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EQUIVARIANT BORDISM AND SMITH THEORY. IV

BY

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ABSTRACT. This paper analyzes two types of characteristic numbers defined for manifolds with Z_4 action, showing their relation and that neither suffices to detect Z_4 equivariant bordism. This extends work of Bix who had given examples not detected by one type of number.

1. **Introduction.** T. tom Dieck [4] has introduced a notion of equivariant Stiefel-Whitney numbers for actions of a finite group G on closed manifolds and has shown that these numbers determine equivariant G bordism for $G = (Z_2)^k$. Recently, Bix [1] has shown that these numbers do not determine the bordism class for $G = Z_4$.

In [2], a notion of equivariant characteristic numbers for Z_2 actions was introduced. In this paper that notion will be extended, in the obvious way, to G actions. It will then be shown that these numbers determine tom Dieck's numbers. For Z_4 they give more information than tom Dieck's numbers, but also do not detect bordism.

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2. **Characteristic numbers.** Let G be a finite group; let $R, V_1, \ldots, V_m$ be the distinct irreducible real representations of G, with R the trivial representation. Form $V = R^\infty \oplus V_1^\infty \oplus \cdots \oplus V_m^\infty$, the direct sum of a countable number of copies of each representation and let BO_n be the Grassmannian of n-planes in V, with the G action ϕ induced by the representation on V. Then (BO_n, ϕ) is a classifying space for n-plane bundles with G action over decent spaces.

Letting $V \longrightarrow R \oplus V$: $v \longrightarrow (0, v)$ and identifying $R \oplus R^\infty$ with R^∞ one assigns to an n-plane α in V the $n + 1$ plane $R \oplus \alpha$ in $R \oplus V \cong V$, which defines an inclusion i: $BO_n \longrightarrow BO_{n+1}$. This is an equivariant map, and if γ_n is the universal n-plane bundle over BO_n, $i^*(\gamma_{n+1}) = \gamma_n \oplus 1_+$, where 1_+ is the trivial line bundle with trivial G action. Let (BO, ϕ) be the limit of the BO_n's with these maps.

If (M, ψ) is a compact n manifold with boundary with G action, there is a

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classifying map τ_M: $(M, \psi) \longrightarrow (BO_n, \phi)$, unique up to equivariant homotopy. If (M, ψ) is a regularly imbedded invariant submanifold of ∂V, with (V, ψ') a G action, then the tangent bundle of V restricts to $\tau_M \oplus 1_+$ on M, which is classified by $i \circ \tau_M$: $(M, \psi) \longrightarrow (BO_{n+1}, \phi)$. Thus one has a well-defined homotopy class of maps τ_M: $(M, \psi) \longrightarrow (BO, \phi)$ classifying the stable tangent bundle of M, and if (M, Q) is regularly imbedded in the boundary of (V, ψ') by $f: M \longrightarrow \partial V$, then $\tau_V \circ f = \tau_M$.

Being given a pair (X, A, ρ) with G action one then defines a natural transformation τ_*: $\mathfrak{N}_*^G(X, A, \rho) \longrightarrow \mathfrak{N}_*^G(X \times BO, A \times BO, \rho \times \phi)$ by sending the class of f: $(M, \partial M, \psi) \longrightarrow (X, A, \rho)$ to the class of $f \times \tau_M$: $(M, \partial M, \psi) \longrightarrow (X \times BO, A \times BO, \rho \times \phi)$.

For any pair (X, A, ρ), one has a natural transformation

$$\mu: \mathfrak{N}_*^G(X, A, \rho) \longrightarrow H_*^G(X, A, \rho; Z_2)$$

assigning to f: $(M, \partial M, \psi) \longrightarrow (X, A, \rho)$ the image $f_*([M, \partial M, \psi])$ of the fundamental Smith homology class (see [3, §2]).

