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EVOLUTION SYSTEM APPROXIMATIONS OF SOLUTIONS TO CLOSED LINEAR OPERATOR EQUATIONS

BY

SEATON D. PURDOM[1]

ABSTRACT. With S a linearly ordered set with the least upper bound property, with g a nonincreasing real-valued function on S, and with A a densely defined dissipative linear operator, an evolution system M is developed to solve the modified Stieljes integral equation $M(s, t)x = x + A((L)\int_s^t dgM(\cdot, t)x)$. An affine version of this equation is also considered. Under the hypothesis that the evolution system associated with the linear equation is strongly (resp. weakly) asymptotically convergent, an evolution system is used to approximate strongly (resp. weakly) solutions to the closed operator equation $Ay = -z$.

Introduction. If X is a Banach space, if A is a linear function from $D(A)$ in X to X, and if g is a function from the real numbers, R, to R which is of bounded variation on each finite interval of R, then the integral equation

$$M(t, 0)x = x + \int_t^0 dgAM(\cdot, 0)x \tag{1}$$

permits a highly detailed theory. In case A is continuous, the theory for the modified Stieljes integral equation

$$M(t, 0)x = x + (R)\int_t^0 dgAM(\cdot, 0)x \tag{2}$$

is subsumed by MacNerney in [5]. In case A is continuous and each of $(I - (g(s^-) - g(s))A)^{-1}$ and $(I - (g(s) - g(s^+))A)^{-1}$ exists for each s in R, much of the theory for

$$M(t, 0)x = x + (L)\int_t^0 dgAM(\cdot, 0)x \tag{3}$$

is subsumed by results due to Herod in [4]. If A is dissipative, linear, and has dense domain, and if $g = -I$, then equations (1), (2), and (3) are equivalent and the theory has been developed in great detail. Yosida in [9] gives a thorough account. In §II here, under the hypotheses that A is linear, dissipative, and densely defined, and that g is nonincreasing, the theory for equation (3), and for an affine version of (3), is offered.

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The motivation for the development of the detailed theory of §II is the getting in §III of an iterative process to solve the equation

$$(4) \qquad Ay = z$$

for y. In [1], Browder and Petryshyn consider the related equation

$$(5) \qquad y - Ty = z.$$

Under the hypotheses that T is a continuous linear operator from X to X and that $\lim_{n\to\infty} T^n x$ exists for each x in X, it is established in [1] that if z is in the range of $(I - T)$, then the iterative process $x_{n+1} = z + Tx_n$ converges to a solution y of (5). Contained in [2] is a weak convergence version of [1]. In [8], Martin generalizes the Browder-Petryshyn paper to solve (4) with A continuous, using the product integral techniques of MacNerney [5]. What is offered in §III is a generalization of Martin's results to the present setting in which A is linear, dissipative, and densely defined. Strongly and weakly convergent iterative processes for (4) are discussed. Also, a test is given to determine whether z in (4) is in the range of A.

I. Notation and preliminary computations

NOTATION. Let X be a Banach space with norm $|\cdot|$. The norm of a continuous linear transformation B from X to X is also denoted by $|B|$, i.e., $|B| = \sup\{|Bx|: x \text{ is in } X \text{ and } |x| = 1\}$. If A is a linear function from a subset $D(A)$ of X to X, then A is said to be m-dissipative only in case for each $\lambda > 0$,

(i) $(I - \lambda A)^{-1}$ exists and has domain all of X, and

(ii) $|(I - \lambda A)^{-1}| \leq 1$.

Henceforth, A will always denote a m-dissipative linear function from $D(A)$ in X to X such that $D(A)$ is dense in X.

In what follows, elements from $\{(I - \lambda A)^{-1}: \lambda > 0\}$ appear frequently. As in Yosida [9], the notational convention $J_n = (I - n^{-1}A)^{-1}$, each $n > 0$, is made. The Hille-Yosida Theorem gives that, for $t \geq 0$, $\exp(tA)$ exists, is nonexpansive, and has many other well-known properties. Results along the same lines using continued products form the first part of the discussion which follows.

Henceforth, S denotes a linearly ordered set with the least upper bound property; R, the real numbers; and g, a nondecreasing function from S to R. If F is a function from $D(F)$ in X to X, and if $\{r_p\}_{p=0}^{m}$ is a sequence in S, then $\Pi_{p=j}^{m}(I - (g(r_p) - g(r_{p-1}))F)^{-1}$ is defined inductively by

$$\prod_{p=j}^{m} (I - (g(r_p) - g(r_{p-1}))F)^{-1}$$
$$= (I - (g(r_j) - g(r_{j-1}))F)^{-1} \prod_{p=j+1}^{m} (I - (g(r_p) - g(r_{p-1}))F)^{-1}$$

with the agreement that $\Pi_{p=m+1}^{m}(I-(g(r_p)-g(r_{p-1}))F)^{-1}=I$. When the sequence $\{r_p\}_{p=0}^{m}$ is clear, $\Pi_{p=j}^{m}(I-dg_pF)^{-1}$ is written in place of the more cumbersome $\Pi_{p=j}^{m}(I-(g(r_p)-g(r_{p-1}))F)^{-1}$. If each of s and t is in S with $s \geqslant t$, and if x is in X, then by ${}_s\Pi^t(I-dgF)^{-1}x$, one means the limit, in the sense of successive refinements of subdivisions $\{r_p\}_{p=0}^{m}$ of $\{s, t\}$, of estimates $\Pi_{p=1}^{m}(I-dg_pF)^{-1}x$. In case H is a function from S to X, $(L)\int_s^t dgFH(\cdot)$ refers to the limit, in the sense of successive refinements, of estimates $\Sigma_{p=1}^{m} dg_pFH(r_{p-1})$; and $(R)\int_s^t dgFH(\cdot)$, of estimates $\Sigma_{p=1}^{m} dg_pFH(r_p)$.

