , respectively.

DEFINITIONS.

3.6. If E is situated in the plane so that (i)–(iii) above hold, and also so that $|B_R(E)| = |B_L(E)|$, we will say that E is *properly situated*.

3.7. The *angle of E* is $|B_R(E)| + |B_L(E)|$, the maximum angle between axes of E .

3.8. E is *uniaxial* if the angle of E is 0, and *multiaxial* otherwise. For example, a figure whose boundary is the graph of a parabola is uniaxial. One whose boundary is the graph of one branch of a hyperbola is multiaxial.

Note that all the axes of a uniaxial figure are translates of each other. Without loss of generality, we will consider all unbounded convex figures to be properly situated in the plane.

DEFINITIONS.

3.9. An *unbounded arc* is the image of a half-infinite interval under a homeomorphism of the plane onto itself. It is open or closed according to whether the

interval is open or closed. The word *arc* will refer to bounded or unbounded arcs.

3.10. Let V_+ and V_- be eventually convex figures.

(i) ∂V_+ is eventually scissor congruent to ∂V_- if and only if there exist four unbounded open arcs, $C_1, C_2 \in \partial V_+$ which are disjoint, and $C_{-1}, C_{-2} \in \partial V_-$ which are also disjoint, and there exist rigid motions M_1 and M_2 such that $M_i: C_i \cong C_{-i}$, $i = 1, 2$.

(ii) V_+ is *S-equivalent* to V_- if and only if:

(a) ∂V_+ is eventually scissor congruent to ∂V_- , with M_1 and M_2 translations.

Furthermore, if V_+ and V_- are uniaxial, we require that

(b) $M_1(N) = M_2(N)$, where N is a line parallel to an axis of V_+ .

We will show in Theorem 9.1 that for two eventually convex figures to be scissor congruent it is necessary and sufficient that they be *S-equivalent*.

LEMMA 3.11. *S-equivalence is an equivalence relation on the class of eventually convex figures.*

Before proceeding with the proof, we will demonstrate the following lemmas.

LEMMA 3.12. *'Is eventually scissor congruent to' is a transitive relation on the class of boundaries of unbounded figures.*

PROOF. Let E_+, E_-, E_* be unbounded figures with boundaries K_+, K_-, K_* respectively. Assume that K_+ is eventually scissor congruent to K_- and K_- is eventually scissor congruent to K_* . For $i = 1, 2$: There are four unbounded arcs $A_i \subset K_+, A_{-i} \subset K_-$ with $R_i: A_i \cong A_{-i}$, R_i rigid motions, and $A_1 \cap A_2 = A_{-1} \cap A_{-2} = \emptyset$. Similarly, there are four unbounded arcs $A'_{-i} \subset K_-, A'_{*i} \subset K_*$ with $R'_i: A'_{-i} \cong A'_{*i}$, R'_i rigid motions, and $A'_{-1} \cap A'_{-2} = A'_{*1} \cap A'_{*2} = \emptyset$.

Let us assume the arcs are indexed so that either $A_{-i} \subset A'_{-i}$ or else $A'_{-i} \subset A_{-i}$. Let $T^i = R'_i R_i$, $B_i = (T^i)^{-1} A'_{*i}$. If $B_i \subset A_i$, let $U_i = B_i$. If $B_i \supset A_i$, let $U_i = A_i$. Let $U_{*i} = T^i(U_i)$.

We have thus defined four unbounded arcs, U_1, U_2 of K_+ which are disjoint, and U_{*1}, U_{*2} of K_* which are disjoint with $T^i: U_i \cong U_{*i}$. Thus K_+ is eventually scissor congruent to K_* . \square

LEMMA 3.13. *If two unbounded convex figures are S-equivalent they have equal angles.*

PROOF. Using the notation of 3.10, suppose L is an axis of V_+ with endpoint the same as that of C_1 , say c_1 . Obviously, $M_1(L) \cap C_{-1} = M_1(c_1)$. Suppose $M_1(L) \cap C_{-2} \neq \emptyset$, say equals c_{-2} . Let $L_- = M_1(L) \cap \text{ext } V_-$. Then $M_2^{-1}(L_-) \in \text{ext } V_+$. But $M_2^{-1}(L_-)$ has endpoint belonging to C_2 and, since M_1 and M_2 are translations, by Lemma 3.5 is an axis of V_+ . Contradiction.

Therefore $M_1(L) \cap C_{-2} = \emptyset$, i.e. $M_1(L)$ is an axis of V_- . Similarly we can

show that M_1^{-1} of any axis of V_- is also an axis of V_+ . It follows from Lemma 3.5 that the angles of V_+ and V_- are equal. \square

PROOF OF LEMMA 3.11. S -equivalence is obviously a reflexive and symmetric relation. Using the notation of Lemma 3.12, assume that E_+ is S -equivalent to E_- and E_- is S -equivalent to E_* .

Then by Lemma 3.12, K_+ is eventually scissor congruent to K_* .

T^1 and T^2 are translations since R_1, R_2, R'_1 and R'_2 are. If E_+ is uniaxial then so are E_- and E_* by Lemma 3.13. If L is a line parallel to an axis of E_+ , then

$$T^1(L) = R'_1 R_1(L) = R'_1 R_2(L) = R'_2 R_2(L) = T^2(L).$$

Therefore E_+ is S -equivalent to E_* . \square

4. S -equivalence implies scissor congruence.

DEFINITIONS.

