TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 216, 1976

SOME SPECIAL DECOMPOSITIONS OF E^3

BY

CHARLES D. BASS

ABSTRACT. A great deal of attention has been given to the question: which upper semicontinuous decompositions of E^3 into pointlike continua give E^3. It has recently been determined that some decompositions of E^3 into points and straight line segments give decomposition spaces which are topologically distinct from E^3. In this paper we apply a new condition to the set of nondegenerate elements of a decomposition which enables one to conclude that the resulting decomposition space is homeomorphic to E^3.

1. **Introduction.** In the attempt to determine which monotone decompositions of E^3 yield E^3, some authors have studied decompositions having very special sets of nondegenerate elements. For instance, McAuley has shown in [6] that a u.s.c. decomposition of E^3 yields E^3 if the set of nondegenerate elements consists of straight line segments, each of which is parallel to one of a countable family of lines. However, Eaton has shown [5] that an example due to Bing of a decomposition whose nondegenerate elements are line segments gives a decomposition space that is topologically different from E^3. This result indicates that simplicity of the elements alone is insufficient to insure that a u.s.c. decomposition of E^3 yields E^3. In this paper we prove that decompositions of E^3 into a special class of compact sets which are similarly positioned gives E^3.

In order to describe the type of decomposition in which we are interested, it is helpful to introduce some preliminary notation. First we define a partial ordering $\preceq$ on E^3 as follows. Suppose that $p, q \in E^3$; $p = (x_1, x_2, x_3)$; and $q = (y_1, y_2, y_3)$. Then $p \preceq q$ if and only if $x_i \leqslant y_i$ for each i. If p and q are distinct points of E^3, we see that the set of points "between" p and q with respect to this ordering is a line segment, a rectangle, or a rectangular solid. Define

$$R(p, q) = \{p' \in E^3 | p \preceq p' \preceq q\}.$$

Let X be a compact subset of E^3. Suppose that X contains a point $\bar{p}$ such that for every $p \in X$, $R(\bar{p}, p) \subset X$. In this case we shall refer to X as a *universally monotone set.* Our primary goal is to prove

Received by the editors September 4, 1974.

AMS (MOS) subject classifications (1970). Primary 57A10, 54B15; Secondary 54C10.

Key words and phrases. Upper semicontinuous decomposition, pointlike, E^3, universally monotone set, vertical diameter, ϵ-compression, shrinkable collection.

THEOREM 1. *If G is a u.s.c. decomposition of E^3 into universally monotone sets, then E^3/G is homeomorphic to E^3.*

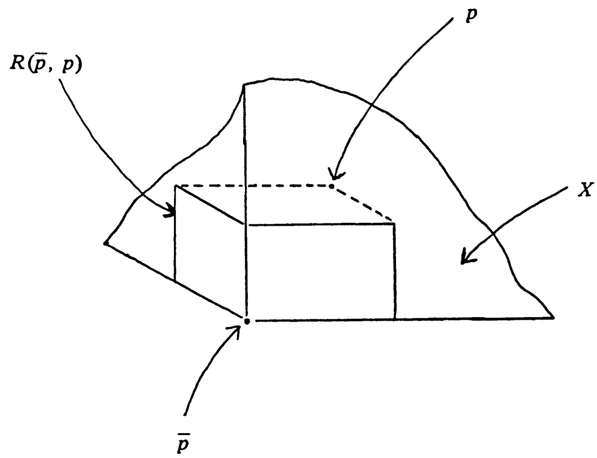

FIGURE 1

It should be observed that the hypothesis of this theorem applies collectively as well as individually to the elements of G. A u.s.c. collection consisting of homeomorphs of universally monotone sets or even of isometric images of such sets may not be topologically equivalent to E^3, as noted above. If the set of nondegenerate elements of G is required to be countable, then this distinction disappears [2, Theorem 2]. In this case the geometric simplicity of the elements alone insures that the decomposition space is equivalent to E^3.

In order to prove Theorem 1 we will show that certain bounded subcollections of the set of nondegenerate elements of a decomposition G are shrinkable in the sense described by McAuley [7]. Roughly speaking, this amounts to proving that there exist homeomorphisms of E^3 onto E^3 which (i) shrink each nondegenerate element of a sufficiently large subcollection of G to a set having very small diameter, and (ii) do not move elements very far with respect to the topology of E^3/G. This is made precise by Lemma 4.1. The lemmas of §3 are designed to facilitate the proof of this key lemma.

The author is indebted to August Lau, who proposed the problem under investigation here, and to Robert Daverman for helpful conversations.

2. **Definitions and notation.** We assume familiarity with basic concepts and terminology of the theory of decomposition spaces. We write E^3/G for the space associated with a decomposition G of E^3. We let π denote the canonical projection map from E^3 onto E^3/G. We will use H^* to denote the set $\pi^{-1}(H)$ for each subset H of G.