The composite

$$\mu \circ \tau_*: \mathfrak{N}_*^G(X, A, \rho) \longrightarrow H_*^G(X \times BO, A \times BO, \rho \times \phi; Z_2)$$

gives rise to equivariant characteristic numbers; i.e. every element of the dual Smith cohomology group $H_G^*(X \times BO, A \times BO, \rho \times \phi; Z_2)$ gives a characteristic number for the G bordism of (X, A, ρ). Restricting to X a point and A empty gives $\mu \circ \tau_*: \mathfrak{N}_*^G \longrightarrow H_*^G(BO, \phi; Z_2)$.

3. **Tom Dieck's numbers.** To describe tom Dieck's characteristic numbers, one may follow Bix's approach. Let π: $EG \longrightarrow BG$ be a universal principal G bundle and let (M, ψ) be a G manifold with $(\tau(M), \psi_*)$ its tangent bundle with G action induced by the differential. Then $EG \times_G \tau(M) \longrightarrow EG \times_G M$ is a vector bundle classified by a map $EG \times_G M \xrightarrow{\alpha} BO$, inducing a homomorphism

$$\alpha^*: H^*(BO; Z_2) \longrightarrow H^*(EG \times_G M; Z_2).$$

Integration along the fibers defines a homomorphism

$$\natural: H^*(EG \times_G M; Z_2) \longrightarrow H^*(BG; Z_2)$$

of degree $-(\dim M)$. Assigning to (M, ψ) the homomorphism $\natural \circ \alpha^*$ defines a homomorphism

$$\chi: \mathfrak{N}_*^G \longrightarrow \mathrm{Hom}(H^*(BO; Z_2), H^*(BG; Z_2))$$

which gives characteristic numbers.

To relate this to Smith homology, consider a space with G action (X, ϕ). Let $C(X)$ denote the mod 2 chains of X and $C^\circ(X)$ the subcomplex of chains invariant under G. Then $H^G_*(X; Z_2)$ is the homology of the complex $C^\circ(X)$. In particular, if G acts freely on X, $C^\circ(X)$ is isomorphic to $C(X/G)$ and $H^G_*(X; Z_2) \cong H_*(X/G; Z_2)$.

Now consider a class $a \in H^G_*(X; Z_2)$ and a class $b \in H_*(BG; Z_2)$. Representative cycles a' and b' may be chosen in $C^\circ(X)$ and $C^\circ(EG)$, and their product $a' \otimes b'$ is a cycle in $C^\circ(X \times EG)$, giving a class in $H_*(X \times_G EG; Z_2)$. Thus one has a product

$$H^G_*(X; Z_2) \otimes H_*(BG; Z_2) \longrightarrow H_*(X \times_G EG; Z_2)$$

or a homomorphism

$$H^G_*(X; Z_2) \longrightarrow \mathrm{Hom}(H_*(BG; Z_2), H_*(X \times_G EG; Z_2))$$

and applying duality of homology and cohomology gives

$$\bigcap\colon H^G_*(X; Z_2) \longrightarrow \mathrm{Hom}(H^*(X \times_G EG; Z_2), H^*(BG; Z_2)).$$

Notice that if X is given by a G manifold (M, ψ), that $\bigcap([M])$ is just $\natural$, i.e. integration along the fibers is obtained in this way from the fundamental class of M.

Now letting (BO, ϕ) be the universal action with universal G bundle (γ, ϕ_*), $EG \times_G \gamma \longrightarrow EG \times_G BO$ is a vector bundle, classified by a map $c\colon EG \times_G BO \longrightarrow BO$. Assigning to $x \in H^G_*(BO, \phi; Z_2)$ the composite $\bigcap(x) \circ c^*$ defines a homomorphism

$$\gamma^*\colon H^G_*(BO, \phi; Z_2) \longrightarrow \mathrm{Hom}(H^*(BO; Z_2), H^*(BG; Z_2)).$$

PROPOSITION 3.1. *The homomorphism χ is the composite of*

$$\mu \circ \tau_*\colon \mathfrak{N}^G_* \longrightarrow H^G_*(BO, \phi; Z_2)$$

and

$$\gamma^*\colon H^G_*(BO, \phi; Z_2) \longrightarrow \mathrm{Hom}(H^*(BO; Z_2), H^*(BG; Z_2)).$$