4.1. A *special wedge*, E , is an unbounded convex figure bounded by two rays with the same endpoint, x . The angle, θ , between the rays is the angle of E . $0 < \theta \leq \pi$. x is the *apex* of E . The axis of E with endpoint x which bisects θ is called the *center line* of E .

4.2 Let E be a special wedge and P a closed half-plane so that $E - E \cap P$ is bounded. Let D be a bounded figure with $\text{int } D \cap \text{int } P = \emptyset$. Then $W = D + E \cap P$ is a *wedge*. Obviously, the angle of W is the angle of E . (D, E, P) will denote the triple associated with W .

LEMMA 4.3. *If two wedges have equal angles they are scissor congruent.*

PROOF. Let W_+ and W_- be two wedges with equal angles. Let (D_+, E_+, P_+) and (D_-, E_-, P_-) be the triples associated with W_+ and W_- , respectively.

Case (a). Angle of $E_+ = \text{angle of } E_- \neq \pi$. E_+ and E_- are congruent special wedges which we can assume are properly situated. Therefore there is a closed upper half-plane, P , such that:

(i) $H_+ = E_+ \cap P_+ \cap P \cong E_- \cap P_- \cap P = H_- \neq \emptyset$ and

(ii) $E_+ - H_+$ and $E_- - H_-$ are bounded and nonempty.

Let x_i (x_{-i}) be the endpoint of the ray X_i (X_{-i}) of the boundary of H_+ (H_-), for $i = 1, 2$. Let Y_1 (Y_{-1}) be an axis of H_+ (H_-) parallel to X_2 (X_{-2}) with endpoint x_1 (x_{-1}). The special wedge, E'_+ , bounded by X_1 and Y_1 is congruent to the special wedge, E'_- , bounded by X_{-1} and Y_{-1} .

Let $I_+ = \text{the strip } (H_+ - E'_+)$ and $I_- = \text{the strip } (H_- - E'_-)$. I_+ and I_- are congruent strips, so the strip $I_+ + (W_+ - H_+)$ is scissor congruent to the strip $I_- + (W_- - H_-)$, by Lemma 2.25.

Case (b). If E_+ and E_- are closed half-planes, let $I_+ \subset E_+$ and $I_- \subset E_-$ be congruent half-infinite strips whose boundary rays are perpendicular to the

boundary of E_+ and E_- , respectively. Then $I_+ + D_+$ and $I_- + D_-$ are scissor congruent strips and $E_+ - I_+$ and $E_- - I_-$ are congruent.

In either case, W_+ is scissor congruent to W_- . \square

THEOREM 4.4. *If two eventually convex figures are S -equivalent they are scissor congruent.*

PROOF. We will use the notation of Definition 3.10. For $i = 1, 2$: Let c_i , c_{-i} be the endpoint of C_i and C_{-i} , respectively. We can assume $c_1 \neq c_2$ and $c_{-1} \neq c_{-2}$. Let R_i be an axis of V_+ with endpoint c_i , and for V_+ multiaxial we require that R_1 and R_2 not be parallel or collinear and $R_1 \cap R_2 = \emptyset$.

Let $R_{-i} = M_i(R_i)$. Then D_i , with boundary $C_i + R_i$ and D_{-i} , with boundary $C_{-i} + R_{-i}$ are (possibly empty) topological half-planes and $D_{-i} = M_i(D_i)$. We have defined D_i and D_{-i} so that $\text{int } D_1 \cap \text{int } D_2 = \text{int } D_{-1} \cap \text{int } D_{-2} = \emptyset$. Thus $D_1 + D_2$ is scissor congruent to $D_{-1} + D_{-2}$.

$V_+ - (D_1 + D_2)$ and $V_- - (D_{-1} + D_{-2})$ are:

- (i) wedges, if V_+ and V_- are multiaxial. They have the same angle since M_1 and M_2 are translations, and are therefore scissor congruent by Lemma 4.3.
- (ii) They are strips of the same width if V_+ and V_- are uniaxial, since by 3.10 (ii)(b)

$$d(R_{-1}, R_{-2}) = d(M_1(R_1), M_2(R_2)) = d(M_2(R_1), M_2(R_2)) = d(R_1, R_2).$$

Thus they are scissor congruent by Lemma 2.25. For either case V_+ and V_- are scissor congruent. \square

PROOF OF THEOREM 1.1. An unbounded convex figure is obviously S -equivalent to the union of a congruent figure with a bounded figure.

Thus this theorem follows from Theorem 4.4. \square

5. Wedges.

LEMMA 5.1. *Two special wedges which are scissor congruent have equal angles.*

We will establish some lemmas needed for the proof of this proposition.

DEFINITION 5.2. A *section*, Z , of a special wedge, E , is a trapezoid contained in E such that:

- (i) the parallel sides of Z , called the base and upper boundary, are perpendicular to the center line of E , with $|\text{base}| < |\text{upper boundary}|$.
- (ii) The nonparallel sides are contained in the boundary of E .

Let E_+ and E_- be scissor congruent special wedges. Using the notation of Definition 2.18, we can state the following.

DEFINITION 5.3. Two sections, z_+ of E_+ and z_- of E_- , are a *pair of sections* if and only if:

- (i) the closed half-plane containing z_+ with boundary containing the base line of z_+ , contains no topological discs of the scissor congruence of E_+ and E_- .
(ii) $S_- = \bigcup_{1 \leq i \leq m} T_i(D_i \cap z_+)$ contains z_- .