We will use d to denote the usual metric on E^3. The set of points within a distance ϵ from a set X will be denoted $N(X, \epsilon)$. The diameter of X is the maximum distance between any two points of X and is denoted by diam X. We will use I, I^2, and I^3 to denote the unit interval $[0, 1]$, the unit square $I \times I$, and the unit cube $I \times I \times I$ respectively. For each $p \in I^2$ put $I_p = p \times I$. For each point $z \in I$ put $P(z) = I^2 \times z$. For each closed subset X of I^3 define:

$$X_p = X \cap I_p; \qquad X(z) = X \cap P(z).$$

If G is a decomposition of I^3 put

$$G(z) = \{g \in G | G \cap P(z) \neq \emptyset\}.$$

Let σ denote the projection of I^3 onto I^2 defined by $\sigma(x_1, x_2, x_3) = (x_1, x_2)$.

Some additional definitions, though somewhat technical, are useful for describing homeomorphisms which shrink the elements of a u.s.c. decomposition G in a very precise manner, and thus embody key concepts of subsequent proofs.

Let $\{h_p\}_{p \in I^2}$ be a family of homeomorphisms h_p of I onto I indexed by the points of I^2. The family $\{h_p\}$ will be called a *continuous family* if and only if, for each $z \in I$, the map h_z defined by $h_z(p) = (p, h_p(z))$ is continuous. If $\{h_p\}_{p \in I^2}$ is a continuous family of homeomorphisms of I, then there is a "canonical" homeomorphism h of I^3 onto I^3 whose restriction to each cross section $P(z)$ is $h_z \circ \sigma$. This homeomorphism will be called a *homeomorphism defined by a continuous family.*

Let X be a compact set in I^3. Put

$$B(X) = \{(s, t) | s \leqslant t \text{ and } X \subset I^2 \times [s, t]\}.$$

The *vertical diameter* of X is defined by the formula:

$$\operatorname{diam}_z X = \operatorname*{Inf}_{(s,t) \in B(X)} t - s.$$

Suppose that G is a u.s.c. decomposition of I^3 into universally monotone sets, ϵ is a positive number, and h is a homeomorphism of I^3 onto I^3 such that

(1) h is defined by a continuous family $\{h_p\}$ of homeomorphisms of I onto I;

(2) for each $g \in G$, if $\operatorname{diam}_z h(g) \geqslant \epsilon$, then $h(g) \subset g$;

(3) for each $p \in I^2$ and $z \in I$, $h_p(z) \leqslant z$.

Then we will refer to h as an *ϵ-compression* of G.

3. **Shrinking the vertical diameter of a compact set.** The lemma below provides a rudimentary tool for shrinking the size of universally monotone sets comprising a decomposition G of I^3.

LEMMA 3.1. *Let G be a u.s.c. decomposition of I^3 into points and vertical line segments such that the sum of the set H of all nondegenerate elements of G is contained in* Int I^3. *Let a and b be real numbers such that* $0 < a < b < 1$. *Let K be a closed subcollection of G and U an open* (*relative to* I^3) *set in* Int I^3, *containing K^*, such that $K \subset G(a)$. Then there exists a homeomorphism h*

defined by a continuous family h_p *such that*

(1) $h(K^*) \subset I^2 \times [0, b]$;

(2) $h|(I^2 \times [0, a]) \cup (I^3 - U)$ *is the identity*; *and*

(3) $h_p(z) \leqslant z$ *for each* $p \in I^2$.

PROOF. We first show that the required homeomorphism h can be defined by means of a continuous function ρ: $I^2 \to I$ having the following properties:

(4) $\rho(x, y) \geqslant a$ for each $(x, y) \in I^2$,

(5) $K^* \cap (I^2 \times [a, 1]) \subset \{(x, y, z)|a \leqslant z \leqslant \rho(x, y)\}$, and

(6) $\{(x, y, z)|a \leqslant z \leqslant \rho(x, y)\} \subset U$.

Let us assume that such a map ρ exists. Then there exist numbers s and t such that $0 < s < 1$, $t > 1$, $\{(x, y, z)|a \leqslant z \leqslant a + s[\rho(x, y) - a]\} \subset I^2 \times [a, b]$, and $\{(x, y, z)|a \leqslant z \leqslant a + t[\rho(x, y) - a]\} \subset U$.

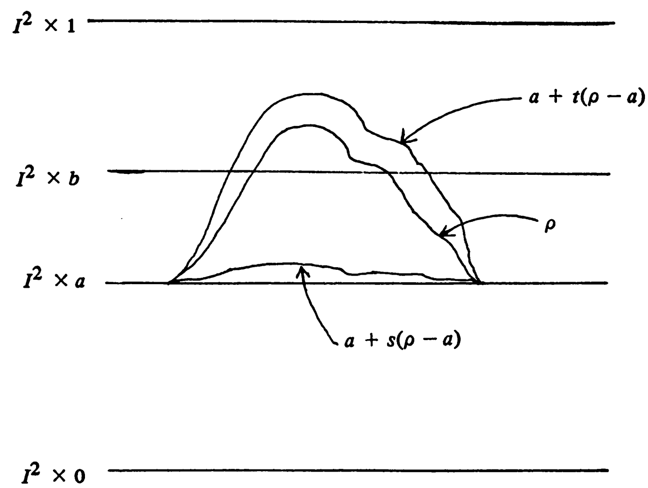

FIGURE 2

Now, for each $p = (x, y)$, h_p is defined by the rule:

$$h_p(z) = \begin{cases} a + s(z - a) & \text{if } a < z \leqslant \rho(x, y), \\ a + s(\rho - a) + (z - \rho)(t - s)/(t - 1) & \\ \qquad \text{if } \rho(x, y) \leqslant z < a + t[\rho(x, y) - a], & \\ z \quad \text{otherwise.} & \end{cases}$$

It is an easy task to verify that we have thus defined a continuous family satisfying the required properties (1), (2), and (3).