PROOF. Given (M, ψ) with $\tau\colon (M, \psi) \longrightarrow (BO, \phi)$ classifying the tangent bundle, the composite

$$EG \times_G M \xrightarrow{1 \times \tau} EG \times_G BO \xrightarrow{c} BO$$

is the map α. Then $\natural \circ \alpha^* = \bigcap([M]) \circ \alpha^* = \bigcap([M]) \circ (1 \times \tau)^* \circ c^* = \bigcap(\tau_*[M]) \circ c^* = \bigcap(\mu \circ \tau_*(M, \psi)) \circ c^* = \gamma^*(\mu \circ \tau_*(M, \psi))$, using the obvious naturality property of $\bigcap$.*

4. **Actions of Z_4.** First one needs to know the structure of $H^{Z_4}_*(BO, \phi; Z_2)$. As noted in [3, Lemma 2.1], if $T \subset G$ is a central subgroup of order 2, then

$$H_*^G(X, A, \rho; Z_2) \cong H_*^G(X, F_T \cup A, \rho; Z_2) \oplus H_*^G(F_T, F_T \cap A, \rho; Z_2)$$

where F_T is the fixed set of T, and

$$H_*^G(X, F_T \cup A, \rho; Z_2) \cong H_*^{G/T}(X/T, (F_T \cup A)/T, \rho'; Z_2)$$
$$H_*^G(F_T, F_T \cap A, \rho; Z_2) \cong H_*^{G/T}(F_T, F_T \cap A, \rho'; Z_2)$$

where ρ' is induced by ρ. In particular, for $G = Z_4$, one may take $T = Z_2$, and one needs to know about the fixed set of Z_2 on BO.

Now the irreducible representations of Z_4 are R, R_-, and C, where R_- is the reals with Z_4 acting as multiplication by -1 and C is the complex numbers with Z_4 acting as multiplication by i. The fixed set of Z_2 on BO_n is then $\bigcup_k BO_{n-k}(R^\infty \oplus R_-^\infty) \times BO_k(C^\infty)$ and taking the limit, the fixed set F of Z_2 on BO is $\bigcup BO(R^\infty \oplus R_-^\infty) \times BO_k(C^\infty)$. The induced Z_2 action on F preserves these components and in particular, $BO(R^\infty \oplus R_-^\infty) \times BO_0(C^\infty) = BO$ is the universal space for Z_2 bundles.

The pair (BO, F) is relatively a free action, hence may be crossed with EZ_4 giving an isomorphism

$$\begin{aligned} H_*^{Z_4}(BO, F, \phi; Z_2) &\cong H_*^{Z_4}(BO \times EZ_4, F \times EZ_4, \phi \times u; Z_2) \\ &\cong H_*^{Z_2}((BO \times EZ_4)/Z_2, F \times (EZ_4/Z_2), (\phi \times u)'; Z_2) \\ &\cong H_*((BO \times EZ_4)/Z_4, (F \times EZ_4/Z_2)/Z_2; Z_2). \end{aligned}$$

Now $BO \times EZ_4/Z_4$ maps into $BZ_4 = EZ_4/Z_4$ by projection with fiber BO, and maps by c (of §3) into BO. Looking at the fixed component $BO \times BO_0$ of F, the map $(BO \times BO_0 \times (EZ_4/Z_2))/Z_2 \longrightarrow BO \times EZ_4/Z_4$ is a homotopy equivalence, both being compatibly homotopy equivalent to $BO \times BZ_4$. Thus $BO \times BO_0 \times EZ_4 \longrightarrow BO \times EZ_4$ is an equivariant homotopy equivalence. Thus the exact sequence for the pair $(BO \times EZ_4, F \times EZ_4)$ decomposes, and the boundary homomorphism

$$\partial\colon H_*^{Z_4}(BO, F, \phi; Z_2)$$
$$\longrightarrow \bigoplus_{k \neq 0} H_{*-1}^{Z_2}(BO(R^\infty \oplus R_-^\infty) \times BO_k(C^\infty) \times (EZ_4/Z_2), \phi' \times u'; Z_2)$$

is an isomorphism.