Preliminary computations. Three observations needed in Yosida's development of $\exp(tA)$ are useful here.

LEMMA 0.1. *Suppose that B is a continuous linear transformation from X to X. These are equivalent*:

(i) *For each* $\lambda > 0$, $(I-\lambda B)^{-1}$ exists, has domain all of X, and $|(I-\lambda B)^{-1}| \leqslant 1$,

(ii) *For each* $t \geqslant 0$, $|\exp(tB)| \leqslant 1$.

LEMMA 0.2. *For each x in X,* $\lim_{n\to\infty} J_n x = x$.

LEMMA 0.3. *If B is a linear function from $D(B)$ in X to X, and if, for some $\lambda > 0$, $(I-\lambda B)^{-1}$ exists, has domain all of X, and is continuous, then B is closed.*

Several computations which are needed frequently in the development of $\Pi(I-dgA)^{-1}$ in the present case are now summarized in Lemma 1.

LEMMA 1. *Suppose that x_0 is in $D(A)$, y is in X, $n > 0$, and $\{\lambda_k\}_{k=1}^{m}$ is a sequence of nonnegative numbers. Then*

(i) *For each* $k = 1, 2, \ldots, m$, $(I-\lambda_k AJ_n)^{-1}$ *exists, has domain all of X, is continuous, and* $|(I-\lambda_k AJ_n)^{-1}| \leqslant 1$,

(ii) *For each* $k = 1, 2, \ldots, m$,

$$|(I-\lambda_k AJ_n)^{-1}x_0-(I-\lambda_k A)^{-1}x_0| \leqslant \lambda_k |(J_n-I)Ax_0|,$$

(iii) $$\left|\prod_{k=1}^{m}(I-\lambda_k AJ_n)^{-1}x_0-\prod_{k=1}^{m}(I-\lambda_k A)^{-1}x_0\right| \leqslant \left(\sum_{k=1}^{m}\lambda_k\right)|(J_n-I)Ax_0|,$$

and

(iv) $$\left|\prod_{k=1}^{m}(I-\lambda_k AJ_n)^{-1}y-\prod_{k=1}^{m}(I-\lambda_k A)^{-1}y\right| \leqslant \inf_{x\in D(A)}\left\{2|x-y|+\left(\sum_{k=1}^{m}\lambda_k\right)|(J_n-I)Ax|\right\}.$$

PROOF. First, $AJ_n = n(J_n-I)$ so AJ_n is continuous. Hence, $\exp(tAJ_n) = \exp(tn(J_n-I)) = \exp(-nt)\exp(tnJ_n)$. Now

$$|\exp(tnJ_n)| = \left|\sum_{k=0}^{\infty} (k!)^{-1}(nt)^k J_n^k\right| \leq \exp(nt)$$

and (i) follows from Lemma 0.1.

The identity

$$(I - \lambda_k AJ_n)^{-1}x_0 - (I - \lambda_k A)^{-1}x_0 = (I - \lambda_k AJ_n)^{-1}(I - \lambda_k A)^{-1}\lambda_k(J_n - I)Ax_0$$

gives (ii).

For differences of products, one has

$$\left|\prod_{k=1}^{m} (I - \lambda_k AJ_n)^{-1}x_0 - \prod_{k=1}^{m} (I - \lambda_k A)^{-1}x_0\right|$$

$$= \left|\sum_{j=1}^{m} \left\{\prod_{k=1}^{j} (I - \lambda_k AJ_n)^{-1} \prod_{k=j+1}^{m} (I - \lambda_k A)^{-1}x_0 - \prod_{k=1}^{j-1} (I - \lambda_k AJ_n)^{-1} \prod_{k=j}^{m} (I - \lambda_k A)^{-1}x_0\right\}\right|$$

$$\leq \sum_{j=1}^{m} |(I - \lambda_j AJ_n)^{-1}x_0 - (I - \lambda_j A)^{-1}x_0|,$$

and (iii) follows from the estimate in (ii).

The inequality of (iv) follows immediately from (iii).

II. Development of the product integrals.

The linear case. A linear realization of a nonlinear result due to Herod [4] is the following, which will be generalized in this paper:

THEOREM 2. *Suppose that g is a nonincreasing function from S to R and that B is a continuous, dissipative, affine transformation from X to X. If each of s and t is in S with $s \geq t$, then*

(i) *$M(s, t) = {}_s\Pi^t(I - dgB)^{-1}$ exists,*

(ii) *If each of x and y is in X, then $|M(s, t)x - M(s, t)y| \leq |x - y|$,*

(iii) *If r is in S and $s \geq r \geq t$, then $M(s, r)M(r, t) = M(s, t)$,*

(iv) *If x is in X, then $M(\cdot, t)x$ is the only function F which is of bounded variation on each finite interval of S and which is a solution of the integral equation $F(s) = x + (L)\int_s^t dgBF(\cdot)$.*

REMARK 2.1. In case B is linear in Theorem 2, it should be noted that the convergence of the estimates to ${}_s\Pi^t(I - dgB)^{-1}$ is uniform in the following sense:

Let each of s and t be in S with $s \geq t$, and let each of ϵ and c be positive. There exists a subdivision $\{r_p\}_{p=0}^n$ of $\{s, t\}$ such that if x is in X with $|x| \leq c$, and if $\{v_p\}_{p=0}^m$ is a refinement of $\{r_p\}_{p=0}^n$, then

$$\left| \prod_{v_k}^{t} (I - dgB)^{-1}x - \prod_{p=k+1}^{m} (I - (g(v_p) - g(v_{p-1}))B)^{-1}x \right| < \epsilon$$

and

$$\left| \prod_{s}^{v_k} (I - dgB)^{-1}x - \prod_{p=1}^{k} (I - (g(v_p) - g(v_{p-1}))B)^{-1}x \right| < \epsilon$$

for $k = 0, 1, \ldots, m$.