In order to define ρ we consider the cross sections $K^*(z)$ for $a \leqslant z < 1$.

For each z there exists a 2-manifold $M(z)$ in Int I^2 and a positive number δ such that

$$K^*(z) \subset \operatorname{Int} M(z) \times z; \qquad M(z) \times [z - \delta, z + \delta] \subset U.$$

Since $K^* \cap (I^2 \times [a, 1])$ is compact, there exist a finite sequence of numbers $a = z_0 < z_1 < z_2 < \cdots < z_{n-1} < z_n < 1$ and a collection of 2-manifolds M_1, $M_2, \ldots, M_n$ such that

$$\bigcup_{a \leqslant z < 1} K^*(z) \subset \bigcup_{i=1}^{n} (\operatorname{Int} M_i \times [z_{i-1}, z_i]) \subset U.$$

In view of the property, $\sigma(K^*(t)) \subset \sigma(K^*(s))$ whenever $t \geqslant s$, we may assume: Int $M_{i-1} \supset M_i$ for $i = 2, \ldots, n$. The sets $M_i \times [z_{i-1}, z_i]$ may form a configuration such as is shown in Figure 3 below. For notational convenience put $M_{n+1} = \varnothing$.

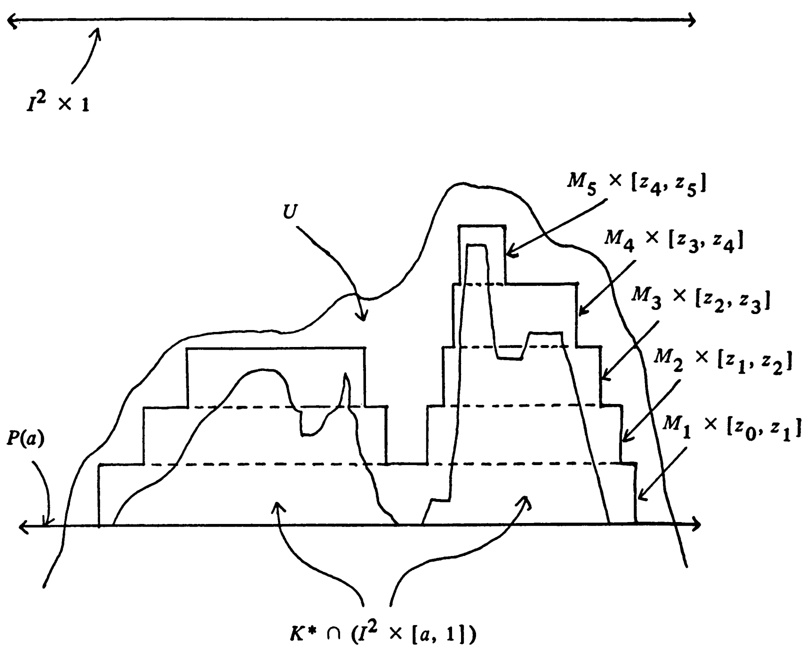

FIGURE 3

For each $i = 1, 2, \ldots, n - 1$ choose a neighborhood R_i of Bd M_i (relative to M_i) such that $R_i \subset M_i - M_{i+1}$; R_i is homeomorphic to (Bd M_i) $\times I$; and $(R_i \times [z_{i-1}, z_i]) \cap K^* = \varnothing$. For each $i = 1, 2, \ldots, n$ let α_i be a homeomorphism of R_i onto (Bd M_i) $\times I$ such that α_i(Bd M_i) = (Bd M_i) $\times 0$. Define β_i on (Bd M_i) $\times I$ by the rule

$$\beta_i(p, r) = r(z_i - z_{i-1}) + z_{i-1}.$$

Now, a map γ_i is defined on each $M_i - M_{i+1}$ by the rule

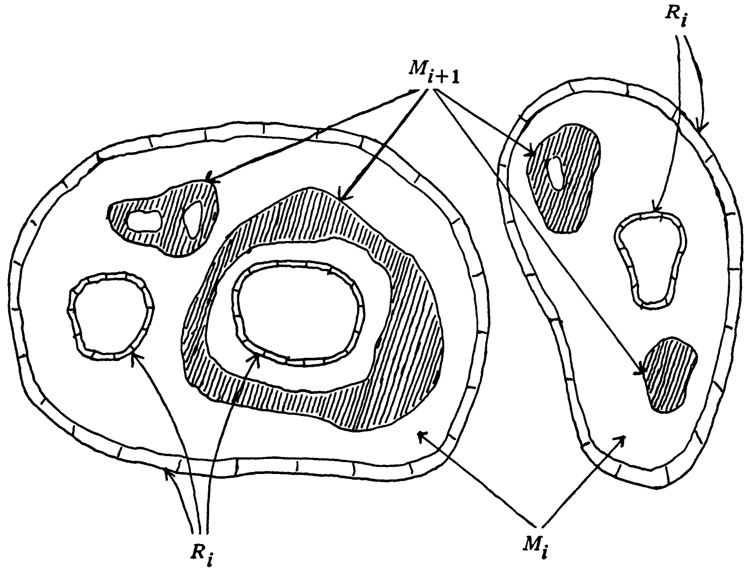

FIGURE 4

$$\gamma_i(p) = \begin{cases} \beta_i \circ \alpha_i(p) & \text{if } p \in R_i, \\ z_i & \text{if } p \notin R_i. \end{cases}$$