Now one analyzes $\mathfrak{N}_*^{Z_4}$. One has an exact sequence

$$\mathfrak{N}_*^{Z_4} \xrightarrow{i} \mathfrak{N}_*^{Z_4}(\text{All, Free}) \xrightarrow{\partial} \mathfrak{N}_*^{Z_4}(\text{Free}) \to \mathfrak{N}_*^{Z_4}$$

and a free action (M, ψ) bounds the mapping cylinder of $M \longrightarrow M/Z_2$. Now $\mathfrak{N}_*^{Z_4}$ (All, Free) may be computed by taking the fixed set of Z_2. The fixed set of (M^n, ψ) is a union of closed submanifolds F^{n-k} imbedded in the interior of M with induced $Z_2 = Z_4/Z_2$ action. The normal bundle of F^{n-k} is a bundle with Z_4 action so that Z_2 acts as -1 in the fibers; i.e. ν is classified by an equivariant map $\nu: F^{n-k} \longrightarrow BO_k(C^\infty)$. Thus

$$\mathfrak{N}_*^{Z_4}(\text{All, Free}) \cong \bigoplus_k \mathfrak{N}_{*-k}^{Z_2}(BO_k(C^\infty)).$$

The mapping cylinder splitting and ∂ give an isomorphism of $\mathfrak{N}_{*-1}^{Z_4}$ (Free) with the summand $\mathfrak{N}_{*-1}^{Z_2}(BO_1(C^\infty))$.

From the splitting, one obtains an isomorphism

$$P: \bigoplus_{k \neq 1} \mathfrak{N}_{*-k}^{Z_2}(BO_k(C^\infty)) \xrightarrow{\cong} \mathfrak{N}_*^{Z_4}$$

assigning to $\nu: F^{n-k} \longrightarrow BO_k(C^\infty)$ the induced Z_4 action on $D(\nu)/(x \sim -x | x \in S(\nu))$, the real projective space bundle of $\nu \oplus 1$. On the summand $\mathfrak{N}_*^{Z_2}(BO_0(C^\infty)) \cong \mathfrak{N}_*^{Z_2}P$ assigns to the involution (M^n, t) the induced Z_4 action (M, ϕ_t) with $Z_2 \subset Z_4$ acting trivially ($D(\nu) = M$ and $S(\nu)$ is empty).

First considering (M^n, t) in $\mathfrak{N}_*^{Z_2}(BO_0(C^\infty))$. $P(M^n, t) = (M, \phi_t)$ has all simplices fixed by Z_2 and hence $\mu \circ \tau_* \circ P(M^n, t)$ lies in the summand $H_*^{Z_4}(F_t, \phi; Z_2)$ of $H_*^{Z_4}(BO, \phi; Z_2)$. The classifying map $\tau_M: (M, \phi_t) \longrightarrow (BO, \phi)$ maps into the fixed set of Z_2, and may be obtained by composing the classifying map for the Z_2 tangent bundle $\tau_M: (M^n, t) \longrightarrow (BO, \phi')$ with the inclusion i of (BO, ϕ') as $BO(R^\infty \oplus R_-^\infty) \times BO_0(C^\infty)$ in F, and then with the inclusion j of F in BO. It is then immediate that the diagram

$$\begin{array}{ccc}
\mathfrak{N}_*^{Z_2} & \xrightarrow{\mu \circ \tau_*} & H_*^{Z_2}(BO, \phi'; Z_2) \\
\wr\| & & \downarrow i_* \\
\mathfrak{N}_*^{Z_2}(BO_0(C^\infty)) & & \bigoplus H_*^{Z_2}(BO \times BO_k, \phi'; Z_2) \\
 & & \wr\| \\
\downarrow P & & H_*^{Z_2}(F, \phi'; Z_2) \\
 & & \wr\| \\
 & & H_*^{Z_4}(F, \phi; Z_2) \\
 & & \downarrow j^* \\
\mathfrak{N}_*^{Z_4} & \xrightarrow{\mu \circ \tau_*} & H_*^{Z_4}(BO, \phi; Z_2)
\end{array}$$

commutes. Since $\mu \circ \tau_*$ is monic for Z_2 by [2, Proposition 3.1] and i_*, j_* are monic, it follows that $\mu \circ \tau_* \circ P$ is monic on $\mathfrak{N}_*^{Z_2}(BO_0(C^\infty))$.