LEMMA 5.4. *There exists a pair of sections.*

PROOF. Let z_+ be a section satisfying 5.3(i). Such a section exists since there are only a finite number of topological discs in a scissor congruence. Let b_+ be the base and b'_+ the upper boundary of z_+ . Intuitively, if the endpoints of $T_i(b_+)$ for each i are closer to the apex of E_- than the endpoints of $T_i(b'_+)$ for each i , then there are certainly sections in S_- . These notions will now be described more rigorously. Let x be the apex of E_- , X its center line. Define $\Pi: E_- \rightarrow X$ to be the perpendicular projection of E_- onto X . Let b be any line segment in E_+ perpendicular to its center line.

For $1 \leq i \leq m$ with $b \cap D_i \neq \emptyset$: Let $c_i(b)$ and $f_i(b)$ be the endpoints of $\Pi(T_i(b) \cap E_-)$ where $d(c_i(b), x) \leq d(f_i(b), x)$. Let

$$c(b) = \min_i d(c_i(b), x),$$

$$f(b) = \max_i d(f_i(b), x).$$

If $f(b_+) < c(b'_+)$ then S_- will contain at least one section. Since E_+ is unbounded we can always find a b'_+ parallel to b_+ satisfying this inequality. Let $z_- \subset S_-$ have base b_- and upper boundary b'_- where $d(b_-, x) = f(b_+)$ and $d(b'_-, x) = c(b'_+)$, with the sides of $z_- \subset \partial E_-$. Thus z_- is a section of E_- . Thus z_+ and z_- satisfying 5.3 (i) and (ii) have been constructed and so are a pair of sections. \square

DEFINITION 5.5. Using the notation of 5.3 and 5.4, a *pair of sections* z_+ and z_- is *allowable* if and only if 5.3 (i) and (ii) are satisfied and $d(b_-, x) = f(b_+)$ and $d(b'_-, x) = c(b'_+)$. Obviously there is an allowable pair of sections, Z_+ and Z_- , with base b_+ and b_- , respectively, and with $z_+ \subset Z_+$ and $z_- \subset Z_-$, having height as large as we wish.

Using the notation we have just developed, we will proceed with the proof of Lemma 5.1.

PROOF OF LEMMA 5.1. Let $2\theta_+$ be the angle of E_+ , $2\theta_-$ that of E_- , with $0 < \theta_+ < \pi/2$; $0 < \theta_- \leq \pi/2$. Let $W = |b_+|$, $W' = |b_-|$. W' is determined by our choice of W and the scissor congruence. Let l_+, l_-, L_+ and L_- be the height of the sections z_+, z_-, Z_+ and Z_- , respectively. Then, if we let $L_+ - l_+ = \gamma$ and $l_+ - l_- = \mu$, with $\mu \geq 0$, $L_+ - L_- = \gamma(1 - \cos \theta_-) + \mu$. Let $N = \gamma(1 - \cos \theta_-) + \mu$. Since $\text{area } Z_+ \geq \text{area } Z_-$ we have

$$WL_+ + L_+^2 \tan \theta_+ \geq W'(L_+ - N) + (L_+ - N)^2 \tan \theta_-.$$

Therefore,

$$W'N - N^2 \tan \theta_- \geq L_+^2(\tan \theta_- - \tan \theta_+) + N'L_+$$

where $N' = W' - 2N \tan \theta_- - W$. Completing the square and taking square roots we obtain

$$\left\{ W'N - N^2 \tan \theta_- + \frac{N'^2}{4(\tan \theta_- - \tan \theta_+)} \right\}^{\frac{1}{2}} - \frac{N'}{2(\tan \theta_- - \tan \theta_+)^{\frac{1}{2}}} \geq L_+(\tan \theta_- - \tan \theta_+)^{\frac{1}{2}}.$$

This contradicts the unboundedness of L_+ unless $\tan \theta_- = \tan \theta_+$, i.e. $\theta_- = \theta_+$. \square

LEMMA 5.6. *If two wedges are scissor congruent their angles are equal. (The converse of Lemma 4.3.)*

PROOF. Let W_+ and W_- be scissor congruent wedges with associated triples (D_+, E_+, P_+) and (D_-, E_-, P_-) , respectively. Then, without loss of generality, we can make the additional requirement that z_+ and z_- lie in $E_+ \cap P_+$ and $E_- \cap P_-$, respectively, in the proof of Lemma 5.1. Then the proof of Lemma 5.1 is also valid for wedges. \square

6. The boundaries K_+ and K_- . In this and the following two sections, the assumption that E_+ is scissor congruent to E_- is implicit. We choose a particular scissor congruence and use the notation of Definition 2.18.

The definition of scissor congruence in [5] only allows the intersection of 2.18 (iii) to be finite, rather than bounded. The first six lemmas proved in [5] are based on properties of topological discs that are also properties of topological half-planes. They will be labeled Lemma 1 [5], etc. to show that they are taken from [5]. The lemmas that do not use assumption 2.18 (iii) will be stated without proof.

LEMMA 1 [5]. *For all i , $(J_i - \bigcup_{j \neq i} J_j) \subset K$.*

DEFINITION [5].

$$V = \bigcup_{i \neq j \neq k; i \neq k} (J_i \cap J_j \cap J_k) \cup \bigcup_{i \neq j} (J_i \cap J_j \cap K).$$

Using (ii) and (iii) of 2.18, it can be easily seen that V is a bounded set.