Finally, we define ρ by

$$\rho(x, y) = \begin{cases} z_0 & \text{if } (x, y) \in I^2 - M_1, \\ \gamma_i(x, y) & \text{if } (x, y) \in M_i - M_{i+1},\ 1 \leqslant i \leqslant n. \end{cases}$$

It is easy to check that $\rho(x, y)$ is a continuous real function satisfying (4), (5), and (6). This completes the proof.

The next two lemmas are essentially refinements of the first. They provide a way of stating how the elements of a decomposition may be shrunk in the vertical direction in a way that lends itself to repetition. It is interesting to note that the images of universally monotone sets under an ϵ-compression may not themselves be universally monotone. Yet, the images of elements which are sufficiently large will still be somewhat similar to the original elements. One might wonder if there is a way to shrink the elements without distorting them. That is, is there a homeomorphism h of I^3 onto I^3 such that for each g, $\operatorname{diam}_z h(g) < \epsilon$ and $g \cup h(g)$ is in the ϵ-neighborhood of some $g' \in G$, yet $h(g)$ is still a universally monotone set?

LEMMA 3.2. *Suppose that U is an open set and G is a u.s.c. monotone decomposition of I^3 into universally monotone sets such that, for H the set of nondegenerate elements, $H^* \subset U \subset \operatorname{Int} I^3$ and* $\sup_{g \in G}(\operatorname{diam}_z g) \geqslant 2\epsilon$, $\epsilon > 0$; $\delta > 0$; *and suppose that h_0 is a homeomorphism of I^3 onto I^3 such that*

(1) *h_0 is an ϵ-compression of G and*

(2) *$h_0|I^3 - U$ is the identity.*

Then there exists a homeomorphism h of I^3 onto I^3 such that

(3) *$h|I^3 - U$ is the identity;*

(4) *$h \circ h_0$ is an ϵ-compression of G;*

(5) *whenever $h(h_0(g)) \neq h_0(g)$, there exists $g' \in G$ such that $h_0(g) \cup h(h_0(g)) \subset N(h_0(g'), \delta)$; and*

(6) *for each $g \in G$, either* $\operatorname{diam}_z h(h_0(g)) \leqslant \sup_{g \in G}(\operatorname{diam}_z h_0(g)) - \epsilon/4$ *or* $\operatorname{diam}_z h(h_0(g)) < 2\epsilon$.

PROOF. Lemma 3.1 is applied repeatedly to sections of the cube between horizontal planes to obtain a sequence of homeomorphisms $\alpha_1, \alpha_2, \ldots, \alpha_n$ whose composition is the required homeomorphism h.

In order to describe these homeomorphisms precisely, we will specify some notation. Put

$$G' = \{h_0(g) | g \in G\}; \qquad M = \sup_{g \in G'} (\operatorname{diam}_z g).$$

We may assume that $M \geqslant 2\epsilon$; otherwise, conditions (3)–(6) are trivially satisfied by the identity map. Put

$$K = \{g \in G' | M - \epsilon/4 \leqslant \operatorname{diam}_z g \leqslant M\};$$

$$u = \sup\{z \in I | K^* \subset I^2 \times [z, 1]\}; \quad \text{and}$$

$$v = \inf\{z \in I | K^* \subset I^2 \times [0, z]\}.$$

Let n be the least positive integer such that $u + M + n\epsilon/4 \geqslant v$. For each $k = 0, 1, 2, \ldots, n+1$ put $z_k = u + (k-1)\epsilon/4 + (M - \epsilon/4)$. Put $z_{n+2} = v$ and choose z_{n+3} such that $v < z_{n+3} < \min\{1, v + \epsilon/4\}$. For each $j = 1, 2, \ldots, n$ put

$$K_j = \{g \in K | g \subset I^2 \times [z_j - M + \epsilon/4, z_{j+2}]\}.$$

It follows that $\bigcup_{j=1}^n K_j = K$.

We are now in a position to define the sequence $\{\alpha_k\}$ alluded to above. Put $\delta_1 = \delta/n$. Choose an open subset U_1 of U such that

(1.1) $\mathrm{Cl}K_1^* \subset U_1$;

(1.2) if $g \in G'$ and $g \cap U_1 \neq \varnothing$, then there is some $g' \in \mathrm{Cl}K_1$ such that $g \subset N(g', \delta/n)$; and

(1.3) $U_1 \cap g = \varnothing$ for each $g \in G'(z)$, $z \geqslant z_4$.