For any class in the remaining summands, $\mathfrak{N}_{*-k}^{Z_2}(BO_k(C^\infty))$, $k \neq 0, 1$, the action $P(\alpha)$ has no top dimensional simplices fixed under the action of Z_2. Thus $\mu \circ \tau_* \circ P$ sends $A = \bigoplus_{k \neq 0,1} \mathfrak{N}_{*-k}^{Z_2}(BO_k(C^\infty))$ into the summand

$H_*^{Z_4}(BO, F, \phi; Z_2)$ of $H_*^{Z_4}(BO, \phi; Z_2)$, which in turn maps isomorphically by ∂ to

$$\bigoplus_{k\neq 0} H_{*-k}^{Z_2}(BO \times BO_k(C^\infty) \times EZ_4/Z_2, \phi' \times u'; Z_2).$$

Now considering ν: $F^{n-k} \longrightarrow BO_k(C^\infty)$, the fixed set of Z_2 in $D(\nu)/(x^\sim - x \mid x \in S(\nu))$ consists of F^{n-k} and $S(\nu)/Z_2 = RP(\nu)$. The image of this class under $\partial \circ \mu \circ \tau_* \circ P$ is the sum of the classes:

(a$'$) the image of the fundamental class of $RP(\nu)$ in

$$H_{n-1}^{Z_2}(BO \times BO_k(C^\infty) \times (EZ_4/Z_2); \phi' \times \mu'; Z_2)$$

obtained by

$$\tau \circ \pi \times \nu \circ \pi \times c\colon RP(\nu) \longrightarrow BO \times BO_k \times (EZ_4)/Z_2$$

classifying the pull back of the tangent bundle, the pull back of ν, and the double cover of $RP(\nu)$ by $S(\nu)$.

(b$'$) The image of the fundamental class of $RP(\lambda)$ in

$$H_{n-1}^{Z_2}(BO \times BO_1(C^\infty) \times (EZ_4/Z_2); \phi' \times \mu'; Z_2)$$

obtained by $\tau \circ \pi \times \nu \circ \pi \times c$: $RP(\lambda) \longrightarrow BO \times BO_1 \times (EZ_4/Z_2)$ where π is projection on the fixed component $RP(\nu)$ with normal bundle λ.

To see this, τ: $M^n \longrightarrow BO$ sends a fixed component F^{n-k} with normal bundle ν^k into $BO \times BO_k(C^\infty)$ classifying $\tau \oplus \nu$. On a tubular neighborhood $D(\nu)$, τ is homotopic to $\tau/F \circ \pi$, i.e. τ is the pull back of $\tau \oplus \nu$ and similarly on the boundary $S(\nu)$. Sending $M - F$ into EZ_4 to classify $M - F \longrightarrow (M - F)/Z_4$ and dividing out Z_2 gives $RP(\nu) \longrightarrow BO \times BO_k(C^\infty) \times EZ_4/Z_2$ classifying $\tau \circ \pi$, $\nu \circ \pi$, and the cover by $S(\nu)$.

Over $BO \times BO_k(C^\infty) \times EZ_4/Z_2$ one has a stable bundle γ with involution, a k plane bundle ρ with Z_4 action covering the involution on the base and a double cover or line bundle λ with similar Z_4 action. Then $\rho \otimes \lambda$ is a k plane bundle with involution and

$$\alpha\colon BO \times BO_k(C^\infty) \times EZ_4/Z_2 \longrightarrow BO \times BO_k(C^\infty) \times EZ_4/Z_2$$

where $\pi_1 \circ \alpha$ classifies $\gamma \oplus (\rho \otimes \lambda)$, $\pi_2 \circ \alpha = \pi_2$, $\pi_3 \circ \alpha = \pi_3$ is an equivariant homotopy equivalance, with inverse obtained by subtracting $\rho \otimes \lambda$.