DEFINITION [5]. For $i > 0$, $T_{-i} = T_i^{-1}$. Let us introduce two transformations T and R defined for certain ordered couples (x, i) . For $x \in J_i$, let $T(x, i) = (T_i x, -i)$. For $x \in J_i \cap J_j - V$, with $i \neq j$, let $R(x, i) = (x, j)$. Since $x \notin V$, j is unique, so R is well defined.

For any function f , let $\mathcal{D}(f)$ denote its domain and $R(f)$ its range. Let 1_X denote the identity map of a set X onto itself.

LEMMA 2 [5]. *Let A be an arc, $A \subset (J_i - V)$. Then $A \subset K$ or $A \subset J_j$ for some $j \neq i$.*

- LEMMA 3 [5]. (a) $\mathcal{D}(R) = \{(x, i): x \in J_i \text{ and } x \notin K \cup V\}$.
 (b) $\mathcal{R}(R) = \mathcal{D}(R)$, and $R^2 = 1_{\mathcal{D}(R)}$.
 (c) $\mathcal{R}(T) = \mathcal{D}(T)$, and $T^2 = 1_{\mathcal{D}(T)}$.
 (d) $\mathcal{R}((TR)^k T) = \mathcal{D}((TR)^k T)$, and $((TR)^k T)^2 = 1_{\mathcal{D}((TR)^k T)}$, for each $k \geq 0$.

LEMMA 4 [5]. Assume (a) $x, y \in K$, (b) $(x, i) \in \mathcal{D}((TR)^k T)$, and (c) $(y, j) \in \mathcal{D}((TR)^h T)$.

Then $(TR)^k T(x, i) = (TR)^h T(y, j)$ only if $k = h$ and $(x, i) = (y, j)$.

DEFINITION [5]. $W = \{x: x \in K \text{ and there exist } i, k, v, i' \text{ such that } v \in V, (x, i) \in \mathcal{D}((TR)^k T) \text{ and } (TR)^k T(x, i) = (v, i')\}$.

LEMMA 5 [5]. W is bounded.

PROOF. For every $x \in W$, the set of all (v, i') such that for some (i, k) (i, k, v, i') satisfies the definition is nonempty. Distinct x 's correspond to disjoint sets by Lemma 4 [5]. $V \cap W$ is bounded since V is bounded. $(W - V) \cap \bigcup_{i > m} J_i$ is bounded since each such J_i is bounded. Suppose $x \in (W - V) \cap \bigcup_{i \leq m} J_i$. It follows from 2.18 (iii) that for all but a bounded set of such x 's, $x \in J_i \cap W$ for exactly one i . Therefore, for each such x there is exactly one pair (x, i) and hence exactly one (i, k, v, i') satisfying the definition of W . (If there were another, say (i, h, v', j') , with $k < h$, then $v \in \mathcal{D}(R)$ which is impossible.) But for all but a bounded set of x 's, if $x \in J_i \cap W$, there is a $k = k'$ such that $(x, i) \in \mathcal{D}((TR)^{k'} T)$. Thus, except for a bounded set of x 's, there are only a finite number of rigid motions mapping the (x, i) pairs onto the (v, i') pairs. Since V is bounded and there are only a finite number of values of i and i' , W must be bounded. \square

LEMMA 6 [5]. $K - (V \cup W)$ is a union of disjoint open arcs, each included in some J_i , and this decomposition is unique. Moreover, this decomposition contains a finite number of unbounded arcs.

PROOF. The proof of the first part of this lemma follows that of Lemma 6 in [5]. For the last part note that $K \subset \bigcup_i J_i$, so $K - (V \cup W) = \bigcup_i (K \cap (J_i - (V \cup W)))$. $V \cup W$ is bounded, so $J_i - (V \cup W)$ is uniquely expressible as a union of disjoint open arcs. If D_i is a topological half-plane, this union contains at most two unbounded open arcs of J_i and thus only a finite number of unbounded arcs. \square

DEFINITION [5]. $p(x) = x'$ if $x, x' \in K - (V \cup W)$ and there exist i, i' and k such that $(x, i) \in \mathcal{D}((TR)^k T)$ and $(TR)^k T(x, i) = (x', i')$.

It is shown in [5] that p is well defined and $p^2 = 1_{\mathcal{D}(p)}$.

DEFINITION [5]. Let A be the set of open arcs described in Lemma 6 [5].

It is proved in Lemma 7 [5] that, for bounded figures, the domain of p is all of $K - (V \cup W)$. This is not necessarily true for unbounded figures even if we

restrict V and W , and thus A , to be finite sets.

Intuitively, some arcs of A are mapped onto the boundary, K , by $(TR)^r T$ for some integer r . The other arcs of A are mapped into the interior of E_+ or E_- by T and their images under T are mapped out of the interior by no $(TR)^r$. The former belong to the domain of p , and the latter do not. For C a bounded open arc of A with $C \not\subset \mathcal{D}(p)$, there exists an i such that $(TR)^r T(C, i_r) \subset J_i$ for infinitely many values of r , say r_1, r_2, \dots , and $(TR)^{r_s} T(C, i_{r_s}) \cap (TR)^{r_j} T(C, i_{r_j}) = \emptyset$, with $s \neq j$. I.e. J_i contains infinitely many disjoint open subarcs each of which is congruent to C .

The goal now is to show that the unbounded arcs of $K - (V \cup W)$ are in the domain of p . Let us now restrict E_+ and E_- to be unbounded figures. If the set V is empty, then E_+ and E_- are congruent figures and we need proceed no further. So we can assume $V \cup W$ has at least one element which belongs to K_+ , and consequently another element which belongs to K_- . Therefore, there are precisely four arcs $A^1, A^2, A^{-1}, A^{-2} \in A$ which are unbounded.