These properties are routine except for (1.2). To obtain this condition we can use the compactness of $(\mathrm{Cl}K_1)^*$ together with the fact that G' is upper semicontinuous. U_1 is taken to be the sum of neighborhoods of (finitely many) elements $g_1, g_2, \ldots, g_m$, chosen in such a way that, whenever an arbitrary element g intersects

U_1, then g must lie within the distance δ/n of one of the g_i's.

Let G'' denote the decomposition obtained by decomposing $\mathrm{Cl}K_1^*$ into vertical line segments. Now, applying Lemma 3.1 with the collection K of that lemma identified with the collection of all vertical line segments of G'' in $\mathrm{Cl}K_1^*$, U identified with U_1, a with z_0, and b with z_1, we obtain a homeomorphism α_1 such that

(1.4) α_1 is defined by a continuous family $\{\alpha_{1,p}\}$ of homeomorphisms of I;

(1.5) $\alpha_1(K_1^*) \subset I^2 \times [0, z_1]$;

(1.6) $\alpha_1|(I^2 \times [0, z_0]) \cup (I^3 - U_1)$ is the identity;

(1.7) $\alpha_{1,p}(z) \leqslant z$ for each $p \in I^2$ and $z \in I$.

For notational convenience we can also specify: β_0 = identity and

(1.8) $\beta_1 = \alpha_1 = \alpha_1 \circ \beta_0$.

Since α_1^{-1} is uniformly continuous there exists a positive number $\delta_2 \leqslant \delta/n$ such that for any two points p and q of I^3

(2.0) $d(p, q) < \delta/n$ whenever $d(\alpha_1(p), \alpha_1(q)) < \delta_2$.

Proceeding in the same way as above, we can choose an open set U_2 in U such that

(2.1) $\alpha_1(\mathrm{Cl}K_2^*) \subset U_2$;

(2.2) if $\alpha_1(g) \cap U_2 \neq \varnothing$, then there is some $g' \in \mathrm{Cl}K_2$ such that $\alpha_1(g) \subset N(\alpha_1(g'), \delta_2)$; and

(2.3) $U_2 \cap \alpha_1(g) = \varnothing$ if $\alpha_1(g) \cap P(z) \neq \varnothing, z \geqslant z_5$.

Apply Lemma 3.1 again with K identified with the collection consisting of all vertical line segments in $\alpha_1(\mathrm{Cl}K_2^*)$ which intersect $P(z_1)$; $U = U_2$; $a = z_1$; and $b = z_2$. We thereby obtain a homeomorphism α_2 such that

(2.4) α_2 is defined by a continuous family of homeomorphisms $\{\alpha_{2,p}\}$ of I;

(2.5) $\alpha_2(\alpha_1(K_2^*)) \subset I^2 \times [0, z_2]$;

(2.6) $\alpha_2|(I^2 \times [0, z_1]) \cup (I^3 - U_2)$ is the identity;

(2.7) $\alpha_{2,p}(z) \leqslant z$ for each $p \in I^2$ and $z \in I$.

In addition we can define

(2.8) $\beta_2 = \alpha_2 \circ \alpha_1$.

Let m be an integer such that $2 \leqslant m \leqslant n$. Inductively suppose that for each $k = 1, 2, \ldots, m-1$ we have selected a positive number δ_k, an open set U_k in U, and homeomorphisms α_k and β_k such that

(k.0) for each p and q in I^3, $d(p, q) < \delta/n$ whenever $d(\beta_{k-1}(p), \beta_{k-1}(q)) < \delta_k$ and $\delta_k \leqslant \delta/n$;

(k.1) $\beta_{k-1}(\mathrm{Cl}K_k^*) \subset U_k$;

(k.2) if $\beta_{k-1}(g) \cap U_k \neq \varnothing$, then there is some $g' \in \mathrm{Cl}K_k$ such that $\beta_{k-1}(g) \subset N(\beta_{k-1}(g'), \delta_k)$;

(k.3) $U_k \cap \beta_{k-1}(g) = \varnothing$ if $\beta_{k-1}(g) \cap P(z) \neq \varnothing, z \geqslant z_{k+3}$;

(k.4) α_k is defined by a continuous family $\{\alpha_{k,p}\}$ of homeomorphisms of I;

(k.5) $\alpha_k(\beta_{k-1}(K_k^*)) \subset I^2 \times [0, z_k]$;

(k.6) $\alpha_k | (I^2 \times [0, z_{k-1}]) \cup (I^3 - U_k)$ is the identity;

(k.7) $\alpha_{k,p}(z) \leqslant z$ for each $p \in I^2$ and $z \in I$; and

(k.8) $\beta_k = \alpha_k \circ \beta_{k-1}$.

Then we choose $\delta_m > 0$ such that $\delta_m \leqslant \delta/n$ and for each p and q in I^3

(m.0) $d(p, q) < \delta/n$ whenever $d(\beta_{m-1}(p), \beta_{m-1}(q)) < \delta_m$,

and in the same way that we obtained U_1 and U_2, we can choose an open set U_m in U satisfying (m.1), (m.2), and (m.3). By applying Lemma 3.1, with K the collection of vertical segments of $\beta_{m-1}(\mathrm{Cl}K_m^*)$ which intersect $P(z_{m-1})$, $U = U_m$, $a = z_{m-1}$, and $b = z_m$, we obtain a homeomorphism α_m satisfying (m.4)–(m.7). Finally, we define $\beta_m = \alpha_m \circ \beta_{m-1}$.