Applying α_* sends the class (a$'$) into the class (a) the image of the fundamental class of $RP(\nu)$ in $H_{n-1}^{Z_2}(BO \times BO_k(C^\infty) \times (EZ_4/Z_2), \phi' \times \mu'; Z_2)$ obtained by $\tau \times \nu \circ \pi \times c$: $RP(\nu) \longrightarrow BO \times BO_k \times (EZ_4/Z_2)$ classifying the tangent bundle of $RP(\nu)$, the pull back of ν and the double cover of $RP(\nu)$ by $S(\nu)$.

On the fixed component $RP(\nu)$, the normal bundle λ is a line bundle, so π: $RP(\lambda) \longrightarrow RP(\nu)$ is a diffeomorphism. The map into $BO_1(C^\infty) = EZ_4/Z_2$ classifying

$\nu \circ \pi$ is the same as the map c classifying the double cover. Tensoring this bundle with itself gives a trivial bundle with trivial action (if e is a unit vector over x, $e \otimes e$ is independent of the choice of e and is sent by the Z_4 action into the point $e' \otimes e'$ if e goes to e'). Identifying $RP(\lambda)$ with $RP(\nu)$ via π, $\tau \circ \pi = \tau \cong \tau \oplus (\lambda \otimes \lambda)$, so applying α_* sends the class (b′) into the class

(b) the image of the fundamental class of $RP(\nu)$ in

$$H_{n-1}^{Z_2}(BO \times BO_1(C^\infty) \times (EZ_4/Z_2); \phi' \times \mu'; Z_2)$$

obtained by $\tau \times \hat{\nu} \times c$: $RP(\nu) \longrightarrow BO \times BO_1 \times (EZ_4/Z_2)$ classifying τ, the normal bundle of $RP(\nu)$ in $D(\nu)/Z_2$ on $S(\nu)$, and the double cover by $S(\nu)$. *Note*: The last two maps are the same.

Note: The class given by (b) may be obtained from that in (a) by applying the map

$$BO \times BO_k \times (EZ_4/Z_2) \xrightarrow{\pi} BO \times (EZ_4/Z_2)$$
$$\xrightarrow{\iota \times \Delta} BO \times (EZ_4/Z_2) \times (EZ_4/Z_2).$$

Further, the class (a) may be obtained from the composite of

$$Q: \mathfrak{N}_{*-k}^{Z_2}(BO_k(C^\infty)) \longrightarrow \mathfrak{N}_{*-1}^{Z_2}(BO_k(C^\infty) \times (EZ_4/Z_2))$$

assigning to ν: $F^{n-k} \longrightarrow BO_k$ the map $\nu \circ \pi \times c$: $RP(\nu) \longrightarrow BO_k \times (EZ_4/Z_2)$ and the monomorphism

$$\mu \circ \tau_*: \mathfrak{N}_{*-1}^{Z_2}(BO_k \times (EZ_4)/Z_2) \longrightarrow H_{*-1}^{Z_2}(BO \times BO_k \times EZ_4/Z_2, \phi' \times \mu'; Z_2).$$

Thus, one concludes that $\mu \circ \tau_*$ is monic for $\mathfrak{N}_*^{Z_4}$ if and only if

$$Q: \mathfrak{N}_{*-k}^{Z_2}(BO_k(C^\infty)) \longrightarrow \mathfrak{N}_{*-1}^{Z_2}(BO_k(C^\infty) \times (EZ_4/Z_2))$$

is monic for each $k > 1$.

For k odd, Q is monic. To see this, note that Z_2 acts freely on $BO_k(C^\infty)$ for a fixed point is a k dimensional subspace of C^∞ invariant under i, i.e. a complex subspace. Thus $\mathfrak{N}_{*-k}^{Z_2}(BO_k(C^\infty)) \cong \mathfrak{N}_{*-k}(BO_k/Z_2)$. Further $(BO_k \times (EZ_4/Z_2))/Z_2$ maps into BO_k/Z_2 and $EZ_4/Z_4 = BZ_4$. Given $M^n \xrightarrow{\nu'} BO_k/Z_2$, one has a double cover $\widetilde{M} \xrightarrow{\nu} BO_k$ and over $\widetilde{M}$ the projective space bundle $RP(\nu)$. Dividing out Z_2 on $RP(\nu)$ gives a bundle $E \xrightarrow{\pi} M$ with fiber $RP(k-1)$ and the corresponding class in

$\mathfrak{N}_{*-1}(BO_k/Z_2 \times BZ_4)$ is given by $\nu' \circ \pi \times f$: $E \longrightarrow BO_k/Z_2 \times BZ_4$

where f classifies the cover $S(\nu) \longrightarrow E$.