DEFINITION 6.1. Let $\mathcal{U} = \{A^1, A^2, A^{-1}, A^{-2}\}$.

LEMMA 6.2. If $A \in \mathcal{U}$ then $A \subset \mathcal{D}(p)$. Moreover, there exist i_0, i_1, \dots, i_k such that for $0 \leq r \leq k$: (a) $-i_r \neq i_{r+1}$; (b) $A \subset J_{i_0}$; and, letting A_r denote $T_{i_r} \cdots T_{i_0} A$, (c) $A_r \subset J_{-i_r} \cap J_{i_{r+1}}$; (d) $p(A) = A_k \subset J_{-i_k} \cap (K - (V \cup W))$.

PROOF. Fix $A \in \mathcal{U}$. Then A is a subarc of some unique J_i , say $i = i_0$. Let (A, i) be the set of all (x, i) for which $x \in A$. Then $(A, i_0) \subset \mathcal{D}(T)$ and $T(A, i_0) = (A_0, -i_0)$, where $A_0 = T_{i_0} A$ is congruent to A . It is shown in Lemma 7 [5] that: if $(r, A_r, -i_r)$ satisfies

- (i) $(A, i_0) \subset \mathcal{D}((TR)^r T)$ and $(TR)^r T(A, i_0) = (A_r, -i_r)$, then either
- (a) $A_r \subset J_{-i_r} \cap (K - (V \cup W))$ or
- (b) $(r+1, A_{r+1}, -i_{r+1})$ satisfies (i).

The proof of this given in [5] applies for our unbounded case. As in Lemma 7 [5], our lemma will be proved if we show that (i) cannot hold for all r . For then $k+1$ can be taken as the first r for which (i) does not hold. So assume (i) holds for all r . Then $A_r \subset J_{-i_r}$, and A_r is congruent to A , and hence an unbounded arc, for all $r \geq 0$. Therefore we have a sequence $(A_0, -i_0), (A_1, -i_1), \dots$, where the terms of the sequence are disjoint by Lemma 4 [5]. Then there is a strictly increasing sequence r_1, r_2, \dots with $-i_{r_j} = i$ for some i and all j , and $A_{r_j} \subset J_i$ for $j = 1, 2, \dots$ and the A_{r_j} 's are disjoint and congruent. This is impossible since D_i is a topological half-plane so any three unbounded subarcs of J_i cannot be disjoint. \square

COROLLARY 6.2-1. For every $A^i \in \mathcal{U}$, $p(A^i) \in \mathcal{U}$.

PROOF. See Lemma 8 [5]. \square

Lemmas 9 [5]–13 [5] and the necessary definitions are valid for the unbounded arcs of \mathcal{U} .

DEFINITION [5]. K_{+-} is the set-theoretic union of all $A \in \mathcal{D}(p)$ such that $A \subset K_+$ and $p(A) \subset K_-$; similarly for K_{-+} , K_{++} and K_{--} .

For the remainder of this paper let us restrict E_+ and E_- so that *the arcs in \mathcal{U} are convex*.

THEOREM 6.3. *If two eventually convex figures are scissor congruent their boundaries are eventually scissor congruent.*

PROOF. Suppose $A^1 \in \mathcal{U} \cap K_{++}$.

Case (a). A^1 is convex but not a ray. Let $E^k = T_{i_k} \cdots T_{i_0} E_+$, where $T_{i_k} \cdots T_{i_0}$ is the rigid motion described in Lemma 6.2. By Lemma 12 [5], E^k and E_+ are on opposite sides of $p(A^1)$, but $p(A^1)$ is part of the boundary of E^k which is congruent to E_+ . This is impossible if A^1 is convex and not a ray.

Case (b). A^1 is a ray and $A^{-1}, A^{-2} \in \mathcal{U} \cap K_-$ and are both convex but not rays. Then by Case (a) $A^{-1}, A^{-2} \in \mathcal{U} \cap K_{-+}$. But $p(A^{-1}) \neq p(A^{-2})$ since p is 1-1. Impossible unless one of them, say A^{-1} , is a ray. Therefore A^1 is congruent to A^{-1} . \square

7. Invariance of angle under scissor congruence. In this section and the next we will use the following notation.

DEFINITION 7.1. E_+ and E_- are properly situated. For $j = 1, 2$:

(i) For each $A^j \in \mathcal{U}$, let $M^j = T_{i_k} \cdots T_{i_0}$ described in Lemma 6.2. Let us index the A^j 's so that $M^j(A^j) = p(A^j) = A^{-j}$.

(ii) Let $a_j = (x_j, g_+(x_j))$ be the endpoint of A^j , where K_+ is the graph of g_+ ; and let $a_{-j} = (x_{-j}, g_-(x_{-j}))$ be the endpoint of A^{-j} , where K_- is the graph of g_- .

We can assume that $x_1 < x_2$ and $x_{-1} < x_{-2}$. Let l_1 (l_2) be the ray with endpoint a_1 (a_2) which is tangent to A^1 (A^2) at a_1 (a_2) from the left (right), with $l_1 \cup l_2 \cup \overline{a_1 a_2}$ a wedge containing an unbounded subset of E_+ . Let $l_{-j} = M^j(l_j)$.

THEOREM 7.2. *The rigid motions M^1 and M^2 are translations.*

PROOF. Let B_j be the topological half-plane bounded by l_j and A^j , $j = 1, 2, -1, -2$. (If A^j is a ray then let $B_j = \emptyset$.)