Thus, we have sequences $\{\delta_m\}$, $\{U_m\}$, $\{\alpha_m\}$, and $\{\beta_m\}$ satisfying (m.0)–(m.8) for every $m = 1, 2, \ldots, n$. We may now identify β_n as the promised homeomorphism h. It remains, of course, to be shown that this map has the required properties (3)–(6). Property (3) is immediate. In order to establish (4) we will make use of the following condition:

(7) if $\mathrm{diam}_z\, g < \epsilon$ then $\mathrm{diam}_z\, h(g) < \epsilon$.

Let us assume for a moment that condition (7) applies to $h = \beta_n$. Since β_n is the composition of $\alpha_1, \alpha_2, \ldots, \alpha_n$ it is clear that β_n is defined by the continuous family $\{\beta_{n,p}\}$ of homeomorphisms of I given by

$$\beta_{n,p} = \alpha_{n,p} \circ \alpha_{n-1,p} \circ \cdots \circ \alpha_{1,p}.$$

Since h_0 is defined by a continuous family, it follows that $\beta_n \circ h_0$ is defined by a continuous family also. It is also clear from (1.7), (2.7), . . . , (n.7) that $\beta_{n,p}(z) \leqslant z$ for each $p \in I^2$ and $z \in I$; hence, that condition (3) of the definition of an ϵ-compression applies to $\beta_n \circ h_0$. Suppose that $\mathrm{diam}_z\, \beta_n(g) \geqslant \epsilon$ for some $g \in G'$ and that condition (7) applies to β_n. Then $\mathrm{diam}_z\, g \geqslant \epsilon$. If, in addition, $\beta_n(g) \subset g$, then, since h_0 is an ϵ-compression, we can conclude that $\beta_n(g) \subset h_0^{-1}(g)$ and thereby conclude that $\beta_n \circ h_0$ is an ϵ-compression.

Let us suppose that $\beta_n | g$ is not the identity. Let k be the least integer such that $\alpha_k | g$ is not the identity. By properties (k.6) and (k.3) we must have $g \cap P(z_{k+3}) = \varnothing$. Then since $\mathrm{diam}_z\, g \geqslant \epsilon$ (assumed) we must have $g \cap P(z_{k-1}) \neq \varnothing$. Now, since $g \subset h_0^{-1}(g)$, $h_0^{-1}(g)$ is a universally monotone set, and h_0 does not increase vertical coordinates; there exists a number $w \leqslant z_{k-1}$ such that

$$g \cap P(w) = h_0^{-1}(g) \cap P(w) \neq \varnothing \quad \text{and} \quad g \subset \{(x, y, z) \in I^3 | z \geqslant w\}.$$

Thus, from properties (k.6) and (k.7) it follows that $\alpha_k(g) \subset g$. By similar reasoning we must have

$$\alpha_{k+2} \circ \alpha_{k+1} \circ \alpha_k(g) \subset \alpha_{k+1}(\alpha_k(g)) \subset \alpha_k(g).$$

Then, since $\alpha_r|g$ is the identity for $r \geqslant k+3$, we finally conclude that $\beta_n(g) \subset g$.

To establish property (7) suppose that $\operatorname{diam}_z g < \epsilon$ for some $g \in G'$. We can suppose as above that $\beta_n|g$ is not the identity and that k is the least integer such that $\alpha_k|g$ is not the identity. Therefore $g \cap P(z) = \varnothing$ for each $z \geqslant z_{k+3}$. If $g \cap P(z_{k-1}) \neq \varnothing$ then we can conclude, using an argument similar to that above, that $\operatorname{diam}_z \beta_n(g) < \epsilon$ as desired. If $g \cap P(z_{k-1}) = \varnothing$ then, by (k.3) and (k.6),

$$\alpha_k(g) \subset I^2 \times (z_{k-1}, z_{k+3}).$$

Moreover, by (k + 1.6), (k + 2.6), . . . , (n.6) and by (k + 1.7), (k + 2.7), . . . , (n.7) it follows that

$$\operatorname{diam}_z \beta_n(g) < \epsilon.$$

To establish property (5) we make use of conditions (k.0), (k.2), (k.6), and (k.7) for $k = 1, 2, \ldots, n$ via an induction argument. We may observe that the desired property is a consequence of the following hypothesis for $k = n$.

(k.9) For each $g \in G'$, if $\beta_k(g) \neq g$, then, for some $r \leqslant k$, there exists $g' \in \mathrm{Cl}K_r$ such that $g \cup \beta_k(g) \subset N(g', k\delta/n)$.