The computational results of §I, combined with Theorem 2, now lead to the development of $M(s, t)x = {}_s\Pi^t(I - dgA)^{-1}x$ in the present case that A is linear, dissipative, and densely defined.

THEOREM 3. *Let A and g be as before. If each of s and t is in S with $s \geqslant t$, then*

(i) *$M(s, t)x = {}_s\Pi^t(I - dgA)^{-1}$ exists for each x in X,*

(ii) *$M(s, t)$ is a continuous linear function from X to X and $|M(s, t)| \leqslant 1$,*

(iii) *If r is in S and $s \geqslant r \geqslant t$, then $M(s, r)M(r, t) = M(s, t)$,*

(iv) *If x_0 is in $D(A)$, then $M(s, t)x_0$ is in $D(A)$ and $AM(s, t)x_0 = M(s, t)Ax_0$,*

(v) *If x_0 is in $D(A)$, then $M(s, t)x_0 = x_0 + (L)\int_s^t dgAM(\cdot, t)x_0$,*

(vi) *If x is in X, then $(L)\int_s^t dgM(\cdot, t)x$ is in $D(A)$, and $M(s, t)x = x + A((L)\int_s^t dgM(\cdot, t)x)$, and*

(vii) *If x_0 is in $D(A^2)$, then $M(\cdot, t)x_0$ is the only function $F(\cdot)$ for which $AF(\cdot)$ is of bounded variation on each finite interval of S and which is a solution of the integral equation $F(s) = x_0 + (L)\int_s^t dgAF(\cdot)$.*

INDICATION OF PROOF. If x is in X, $n > 0$, and each of s and t is in S with $s \geqslant t$, then $M_n(s, t)x = {}_s\Pi^t(I - dgAJ_n)^{-1}$ exists by Theorem 2. One shows that $\lim_{n\to\infty} M_n(s, t)x$ exists as follows:

If $\{r_k\}_{k=0}^p$ is a subdivision of $\{s, t\}$ and x_0 is in $D(A)$, then

$$\begin{aligned} &|M_n(s, t)x_0 - M_m(s, t)x_0| \\ &\quad \leqslant \left| M_n(s, t)x_0 - \prod_{k=1}^{p} (I - dg_kAJ_n)^{-1}x_0 \right| \\ &\qquad + \left| \prod_{k=1}^{p} (I - dg_kAJ_n)^{-1}x_0 - \prod_{k=1}^{p} (I - dg_kAJ_m)^{-1}x_0 \right| \\ &\qquad + \left| \prod_{k=1}^{p} (I - dg_kAJ_m)^{-1}x_0 - M_m(s, t)x_0 \right| . \end{aligned}$$

It follows that

$$|M_n(s,t)x_0 - M_m(s,t)x_0| \leqslant \left(\int_s^t dg\right)(|(J_n - I)Ax_0| + |(J_m - I)Ax_0|).$$

Hence, $\lim_{n\to\infty} M_n(s,t)x_0$ exists for each x_0 in $D(A)$; and the limit is uniform on bounded subsets of S. Inequality (iv) of Lemma 1 now gives that $\lim_{n\to\infty} M_n(s,t)y$ exists for each y in X; and again the limit is uniform on bounded subsets of S. Henceforth, let $M(s,t)x = \lim_{n\to\infty} M_n(s,t)x$ for each x in X.

Conditions (ii) and (iii) on M are inherited directly from the corresponding conditions on the M_n. Also, if x is in X and x_0 is in $D(A)$, then

$$\begin{aligned}
&\left|M(s,t)x - \prod_{k=1}^{p} (I - dg_k A)^{-1}x\right| \\
&\qquad \leqslant |M(s,t)(x - x_0)| + |M(s,t)x_0 - M_n(s,t)x_0| \\
&\qquad\quad + \left|M_n(s,t)x_0 - \prod_{k=1}^{p} (I - dg_k AJ_n)^{-1}x_0\right| \\
&\qquad\quad + \left|\prod_{k=1}^{p} (I - dg_k AJ_n)^{-1}x_0 - \prod_{k=1}^{p} (I - dg_k A)^{-1}x_0\right| \\
&\qquad\quad + \left|\prod_{k=1}^{p} (I - dg_k A)^{-1}(x_0 - x)\right| \\
&\qquad \leqslant |x - x_0| + |M(s,t)x_0 - M_n(s,t)x_0| \\
&\qquad\quad + \left|M_n(s,t)x_0 - \prod_{k=1}^{p} (I - dg_k AJ_n)^{-1}x_0\right| \\
&\qquad\quad + \left(\int_s^t dg\right)|(J_n - I)Ax_0| + |x - x_0|.
\end{aligned}$$

Since $D(A)$ is dense in X, the representation in (i) holds. (A uniform condition here is the subject of Corollary 3.2.) Condition (iv) on M follows immediately since A is closed and

$$A \prod_{k=1}^{p} (I - dg_k A)^{-1}x_0 = \prod_{k=1}^{p} (I - dg_k A)^{-1}Ax_0.$$