Then we have that $B_{-j} = M^j(B_j)$, $j = 1, 2$. Therefore we have that $E_+ \cup B_1 \cup B_2$ is scissor congruent to $E_- \cup B_{-1} \cup B_{-2}$ since

$$\begin{array}{ccccc} E_+ & \cup & B_1 & \cup & B_2 \\ \downarrow & & \downarrow & & \downarrow \\ & \text{by} & & & \\ & \text{hypothesis} & M^1 & & M^2 \\ \downarrow & & \downarrow & & \downarrow \\ E_- & \cup & B_{-1} & \cup & B_{-2} \end{array}$$

By Lemma 5.6, the two wedges $E_+ \cup B_1 \cup B_2$ and $E_- \cup B_{-1} \cup B_{-2}$ have equal angles, i.e. the angle, θ , between l_1 and l_2 is the same as the angle between l_{-1} and l_{-2} . Since E_+ and E_- are properly situated and $A^j \cong A^{-j}$, l_j is parallel to l_{-j} , $j = 1, 2$. Therefore M^1 and M^2 are translations. \square

COROLLARY 7.2-1. *Two eventually convex scissor congruent figures have the same angle.*

PROOF. Follows immediately from Lemma 7.2. \square

LEMMA 7.3. *If an arc, A , is congruent to a proper subarc of itself by a plane rigid motion, the rigid motion is the product of a translation and reflections. If A is convex, then it is a ray.*

PROOF. Let M be a plane rigid motion such that $M(A) \cong$ proper subarc of A . Let p_0 be a point of A such that $p_0 \neq M(p_0)$. In general, let $p_i = M(p_{i-1})$, $i = 1, 2, \dots$. Let P be the polygonal arc $\overline{p_0 p_1} + \overline{p_1 p_2} + \dots$. If A is convex it must lie on one side of P . If we show that P is a ray, then a convex arc, A , must be a ray since it intersects P infinitely often. Let θ_i = exterior angle at p_i , measured from $\overline{p_{i-1} p_i}$ (extended) to $\overline{p_i p_{i+1}}$. $0 \leq \theta_1 \leq \pi$, and assume positive θ_1 is measured clockwise.

If $\theta_1 = 0$, then our arc, P , is an infinite ray, and $M = TR$, where T is a translation and R is a reflection such that $R(P) = P$. Therefore assume $\theta_1 > 0$. See Figure 1.

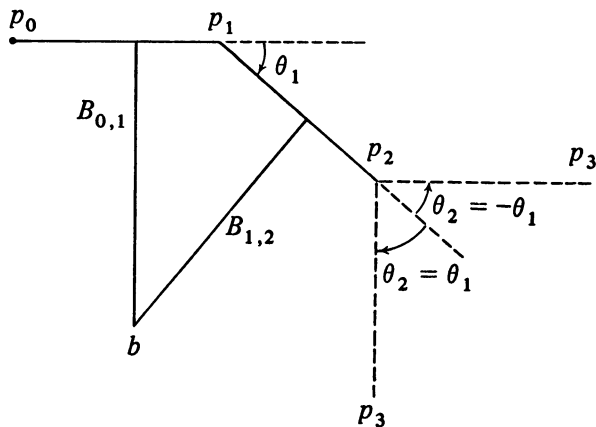


FIGURE 1

Since $M(\overline{p_0 p_1}) = \overline{p_1 p_2}$, M is determined up to a reflection of the plane about $\overline{p_1 p_2}$. (I.e. $M(\text{plane}) = \text{plane}$, a plane is determined by three noncollinear points. Thus p_1, p_2 and p_3 determine $M(\text{plane})$.) Therefore $\theta_2 = -\theta_1$ or else θ_1 .

Case (a). If $\theta_2 = \theta_1$: Let $B_{i,i+1}$ = perpendicular bisector of $\overline{p_i p_{i+1}}$, $i = 0, \dots, M(B_{i,i+1}) = B_{i+1,i+2}$. Let $b = B_{0,1} \cap B_{1,2} \neq \emptyset$ if $\overline{p_0 p_1}$ and $\overline{p_1 p_2}$ are not collinear (i.e. if P is not an infinite ray).

Therefore $b = B_{0,1} \cap M(B_{0,1})$, $M(b) = M(B_{0,1}) \cap M(B_{1,2})$ and b is equidistant from p_0, p_1 and p_2 . b is a fixed point. Thus M reduces to a rotation about the point b through an angle θ_1 . In general, for $i \geq 0$, p_i, p_{i+1} are equidistant from b , length $\overline{p_i p_{i+1}} = \text{length } \overline{p_{i+1} p_{i+2}}$, and $\theta_i = \theta_{i+1} = \theta_1$. For $\theta_1 > 0$, there exists a least n such that $n\theta_1 \geq 2\pi$.

(i) If $n\theta_1 = 2\pi$, then P is the boundary of a regular n -gon, which is not congruent to a proper subarc of itself.

(ii) If $n\theta_1 > 2\pi$, then since length $\overline{b p_0} = \text{length } \overline{b p_1}$, $\overline{p_{n-1} p_n}$ must intersect $\overline{p_0 p_1}$, i.e. A would intersect itself and hence not be an arc.

Therefore $\theta_1 = 0$.