Put $k = 1$. Assume that $\alpha_1(g) \neq g$ for some g. Then by (1.2) and (1.6) there is some $g' \in \mathrm{Cl}K_1$ such that $g \subset N(g', \delta/n)$. More specifically, for each point $p \in g$ there is some $q \in I^2$ such that $p \in N(g'_q, \delta/n)$ and $g'_q \cap P(z_0) \neq \varnothing$. If $\alpha_1(p) \neq p$ then $\alpha_1(p) \in I^2 \times [z_0, z_4]$. Thus by (1.7) we conclude that

$$d(\alpha_1(p), g'_q) \leqslant d(p, g'_q) < \delta/n.$$

Therefore, $\alpha_1(g) \subset N(g', \delta/n)$ as desired.

Suppose for some m that (k.9) is true whenever $1 \leqslant k \leqslant m$. Let $g \in G'$ be such that $\beta_{m+1}(g) \neq g$. If $\beta_{m+1}(g) = \beta_m(g)$ then it must be the case that $\beta_m(g) \neq g$. Then by the induction hypothesis there is some r and an element g' of $\mathrm{Cl}K_r$ such that $g \cup \beta_m(g) \subset N(g', m\delta/n)$. Thus, ($m$ + 1.9) is satisfied in this special case. If $\beta_{m+1}(g) \neq \beta_m(g)$, then ($m$ + 1.2) and (m + 1.6) imply that there is some $g' \in \mathrm{Cl}K_{m+1}$ such that $\beta_m(g) \subset N(\beta_m(g'), \delta_{m+1})$. Using the argument above we can conclude in addition that

$$\alpha_{m+1}(\beta_m(g)) \subset N(\beta_m(g'), \delta_{m+1}).$$

Moreover, (m + 1.0) implies that $g \subset N(g', \delta/n)$. If $\beta_m(g') = g'$ then (m + 1.9) is established. If not, the induction hypothesis implies that there is some $r \leqslant m$ and some $g'' \in \mathrm{Cl}K_r$ such that $g' \cup \beta_m(g') \subset N(g'', m\delta/n)$. Thus, since $g \subset N(g'', (m+1)\delta/n)$ and $\beta_{m+1}(g) \subset N(g'', (m+1)\delta/n)$, this g'' is the required element showing that (m + 1.9) is valid. We conclude that (k.9) is true for each $k = 1, 2, \ldots, n$. As observed earlier, the particular case (n.9) establishes property (5).

To establish property (6) it suffices to show that

$$\operatorname{diam}_z \beta_n(g) \leqslant M - \epsilon/4$$

for each $g \in K$. For all other elements of G' this property is covered by (7) or by the argument above that $\beta_n(g) \subset g$ whenever $\operatorname{diam}_z \beta_n(g) \geqslant \epsilon$. Suppose then that $g \in K_j$ for some j. Then $g \subset I^2 \times [z_j - M + \epsilon/4, z_{j+2}]$. Condition ($k$.6), $k \leqslant j$, implies that $\beta_j(g) \subset I^2 \times [z_j - M + \epsilon/4, 1]$. Now, for $m = j + 1, j + 2, \ldots, n$, condition ($m$.6) implies that $\beta_m(g) = \beta_j(g)$. From this it follows that

$$\beta_n(g) \subset I^2 \times [z_j - M + \epsilon/4, z_j] \quad \text{or} \quad \operatorname{diam}_z \beta_n(g) \leqslant M - \epsilon/4.$$

This completes the proof of Lemma 3.2.

LEMMA 3.3. *Suppose that G is a u.s.c. decomposition of I^3 into universally monotone sets, H is the set of nondegenerate elements of G, and U is an open set such that $H^* \subset U \subset \operatorname{Int} I^3$. Let ϵ be a positive number. Then there exists an $\epsilon/2$-compression h of G such that*

(1) $\operatorname{diam}_z h(g) < \epsilon$ *for each* $g \in G$;

(2) $h|I^3 - U$ *is the identity; and*

(3) *if $h(g) \neq g$ for some $g \in G$, then there exists $g' \in G$ such that $g \cup h(g) \subset N(g', \epsilon)$.*

PROOF. The existence of h is established by an induction argument similar to the approach used in the proof of Lemma 3.2.

Let h_0 denote the identity map. Put

$$M = \sup_{g \in G} (\operatorname{diam}_z g).$$

Let n denote the least positive integer such that $M - n\epsilon/8 < \epsilon$. Suppose that for each $k = 0, 1, 2, \ldots, n$ there exists a homeomorphism h_k having the following properties.

(k.1) $h_k|I^3 - U$ is the identity;

(k.2) h_k is an $\epsilon/2$-compression of G;

(k.3) for each $g \in G$, $\operatorname{diam}_z h_k(g) \leqslant \max\{m - k\epsilon/8; \epsilon\}$;

(k.4) if $h_k(g) \neq g$, then there exists $g' \in G$ such that $g \cup h_k(g) \subset N(g', (k + 1)\epsilon/(n + 1))$. Then the homeomorphism h_n, by properties (n.1)–(n.4), is an $\epsilon/2$-compression satisfying (1), (2), and (3) if it is substituted for h. We use induction to establish the existence of the h_k's satisfying (k.1)–(k.4).