If x_0 is in $D(A)$, then

$$\left|(L)\int_s^t dgM(\cdot,\ t)Ax_0 - x_0 - M(s,\ t)x_0\right|$$

$$\leqslant \left|(L)\int_s^t dgM(\cdot,\ t)Ax_0 - (L)\int_s^t dgM_n(\cdot,\ t)AJ_nx_0\right|$$

$$+\ |M(s,\ t)x_0 - M_n(s,\ t)x_0|$$

$$\leqslant \left|(L)\int_s^t dg(M_n(\cdot,\ t) - M(\cdot,\ t))Ax_0\right| + \left|(L)\int_s^t dgM_n(\cdot,\ t)(J_n - I)Ax_0\right|$$

$$+\ |M(s,\ t)x_0 - M_n(s,\ t)x_0|$$

$$\leqslant \left(\int_s^t dg\right)\sup_{s\leqslant z\leqslant t}\{|M_n(z,\ t)Ax_0 - M(z,\ t)Ax_0|\}$$

$$+ \left(\int_s^t dg\right)|(J_n - I)Ax_0| + \left(\int_s^t dg\right)|(J_n - I)Ax_0|.$$

Now

$$\sup_{s\leqslant z\leqslant t}\{|M_n(z,\ t)Ax_0 - M(z,\ t)Ax_0|\}$$

$$\leqslant \inf_{\xi\in D(A)}\left\{2|\xi - Ax_0| + \left(\int_s^t dg\right)|(J_n - I)A\xi|\right\}.$$

Since A commutes with M on $D(A)$, the integral equation of (v) is satisfied; (vi) follows immediately since A is closed.

Finally, suppose that x_0 is in $D(A^2)$. Since

$$AM(s,\ t)x_0 = Ax_0 + (L)\int_s^t dgM(\cdot,\ t)A^2x_0,$$

$AM(\cdot,\ t)x_0$ is of bounded variation on each finite interval of S. If F is a function from S to $D(A)$ such that $AF(\cdot)$ is of bounded variation and such that the integral equation of (vii) is satisfied, then any subdivision $\{r_p\}_{p=0}^n$ of $\{s,\ t\}$ which gives that

$$\sum_{p=1}^n \left|dg_pAf(r_{p-1}) - (L)\int_{r_{p-1}}^{r_p} dgAf(\cdot)\right|$$

$$+ \sum_{p=1}^n \left|dg_pAM(r_{p-1},\ t) - (L)\int_{r_{p-1}}^{r_p} dgAM(\cdot,\ t)x_0\right| < \epsilon$$

also gives that

$$|f(s) - M(s, t)x_0| \leqslant |f(s) - M(s, t)x_0| + \sum_{p=1}^{n} \{|(I - dg_pA)(f(r_{p-1}) - M(r_{p-1}, t)x_0)|\} - |f(r_{p-1}) - M(r_{p-1}, t)x_0|$$

$$= \sum_{p=1}^{n} \{- |f(r_p) - M(r_p, t)x_0| + |(I - dg_pA)(f(r_{p-1}) - M(r_{p-1}, t)x_0)|\}$$

$$\leqslant \sum_{p=1}^{n} \{|f(r_p) - f(r_{p-1}) - dg_pAf(t_{p-1})| + |M(r_p, t)x_0 - M(r_{p-1}, t)x_0 - dg_pAM(r_{p-1}, t)x_0|\}$$

$$\leqslant \sum_{p=1}^{n} \left\{\left|(L)\int_{r_{p-1}}^{r_p} dgAf(\cdot) - dg_pAf(r_{p-1})\right| + \left|(L)\int_{r_{p-1}}^{r_p} dgAM(\cdot, t)x_0 - dg_pAM(r_{p-1}, t)x_0\right|\right\} < \epsilon.$$

Hence, $f(s) = M(s, t)x_0$.

From the preceding proof, one has the representation formula for M

COROLLARY 3.1. *If x is in X, and if each of s and t is in S with $s \geqslant t$, then*

$$M(s, t)x = \lim_{n\to\infty} M_n(s, t)x = \lim_{n\to\infty} {}_s\prod^t (I - dgAJ_n)^{-1}x,$$

uniformly on bounded subsets of S.

Remark 2.1, coupled with the proof that the representation (i) of Theorem 3 holds, gives

COROLLARY 3.2. *If x is in X, and if each of s and t is in S with $s \geqslant t$, then the limit $M(s, t)x = {}_s\Pi^t(I - dgA)^{-1}x$ is uniform in the following sense:*

If $\epsilon > 0$, then there exists a subdivision $\{r_p\}_{p=0}^n$ of $\{s, t\}$ such that if $\{v_p\}_{p=0}^m$ is a refinement of $\{r_p\}_{p=0}^n$, then

$$\left|{}_{v_k}\prod^t (I - dgA)^{-1}x - \prod_{p=k+1}^{m} (I - (g(v_p) - g(v_{p-1}))A)^{-1}x\right| < \epsilon$$

and

$$\left|\prod_s^{v_k} (I - dgA)^{-1}x - \prod_{p=1}^{k+1} (I - (g(v_p) - g(v_{p-1}))A)^{-1}x\right| < \epsilon$$

for $k = 0, 1, \ldots, m$.

A related integral equation is also satisfied.

COROLLARY 3.3. *If* x_0 *is in* $D(A)$, *and if each of* s *and* t *is in* S *with* $s \geqslant t$, *then*

$$M(s, t)x_0 = x_0 + (R)\int_s^t dgAM(s, \cdot)x_0.$$

INDICATION OF PROOF. Let $\{r_p\}_{p=0}^m$ be a subdivision of $\{s, t\}$ and let x_0 be in $D(A)$. Since

$$\prod_{k=1}^{m} (I - dgA)^{-1}x_0 - x_0 = \sum_{j=1}^{m} dg_j \prod_{k=1}^{j} (I - dg_kA)^{-1}Ax_0,$$

Corollary 3.2 gives that $M(s, t)x_0 - x_0 = (R)\int_s^t dgM(s, \cdot)Ax_0$. Since A is closed, $M(s, t)x - x = A((R)\int_s^t dgM(s, \cdot)x)$ for each x in X.