Case (b). If $\theta_2 = -\theta_1 \neq 0$. This is impossible if A is convex. But if A is not convex, then $\theta_3 = -\theta_2 = \theta_1$, etc. Thus $\overline{p_i p_{i+1}}$ and $\overline{p_{i+2} p_{i+3}}$ are parallel line segments, for $i = 0, 1, \dots$, and p_0, p_2, p_4, \dots lie along a ray and p_1, p_3, \dots all lie on the same side of this ray. Thus there is a translation, T , such that $T = M^2$. So if M contains a rotation factor that is nontrivial, it must be a rotation of π radians about some point x . Call the rotation R_x . Let L_1 and L_2 be any two perpendicular lines intersecting at x . Then R_x is equivalent to a reflection about L_1 followed by a reflection about L_2 . Therefore M is the product of a translation and reflections. \square

THEOREM 7.4 (Classification of the mappings M^1 and M^2). If K_+ and K_- do not contain a ray, and there exist convex arcs $A_*^1, A_*^2 \subset K_+$ and $A_*^{-1}, A_*^{-2} \subset K_-$ and rigid motions $M_*^j: A_*^j \cong A_*^{-j}, j = 1, 2$, then either $M_*^1 = M^1$ and $M_*^2 = M^2$ or else $M^1 = M_*^2(M^2)^{-1}M_*^1$.

PROOF. For definiteness, suppose $A_*^1 \subset A^1$. Then $M^1(A_*^1) \subset A^{-1}$ and $M_*^1(A_*^1) \cong M^1(A_*^1)$.

(i) If $A^{-1} \subset A_*^{-2}$ or $A_*^{-1} \subset A^{-1}$ then $M_*^1 = M^1$ since A_*^{-1} is convex and not a ray and hence does not contain a proper subarc to which it is congruent, by Lemma 7.3. Similarly, $M_*^2 = M^2$.

(ii) If $A^{-1} \subset A_*^{-2}$ or $A_*^{-2} \subset A^{-1}$ there is an arc, $A \subset A_*^1$ which is congruent to a subarc of A_*^j and A^j for all $j = 1, 2, -1, -2$, i.e. $M^1(A) \cong M_*^2(M^2)^{-1}M_*^1(A)$. Again, by Lemma 7.3 $M^1 = M_*^2(M^2)^{-1}M_*^1$. \square

8. An invariant of scissor congruent uniaxial figures. In this section we will assume that E_+ is uniaxial. By Corollary 7.2-1, E_- is also uniaxial.

DEFINITION 8.1. For E a uniaxial convex figure, a *curved rectangle* of E is a bounded, closed, convex subset of E whose boundary consists of two parallel line segments which are perpendicular to the axes of E and the subarcs of the boundary of E joining the endpoints of the line segments. The *height* of a curved rectangle is the distance between the parallel line segments. If a curved rectangle

is an ordinary parallelogram choose the distance between the segments perpendicular to the axes.

THEOREM 8.2. *If Y is a line parallel to an axis of E_+ , $M^1(Y) = M^2(Y)$.*

PROOF. Define $I_+ \subset E_+$ to be a strip with $\partial I_+ = \bigcup_{j=1,2} \text{axis}$ with end-point $a_j \cup \overline{a_1 a_2}$.

Let I_- be similarly defined in E_- . Our theorem will be proved if we show that I_+ and I_- have equal widths. Let $|I_+| = \text{width of } I_+$, $|I_-| = \text{width of } I_-$.

Let H be a closed upper half-plane with boundary l such that: (i) $H \cap D_i \neq \emptyset$ for $m < i \leq n$, and (ii) a_1 and a_2 are not in H . Such an H obviously exists.

Let C_+ be a curved rectangle of $H \cap E_+$ with the lower boundary line of C_+ equal to $l \cap E_+$. Let L denote the height of C_+ .

For $i = 1, \dots, m$: Each T_i is a translation. So with each T_i we can associate two real numbers, x_i and y_i , such that for any point (x, y) of the plane, $T_i(x, y) = (x + x_i, y + y_i)$. Let $\epsilon/2 = \max_i y_i - \min_i y_i$.

Then there is a curved rectangle, C_- of E_- contained in $\bigcup_{1 \leq i \leq m} T_i(C_+ \cap D_i)$ of height $L - \epsilon$, provided $L > \epsilon$. In fact, there are such pairs, C_+ and C_- , with height L and $L - \epsilon$ respectively, as large as we wish. Let R_{-1} and R_{-2} be the two curved rectangles of E_- such that $R_{-1} \cup R_{-2} \cup (C_- \cap I_-) = C_-$.

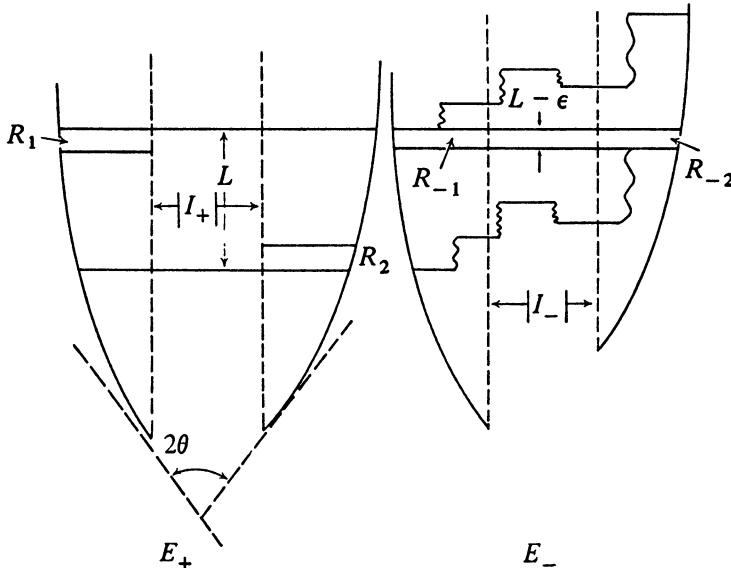


FIGURE 2

Since E_+ and E_- are scissor congruent, their boundaries are eventually scissor congruent by Theorem 6.3. Thus there are two curved rectangles R_1 and R_2 which are contained in C_+ and are congruent to R_{-1} and R_{-2} , respectively.