Now the $RP(k-1)$ bundle $E \xrightarrow{\pi} M$ is not totally nonhomologous to zero, but the map into BZ_4 induces the nontrivial four fold cover of $RP(k-1)$, and since k is odd, this implies that for $\alpha \in H^{k-1}(BZ_4, Z_2)$ the nonzero class, α pulls back to the nonzero class in $H^{k-1}(RP(k-1); Z_2)$.

Let $f_i: M_i^{n_i} \longrightarrow BO_k/Z_2$ be bordism elements giving a base for mod 2 homology, with dual base $\beta^i \in H^{n_i}(BO_k/Z_2; Z_2)$. Then

$$Q': \mathfrak{N}_*(BO_k/Z_2) \longrightarrow \mathfrak{N}_{*+k-1}(BO_k/Z_2 \times BZ_4)$$

sends $[M_i^{n_i}, f_i]$ into linearly independent classes detected by the $\beta^i \otimes \alpha$. Since Q' is an $\mathfrak{N}_*$ module homomorphism, Q' is monic. Since Q' factors through Q, Q is monic.

However, for k even, Q is not monic. The easiest example is to consider $RP(4)$ with the trivial involution T_0 and with $T_1([x_0, x_1, \ldots, x_4]) = [-x_0, -x_1, \ldots, x_4]$, with a point map into $BO_2(C^\infty)$, i.e. with a trivial bundle. These are not bordant in $\mathfrak{N}_4^{Z_2}(BO_2(C^\infty))$. In $\mathfrak{N}_5^{Z_2}(BO_2(C^\infty) \times (EZ_4/Z_2))$ they are represented by maps into point $\times EZ_4/Z_2$, hence come from $\mathfrak{N}_5^{Z_2}(pt \times (EZ_4/Z_2)) \cong \mathfrak{N}_5(BZ_4)$; i.e. one need only consider the bordism classes of the free Z_4 actions $T_0 \times i$ and $T_1 \times i$ on $RP(4) \times S^1$.

First dividing out Z_2 gives $RP(4) \times S^1$ with actions $T_0 \times (-1)$ and $T_1 \times (-1)$ and dividing Z_2 again gives the real projective space bundles $RP(5)$ and $RP(2\lambda \oplus 3)$ over $RP(1)$, λ the nontrivial line bundle. Since 2λ is trivial, both actions are given by $RP(4) \times S^1 \xrightarrow{\pi} S^1 \hookrightarrow BZ_4$ where the map into BZ_4 classifies the standard cover $z \longrightarrow z^4: S^1 \longrightarrow S^1$. This gives

PROPOSITION 4.1. *$\mu \circ \tau_*$ is not monic. It does, however, determine the fixed component and fixed data of odd codimension larger than one.*

COROLLARY (Bix). *$\chi: \mathfrak{N}_*^G \longrightarrow \mathrm{Hom}(H^*(BO; Z_2), H^*(BG; Z_2))$ is not monic if $G = Z_4$.*

To see that $\mu \circ \tau_*$ detects classes not detected by χ one must explicitly compute some examples. One follows Bix's technique.