The example following indicates the need for a uniqueness condition as cumbersome as (vii) of Theorem 3.

EXAMPLE. Let $X =$ {all continuous $f\colon R \longrightarrow R$ such that $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to+\infty} f(x)$ both exist}. If f is in X, let $|f| = \sup\{|f(x)|\colon x \in R\}$. If $(M(s, t)f)(x) = f(x + s - t)$, and if

$$\psi(x) = \begin{cases} 0 & \text{if } x \leqslant 0,\\ x\cos(\pi/x) & \text{if } 0 < x < 2,\\ 0 & \text{if } x \geqslant 2,\end{cases}$$

then there is a y in X such that $Ay = dy/dx = \psi$. Now $AM(\cdot, 0)\psi = M(\,\cdot\,, 0)\psi$ and is of bounded variation on no interval of R.

In terms of the evolution system M, the uniqueness condition of Theorem 3 is not vague, for M is a collection of continuous operators and $D(A^2)$ is dense in X.

The affine case. With A as in Theorem 3 and z in X, let $A + z$ be the affine transformation defined by $(A + z)x = Ax + z$ for each x in $D(A)$. A few computational results facilitate the development of the affine version of Theorem 3.

LEMMA 4. *Let* A *be as before*; *let* $\beta > 0$, $n > 0$, *and* $\{\lambda_k\}_{k=1}^m$ *be a sequence of nonnegative numbers*; *let each of* u *and* w *be in* $D(A)$; *and let* x, y, *and* z *be in* X. *Then*

(i) $$(I - \beta(A + z))^{-1}y = (I - \beta A)^{-1}(y + \beta z),$$

(ii) $$(I-\beta(A+z))^{-1}y-(I-\beta A)^{-1}y=\beta(I-\beta A)^{-1}z,$$

(iii) $$\prod_{k=1}^{m}(I-\lambda_k(A+z))^{-1}y = \prod_{k=1}^{m}(I-\lambda_kA)^{-1}y+\sum_{j=1}^{m}\left\{\prod_{k=1}^{j}(I-\lambda_kA)^{-1}\lambda_jz\right\},$$

(iv) $$\left|\prod_{k=1}^{m}(I-\lambda_k(AJ_m+u))^{-1}w-\prod_{k=1}^{m}(I-\lambda_k(A+u))^{-1}w\right| \leqslant \left(\sum_{k=1}^{m}\lambda_k\right)|(J_n-I)Aw|+\left(\sum_{k=1}^{m}\lambda_k\right)^2|(J_n-I)Au|,$$

(v) $$\left|\prod_{k=1}^{m}(I-\lambda_k(A+z))^{-1}x-\prod_{k=1}^{m}(I-\lambda_k(A+y))^{-1}x\right| \leqslant \left(\sum_{k=1}^{m}\lambda_k\right)|z-y|.$$

PROOF.

$$(I-\beta(A+z))(I-\beta A)^{-1}(y+\beta z)=(I-\beta A)(I-\beta A)^{-1}(y+\beta z)-\beta z=y,$$

giving (i). Statements (ii) and (iii) are immediate applications of (i). Statement (iv) follows from (iii) and from Lemma 1. Finally, (v) follows from (iii) and the fact that $|(I-\lambda A)^{-1}|\leqslant 1$ for each $\lambda>0$.

Estimates obtained in the development of $\Pi(I-dgA)^{-1}$ combined with the above computations now give an affine version of Theorem 3.

THEOREM 5. *Let A and g be as before and let z be in X. If each of s and t is in S with* $s\geqslant t$, *then*

(i) $W(z;s,t)x={}_s\Pi^t(I-dg(A+z))^{-1}x$ *exists for each x in X,*

(ii) $W(z;s,t)$ *is a continuous affine function from X to X such that if each of x and y is in X, then* $|W(z;s,t)x-W(z;s,t)y|\leqslant|x-y|$,

(iii) *If each of x and y is in X, then*

$$|W(z;s,t)x-W(y;s,t)x|\leqslant\left(\int_s^t dg\right)|z-y|,$$

(iv) *If r is in S and* $s\geqslant r\geqslant t$, *then* $W(z;s,r)W(z;r,t)=W(z;s,t)$,

(v) *If each of x and z is in X, then*

$$W(z; s, t)x = M(s, t)x + (R)\int_s^t dgM(s, \cdot)z,$$

(vi) *If x_0 is in $D(A)$ and z is in X, then*

$$W(z; s, t)x_0 = x_0 + (L)\int_s^t dg(A + z)\, W(z; \cdot, t)x_0,$$

(vii) *If each of x and z is in X, then $(L)\int_s^t dgW(z; \cdot, t)x$ is in $D(A)$ and*

$$W(z; s, t)x = x + A\left((L)\int_s^t dgW(z; \cdot, t)x\right) + \int_s^t dg\, z,$$

and

(viii) *If x is in $D(A^2)$ and z is in $D(A)$, then $W(z; \cdot, t)x$ is the only function $F(\cdot)$ for which $AF(\cdot)$ is of bounded variation on each finite interval of S and which solves the integral equation*

$$F(\cdot) = x + (L)\int_s^t dg(A + z)F(\cdot).$$

INDICATION OF PROOF. If $\{r_k\}_{k=0}^m$ is a subdivision of $\{s, t\}$, if each of x_0 and z is in $D(A)$, and if n is a positive integer, then

$$\left|\prod_{k=1}^m (I - dg_k(AJ_n + z))^{-1}x_0 - \prod_{k=1}^m (I - dg_k(A + z))^{-1}x_0\right|$$

$$= \left|\prod_{k=1}^m (I - dg_k AJ_n)^{-1}x_0 - \prod_{k=1}^m (I - dg_k A)^{-1}x_0\right.$$

$$\left. + \sum_{j=1}^m \left\{\prod_{k=1}^j (I - dg_k AJ_n)^{-1}\, dg_j z - \prod_{k=1}^j (I - dg_k A)^{-1}\, dg_j z\right\}\right|$$

$$\leqslant \left(\int_s^t dg\right)|(J_n - I)Ax_0| + \left(\int_s^t dg\right)^2 |(J_n - I)Az|.$$