Let $R_+ = C_+ \cap I_+$.

Let S_1, \dots, S_4 be the four (possibly empty) curved rectangles of $C_+ -$

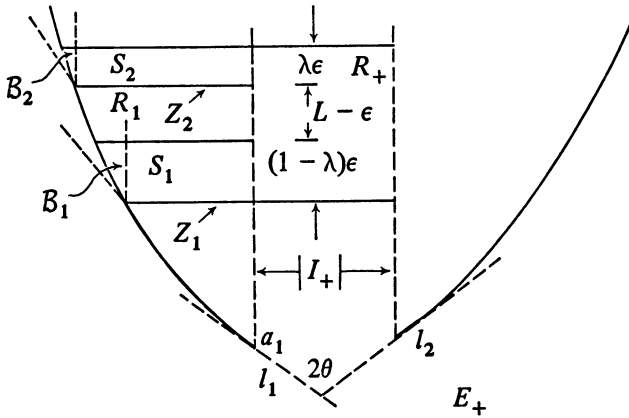


FIGURE 3

$(R_+ \cup R_1 \cup R_2)$. Refer to Figures 2 and 3 where some of the details are shown.

For $j = 1, \dots, 4$: Let Z_j = the length of the lower line segment of ∂S_j , and s_j = the point of intersection of this segment with K_+ . Let K_+ be the graph of g . Let $F(s_j) = \min(|(g'_R(s_j))^{-1}|, |(g'_L(s_j))^{-1}|)$. (See page .) Let $B_j = |\arctan F(s_j)|$. Let 2θ = angle between l_1 and l_2 .

Let $Z = \sum_j Z_j$. We can choose $L \gg \epsilon$, in fact as large as we please. $\text{area } C_+ \geq \text{area } C_-$. Therefore

$$\sum_j \text{area } S_j + \text{area}(R_1 \cup R_2) + L|I_+| > (L - \epsilon)|I_-| + \text{area}(R_{-1} \cup R_{-2}).$$

But $Z_2 \leq L \tan B_2$ and $Z_4 \leq L \tan B_4$, so

$$\begin{aligned} S_1 + S_2 &\leq \lambda \epsilon L \tan B_2 + \frac{1}{2}(\lambda \epsilon)^2 \tan B_2 + \frac{1}{2}(1 - \lambda)^2 \epsilon^2 \tan B_1 + Z_1(1 - \lambda)\epsilon \\ &\leq \epsilon L \tan B_2 + \frac{1}{2}\epsilon^2 \tan B_2 + \epsilon Z_1 + \frac{1}{2}\epsilon^2 \tan \theta \end{aligned}$$

since $B_1 \leq \theta$ because E_+ is eventually convex and $a_1, a_2 \notin \text{int } H$, and $0 \leq \lambda \leq 1$. Similarly, $B_3 \leq \theta$. Therefore,

$$\begin{aligned} \sum_j S_j &\leq \epsilon L(\tan B_2 + \tan B_4) + \epsilon^2(\tan B_2 + \tan B_4) \\ &\quad + \epsilon(Z_1 + Z_2) + \epsilon^2 \tan \theta. \end{aligned}$$

Therefore,

$$\begin{aligned} &\epsilon(\epsilon(\tan B_2 + \tan B_4) + Z_1 + Z_2 + \epsilon \tan \theta + |I_-|) \\ &\geq L(\epsilon(\tan B_2 + \tan B_4) + |I_-| - |I_+|). \end{aligned}$$

But $B_2 \rightarrow 0$ and $B_4 \rightarrow 0$ since E_+ is uniaxial and L is unbounded. Thus $|I_-| \rightarrow |I_+|$. \square

COROLLARY 8.2-1. *Two strips which are scissor congruent have the same width.*

PROOF. Follows immediately from Theorem 8.2. \square

9. Equivalence of scissor congruence and S -equivalence.

THEOREM 9.1. *Two eventually convex figures are scissor congruent if and only if they are S -equivalent.*

PROOF. One way is Theorem 4.4. The other follows from Theorems 6.3, 7.2 and 8.2. \square

THEOREM 9.2. *'Is scissor congruent to' is an equivalence relation on the class of eventually convex figures.*

PROOF. Follows immediately from Theorem 9.1 and Lemma 3.11. \square

We can now prove Theorem 1.2, which we will restate.

THEOREM 1.2. *Two unbounded convex figures, E_+ and E_- , are scissor congruent if and only if the union of E_+ and a finite number of topological discs is scissor congruent to the union of E_- and a finite number of topological discs.*

PROOF. Let F_+ = the union of E_+ and a bounded figure. Let F_- = the union of E_- and a (not necessarily the same) bounded figure. Then E_+ is scissor congruent to F_+ and E_- is scissor congruent to F_- by Theorem 1.1. Thus F_+ and F_- are scissor congruent if and only if E_+ and E_- are since 'is scissor congruent to' is a transitive relation by Theorem 9.2. \square

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