It will be convenient to have a method for describing $\mathrm{Hom}(H^*(BO; Z_2), A)$. Let $f: RP(\infty) \longrightarrow BO$ classify the nontrivial line bundle and $b_i \in H_i(BO; Z_2)$ the image of the nonzero class in $H_i(RP(\infty); Z_2)$. Using the ring structure obtained from the Whitney sum, $H_*(BO; Z_2)$ is the Z_2 polynomial ring with unit $1 = b_0$ on the b_i. A homomorphism $\lambda: H^*(BO; Z_2) \longrightarrow A$ may then be identified as the class in $H_*(BO; Z_2) \hat{\otimes} A$ given by $\Sigma b_{i_1} \cdots b_{i_r} \otimes \lambda(s_{(i_1, \ldots, i_r)})$ where s_ω form the base in $H^*(BO; Z_2)$ dual to the b_ω in $H_*(BO; Z_2)$.

Let u, d be the nonzero classes in $H^1(BZ_4; Z_2)$ and $H^2(BZ_4; Z_2)$ so that $H^*(BZ_4; Z_2) = Z_2[u, d]/u^2 = 0$.

Let $M = RP(2n + 2)$ with the trivial Z_4 action "id" fixing every point.

Then $EG \times_G M = BG \times RP(2n+2)$, and $H^*(EG \times_G M; Z_2)$ is $H^*(BG; Z_2)[a]/a^{2n+3} = 0$, where a is the nonzero class in $H^1(RP(2n+2); Z_2)$. α^* is then the homomorphism sending w to $(1+a)^{2n+3}$ which may be written as $(\Sigma_i b_i a^i)^{2n+3}$. Applying integration along the fibers takes the coefficient of a^{2n+2}, so

$$\chi((RP(2n+2), \mathrm{id})) = \sum b_\omega \otimes s_\omega[RP(2n+2)] \in H_*(BO; Z_2) \otimes H^0(BG; Z_2).$$

Letting $M = RP(2n+2)$ with the involution $t([x_0, x_1, \ldots, x_{2n+2}]) = [-x_0, -x_1, x_2, \ldots, x_{2n+2}]$, $EG \times_G M$ may be identified with the projective space bundle of the vector bundle $2\lambda \oplus (2n+1)$ over BZ_4, where λ is the nontrivial line bundle over BZ_4, $w_1(\lambda) = u$. Letting μ be the bundle of vectors in lines in fibers of $2\lambda \oplus (2n+1)$, and $a = w_1(\mu)$, $H^*(EG \times_G M; Z_2)$ is the free module over $H^*(BG; Z_2)$ on $1, a, \ldots, a^{2n+2}$ with relation $a^{2n+3} = \Sigma w_i(2\lambda \oplus (2n+1))a^{2n+3-i}$, but $w(2\lambda \oplus (2n+1)) = (1+u)^2 = 1$, so $a^{2n+3} = 0$. The bundle $EG \times_G \tau(M)$ is the tangent bundle θ to the fibers of $RP(2\lambda \oplus (2n+1))$ and $\theta \oplus 1$ is $\mu \otimes \pi^*(2\lambda \oplus (2n+1))$ so

$$w(\theta) = (1+a+u)^2(1+a)^{2n+1} = (1+a^2+u^2)(1+a)^{2n+1}$$
$$= (1+a^2)(1+a)^{2n+1} = (1+a)^{2n+3}.$$

Thus α^* can be written as $(\Sigma_i b_i a^i)^{2n+3}$. Applying integration along the fibers takes the coefficient of a^{2n+2} and thus $\chi((RP(2n+2), t)) = \chi((RP(2n+2), \mathrm{id}))$.

Noting that the fixed set of t is $RP(1) \cup RP(2n)$ with $RP(2n)$ not bounding as a manifold, while id fixes $RP(2n+2)$, these are not bordant involutions. Thus one has

PROPOSITION 4.2.

$$\chi \circ P: \mathfrak{N}_*^{Z_2} = \mathfrak{N}_*^{Z_2}(BO_0(C^\infty)) \longrightarrow \mathrm{Hom}(H^*(BO; Z_2), H^*(BZ_4; Z_2))$$

is not monic.

Since P is monic, one sees that $\mu \circ \tau_*$ distinguishes classes in $\mathrm{im}\, P \subset \mathfrak{N}_*^{Z_4}$ which are not distinguished by χ.

COROLLARY. $\ker \psi$ *properly contains* $\ker \mu \circ \tau_*$ *for* $G = Z_4$.

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