Hence,

$$\left|{}_s\prod^t (I - dg(AJ_n + z))^{-1}x_0 - {}_s\prod^t (I - dg(AJ_p + z))^{-1}x_0\right|$$

$$\leqslant \left(\int_s^t dg\right)(|(J_n - I)Ax_0| + |(J_p - I)Ax_0|)$$

$$+ \left(\int_s^t dg\right)^2(|(J_n - I)Az| + |(J_p - I)Az|);$$

so $\lim_{n\to\infty} {}_s\Pi^t(I - dg(AJ_n + z))^{-1}x_0$ exists for each of x_0 and z in $D(A)$. By (iv) of Lemma 4, $\lim_{n\to\infty} {}_s\Pi^t(I - dg(AJ_n + z))^{-1}x$ exists for x in X and z in $D(A)$; and, by (v) of Lemma 4, the limit exists for z in X and x in X. Let

$W(z; s, t)x = \lim_{n\to\infty} {}_s\Pi^t(I - dg(AJ_n + z))^{-1}x$. If each of x_0 and z_0 is in $D(A)$, and if

$$W_n(z; s, t)x = \prod_s^t (I - dg(AJ_n + z))^{-1}x,$$

then

$$\begin{aligned}
&\left|W(z; s, t)x - \prod_{k=1}^m (I - dg(A + z))^{-1}x\right| \\
&\quad\leqslant |W(z; s, t)x - W_n(z; s, t)x| + |W_n(z; s, t)x - W_n(z_0; s, t)x| \\
&\qquad + |W_n(z_0; s, t)x - W_n(z_0; s, t)x_0| \\
&\qquad + \left|W_n(z_0; s, t)x_0 - \prod_{k=1}^m (I - dg_k(AJ_n + z_0))^{-1}x_0\right| \\
&\qquad + \left|\prod_{k=1}^m (I - dg_k(AJ_n + z_0))^{-1}x_0 - \prod_{k=1}^m (I - dg_k(A + z_0))^{-1}x_0\right| \\
&\qquad + \left|\prod_{k=1}^m (I - dg_k(A + z_0))^{-1}x_0 - \prod_{k=1}^m (I - dg_k(A + z_0))^{-1}x\right| \\
&\qquad + \left|\prod_{k=1}^m (I - dg_k(A + z_0))^{-1}x - \prod_{k=1}^m (I - dg_k(A + z))^{-1}x\right| \\
&\quad\leqslant |W(z; s, t)x - W_n(z; s, t)x| + \left(\int_s^t dg\right)|z - z_0| \\
&\qquad + |x - x_0| + \left|W_n(z_0; s, t)x_0 - \prod_{k=1}^m (I - dg_k(AJ_n + z_0))^{-1}x_0\right| \\
&\qquad + \left(\left(\int_s^t dg\right)|(J_n - I)Ax_0| + \left(\int_s^t dg\right)^2 |(J_n - I)Az_0|\right) \\
&\qquad + |x - x_0| + \left(\int_s^t dg\right)|z - z_0|.
\end{aligned}$$

Hence, the representation of W in (i) holds.

Properties (ii) and (iii) follow immediately from the corresponding properties of the approximating products. Property (iv) is inherited from the W_n.

Now, the fact that

$$\prod_{k=1}^{m} (I - dg_k(A + z))^{-1}x$$

$$= \prod_{k=1}^{m} (I - dg_k A)^{-1}x + \sum_{j=1}^{m} dg_j \left(\prod_{k=1}^{j} (I - dg_k A)^{-1} z \right)$$

together with Corollary 3.2 gives that

$$W(z; s, t)x = M(s, t)x + (R)\int_s^t dgM(s, \cdot)z,$$

which is equation (v).

Now if each of x_0 and z_0 is in $D(A)$, then $AW(z_0; s, t)x_0 = W(Az_0; s, t)Ax_0$. As in Theorem 3, one can use $\{W_n\}_{n=1}^{\infty}$ to show that

$$W(z_0; s, t)x_0 = x_0 + (L)\int_s^t dg(A + z)W(z_0; \cdot, t)x_0.$$

Moreover, if z is in X, then $AW(z; s, t)x_0 = AM(s, t)x_0 + M(s, t)z - z$ and is integrable. The fact that

$$|W(z; s, t)x_0 - W(z_0; s, t)x_0| \leqslant \left(\int_s^t dg\right)|z - z_0|$$

then gives that (vi) holds. The integral equation (vii) follows at once since A is closed.

Finally, if z is in $D(A)$ and x is in $D(A^2)$, then $AW(z; \cdot, t)x$ is of bounded variation on each finite interval of S. If F is also, and if F satisfies $F(s) = x + (L)\int_s^t dg(A + z)F(\cdot)$ for each $s \geqslant t$, then

$$F(t) - W(z; t, t)x = 0$$

and

$$F(s) - W(z; s, t)x = 0 + (L)\int_s^t dgA(F(\cdot) - W(z; \cdot, t)x).$$

Theorem 3 gives that $F(s) = W(z; s, t)x$, and the proof is complete.

III. An application to closed operator equations. In special circumstances, the results of the preceding section may be used to solve the operator equation

$$(*) \qquad Ay = z$$

for y.

In addition, now let S be an unbounded set of the nonnegative numbers with 0 in S, and let g now be such that $\lim_{t \to +\infty} g(t) = -\infty$. Theorems 3 and 5 are now applied in Theorems 7 and 8 to give an iterative procedure to approximate solutions to $(*)$.

LEMMA 6. *Let B be a linear function from $D(B)$ in X to X and let $\lambda > 0$ be such that $(I - \lambda B)^{-1}$ exists, has domain all of X, and is continuous. If $\{x_n\}_{n=1}^{\infty}$ is a sequence in $D(B)$ such that $w\text{-}\lim_{n\to\infty} x_n = x$ and $w\text{-}\lim_{n\to\infty} Bx_n = y$, then x is in $D(B)$ and $Bx = y$.*

PROOF. Let $y_n = Bx_n$ and $z_n = x_n - \lambda y_n$ for each n. Then $w\text{-}\lim_{n\to\infty} z_n = x - \lambda y$; and, since continuous linear functions preserve weak limits,

$$(I - \lambda B)^{-1}(x - \lambda y) = \underset{n\to\infty}{w\text{-}\lim}\,(I - \lambda B)^{-1}(x_n - \lambda y_n) = x.$$

Hence, x is in $D(B)$, and $(I - \lambda B)^{-1}(x - \lambda y) = x$ gives that $x - \lambda y = x - \lambda Bx$.

DEFINITION. An evolution system M is strongly (resp., weakly) asymptotically convergent if, and only if, $\lim_{t\to+\infty} M(t, 0)x$ (resp., $w\text{-}\lim_{t\to+\infty} M(t, 0)x$) exists for each x in X.

THEOREM 7. *Let A, g, and M be as in Theorem 3 so that $M(s, t)x = {}_s\Pi^t (I - dgA)^{-1}x$ for each x in X and for each of s and t in S with $s \geqslant t$. If M is strongly (resp., weakly) asymptotically convergent, and if $Qx = \lim_{t\to+\infty} M(t, 0)x$ (resp., $w\text{-}\lim_{t\to+\infty} M(t, 0)x$) for each x in X, then*

(i) *Q is a continuous projection of X onto the null space of A,*

(ii) *$|Q| \leqslant 1$, and*

(iii) *the null space of Q is the closure of the range of A.*

PROOF. If x_0 is in $D(A)$, then the fact that

$$Qx_0 - x_0 = \lim_{s\to+\infty} (L)\int_s^0 dgM(\cdot, 0)Ax_0$$

gives that $QAx_0 = 0$. Also,

$$\lim_{s\to\infty} AM(s, 0)x_0 = \lim_{s\to\infty} M(s, 0)Ax_0 = 0 \quad \text{and} \quad \lim_{s\to\infty} M(s, 0)x_0 = Qx_0,$$

so $AQx_0 = 0$. Since A is closed, $AQx = 0$ for each x in X. That $|Q| \leqslant 1$ follows from the fact that $|M(t, 0)| \leqslant 1$ for each $t \geqslant 0$. Since

$$Q \prod_{k=1}^{m} (I - \lambda_k A)^{-1}x - Qx = 0, \quad QM(s, t) = M(s, t)Q = Q.$$

Hence, $Q^2 = Q$.

Already, one has that if z is in the range of A, then $Qz = 0$. Since Q is continuous, $Qx = 0$ for each x in the closure of the range of A. Since $M(t, 0)x - x = A\,((L)\int_t^0 dgM(\cdot, 0)x)$, $Qx = 0$ only in case x is in the closure of the range of A.

The proof in case M is weakly asymptotically convergent uses Lemma 6 and follows much the same lines.

THEOREM 8. *Let A, g, and M be as in Theorem 3 and suppose that $M(\cdot, 0)$ is strongly (resp., weakly) asymptotically convergent. If each of s and t is in S with $s \geqslant t$, if each of x and z is in X, and if $W(z; s, t)x = {}_s\Pi^t(I - dg(A + z))^{-1}x$ as in Theorem 5, then these are equivalent:*

(i) *z is in the range of A,*

(ii) *For each x in X, $\lim_{t\to+\infty} W(z; t, 0)x$ (resp., $w\text{-}\lim_{t\to+\infty} W(z; t, 0)x$) exists and is a solution y of the equation $Ay = -z$.*

(iii) *There is an x in X and an increasing, unbounded sequence $\{t_k\}_{k=1}^{\infty}$ in S such that $w\text{-}\lim_{k\to+\infty} W(z; t_k, 0)x$ exists.*

PROOF. A proof is offered in case M is strongly asymptotically convergent. First, suppose that $Au = z$. Then

$$W(z; t, 0)x = M(t, 0)x + (R)\int_t^0 dgM(t, \cdot)Au$$

$$= M(t, 0)x + M(t, 0)u - u.$$

Hence, $\lim_{t\to\infty} W(z; t, 0)x = Qx + Qu - u$ and $AQx + AQu - Au = -z$. Since A is linear, (i) implies (ii). That (ii) implies (iii) is clear.

Finally, suppose that (iii) holds so that $w\text{-}\lim_{k\to\infty} W(z; t_k, 0)x = u$. If x_0 is in $D(A)$, then

$$u + Q(x_0 - x) = \underset{k\to\infty}{w\text{-}\lim}\, W(z; t_k, 0)x + \lim_{k\to\infty} M(t_k, 0)(x_0 - x)$$

$$= \underset{k\to\infty}{w\text{-}\lim}\, W(z; t_k, 0)x_0$$

and

$$\underset{k\to\infty}{w\text{-}\lim}\, AW(z; t_k, 0)x_0 = \underset{k\to\infty}{w\text{-}\lim}\, AW(z; t_k, 0)x_0 - AM(t_k, 0)x_0$$

$$= \underset{k\to\infty}{w\text{-}\lim}\, A(R)\int_{t_k}^0 dgM(t_k, \cdot)z$$

$$= \underset{k\to\infty}{w\text{-}\lim}\, M(t_k, 0)z - z$$

$$= Qz - z.$$

By Lemma 6, one has $Au = Qz - z$. Now

$$W(z; t_k, 0)x = M(t_k, 0)x + (R)\int_{t_k}^0 dgM(t_k, \cdot)z$$

$$= M(t_k, 0)x + (R)\int_{t_k}^0 dgM(t_k, \cdot)(-Au) + (R)\int_{t_k}^0 dgM(t_k, \cdot)Qz$$

$$= M(t_k, 0)x + u - M(t_k, 0)u + (R)\int_{t_k}^0 dgQz.$$

Hence, $Qz = 0$; so $Au = -z$ and (iii) implies (i).

In case M is weakly asymptotically convergent, virtually the same proof gives the weak version of Theorem 8.

REMARK. The original requirement that A be dissipative can be weakened somewhat. In particular, suppose that A is a linear function from $D(A)$ in X to X such that $D(A)$ is dense in X and that there is a number C such that if $\{\lambda_k\}_{k=1}^m$ is a sequence of positive numbers, then $|\Pi_{k=1}^m (I - \lambda_k A)^{-1}| \leqslant C$. It follows that $|\exp(tA)| \leqslant C$ for each $t \geqslant 0$. The norm, $|x|_2 = \sup_{t \geqslant 0} |\exp(tA)x|$, is equivalent to the norm $|\cdot|$ on X, and $|\exp(tA)x|_2 \leqslant |x|_2$. Hence, A is dissipative with respect to $|\cdot|_2$. The previous hypothesis that $|(I - \lambda A)^{-1}| \leqslant 1$ for each $\lambda > 0$ can thus be weakened.

Another extension of the integral equation theory of §II can be had. If β and λ are numbers, one has the identity

$$(I - \lambda(A + \beta I))^{-1} = (1 - \lambda\beta)^{-1}(I - \lambda(1 - \lambda\beta)^{-1}A)^{-1}$$

provided that $\lambda\beta \neq 1$. If g is a nonincreasing function from S to R and if β is negative, then ψ, defined by

$$\psi(t) = \int_t^0 dg(1 - \beta dg)^{-1},$$

is nonincreasing. If A is dissipative with respect to some norm equivalent to $|\cdot|$, then $A + \beta I$ is dissipative and Theorem 3 already guarantees the existence of ${}_s\Pi^t(I - dg(A + \beta I))^{-1}$ for $s \geqslant t$ in S. The identity,

$$\prod_s^t (I - dg(A + \beta I))^{-1} = \left(\prod_s^t (1 - \beta dg)^{-1}\right)\left(\prod_s^t (I - d\psi A)^{-1}\right),$$

furnishes better normed estimates in Theorems 3 and 5. The theory of §III is largely had already as a part of the Hille-Yosida Theorem.

Some passage of the theory of §§II and III to the operator $A + \beta I$, $\beta > 0$, can also be had. One requires of the function g that there exist a number P such that if s is in S, then there exist u and v in S such that $v < s < u$ and $g(v) - g(u) \leqslant P$. If $\beta P < 1$, one has $\psi(t) = \int_t^0 dg(1 - \beta dg)^{-1}$ is nonincreasing,

$$\prod_s^t (I - dg(A + \beta I))^{-1} = \left(\prod_s^t (1 - \beta dg)^{-1}\right)\left(\prod_s^t (I - d\psi A)^{-1}\right),$$

and the attendant integral equation theory holds. If, as in §III, $\lim_{t \to +\infty} {}_t\Pi^0(I - dg(A + \beta I))^{-1}x$ exists (or if the weak limit exists) for each x in X and is Qx, then the uniform boundedness theorem gives that Q is continuous. Statements (i) and (iii) of Theorem 7 and the iteration description of Theorem 8 follow with A replaced by $A + \beta I$. Even if one has only that

$\{|{}_t\Pi^0(I - dg(A + \beta I))^{-1}|: t > 0\}$ is bounded, then the iteration theory of §III can be developed for the operators $A + \gamma I$, $0 \leq \gamma < \beta$. If $y(t) = \int_t^0 dg(1 - (\beta - \gamma)dg)^{-1}$, then y is nonincreasing. Also, $g(0) - g(t) \leq y(0) - y(t)$ so $\lim_{t\to+\infty} g(t) = -\infty$ forces $\lim_{t\to+\infty} y(t) = -\infty$. Now

$$\prod_s^t (I - dy(A + \gamma I))^{-1} = \left(\prod_s^t (1 - (\beta - \gamma)dg)\right) \prod_s^t (I - dg(A + \beta I))^{-1}$$

and the fact that $\lim_{t\to+\infty} {}_t\Pi^0(1 - (\beta - \gamma)dg) = 0$ gives that

$$\lim_{t\to+\infty} \prod_t^0 (I - dy(A + \gamma I))^{-1}x = 0 \quad \text{for each } x \text{ in } X.$$

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SCHOOL OF MATHEMATICS, GEORGIA INSTITUTE OF TECHNOLOGY, ATLANTA, GEORGIA 30332