We have specified h_0 already. Suppose that h_k exists, satisfying (k.1)–(k.4) for some k such that $0 < k \leqslant n - 1$. Then we seek a homeomorphism h_{k+1} satisfying ($k + 1$.1)–($k + 1$.4). If

$$\sup_{g \in G} [\operatorname{diam}_z h_k(g)] < \epsilon,$$

then we put $h_{k+1} = h_k$. Suppose that

$$\sup_{g \in G} [\operatorname{diam}_z h_k(g)] \geqslant \epsilon.$$

Using the uniform continuity of h_k^{-1}, choose $\delta_k \leqslant \epsilon/(n+1)$ such that

$$d(p, q) < \epsilon/(n+1) \quad \text{whenever } d(h_k(p), h_k(q)) < \delta_k.$$

Now, apply Lemma 3.2 to the decomposition G, putting h_k for h_0, $\epsilon/2$ for ϵ, and δ_k for the number δ. By Lemma 3.2 there is a homeomorphism α_k having the properties:

(k.5) $\alpha_k | I^3 - U$ is the identity;

(k.6) $\alpha_k \circ h_k$ is an $\epsilon/2$-compression of G;

(k.7) if $\alpha_k(h_k(g)) \neq h_k(g)$, $g \in G$, then there exists $g' \in G$ such that $h_k(g) \cup \alpha_k(h_k(g)) \subset N(h_k(g'), \delta_k)$; and

(k.8) for each $g \in G$, $\operatorname{diam}_z(\alpha_k h_k(g)) \leqslant M - (k+1)\epsilon/8$.

Put $h_{k+1} = \alpha_k \circ h_k$. It is immediate from ($k$.1), ($k$.5), ($k$.6), and ($k$.8) that h_{k+1} satisfies (k + 1.1), (k + 1.2), and (k + 1.3). Property (k + 1.4) can be verified by making use of (k.7) and (k.4) in a manner very similar to a technique used above in the proof of Lemma 3.2. We omit further details of this.

4. **The proof of Theorem 1.** We take full advantage of the geometric simplicity of universally monotone sets in order to prove the conclusive lemma below, which sets the stage for the proof of our main result.

Lest the technical aspects of the proof obscure the main idea, we describe it briefly in advance. Lemma 3.3 provides a homeomorphism which will shrink vertical line segments contained in a universally monotone set to very short segments. By rotating the cube I^3 we can cause line segments parallel to the x-axis or the y-axis to be shrunk as well. However, we must find a way to produce each of these effects by one map. If, say, a homeomorphism h_1 has been defined that will shrink line segments parallel to the z-axis, then we wish to define homeomorphisms h_2 and h_3 that will shrink line segments parallel to the y-axis and x-axis respectively in such a way that $h_1 \circ h_2 \circ h_3$ will shrink line segments of all three kinds. We utilize a sort of "satellite" effect, taking care to see that each element g moved by h_2 remains very close to some other element g' of the decomposition or close to itself, so close, in fact, that the first homeomorphism h_1 has nearly the same effect upon $h_2(g)$ as it has on g'. A similar consideration is apparent in the application of Lemma 3.3 to obtain h_3.

Before we state Lemma 4.1 it is convenient to introduce some additional notation. Let γ_1 and γ_2 denote the isometries of I^3 defined by: $\gamma_1(x, y, z) = (x, z, y)$; $\gamma_2(x, y, z) = (z, y, x)$. Let X be any closed subset of I^3. Put

$$\operatorname{diam}_y X = \operatorname{diam}_z \gamma_1(X) \quad \text{and} \quad \operatorname{diam}_x X = \operatorname{diam}_z \gamma_2(X).$$

The map γ_1 rotates segments, parallel to the y-axis, onto vertical segments; γ_2 converts segments, parallel to the x-axis, into vertical segments. Under either of the two maps the image of a universally monotone set is also a universally monotone set.

LEMMA 4.1. *Let G, H, U, and ϵ be as in the hypothesis of Lemma* 3.3. *Then there exists a homeomorphism h of I^3 onto I^3 such that*

(1) *$h|I^3 - U$ is the identity;*

(2) diam *$h(g) < \epsilon$ for every $g \in G$; and*

(3) *if $h(g) \neq g, g \in G$, there exists $g' \in G$ such that $g \cup h(g) \subset N(g', \epsilon)$.*

PROOF. The homeomorphism h is a composition of three homeomorphisms h_1, h_2, and h_3, each of which squeezes the elements of G in one of the three directions perpendicular to the faces of the cube. Put $\epsilon_1 = \sqrt{3}\epsilon/15$. Apply Lemma 3.3 to obtain an $\epsilon_1/2$-compression h_1 of G such that

(4) $h_1|I^3 - U$ is the identity;

(5) $\operatorname{diam}_z h_1(g) < \epsilon_1$ for each $g \in G$;

(6) if $h_1(g) \neq g$, then there exists $g' \in G$ such that $g \cup h_1(g) \subset N(g', \epsilon_1)$.

Choose ϵ_2 such that $0 < \epsilon_2 \leqslant \epsilon_1$ and for $p, q \in I^3$

$$d(h_1(p), h_1(q)) < \epsilon_1 \quad \text{whenever } d(p, q) < \epsilon_2.$$

Apply Lemma 3.3 to the decomposition G' whose elements are of the form $\gamma_1(g), g \in G$, and the number ϵ_2. We obtain an $\epsilon_2/2$-compression h_2' of G' such that

(7) $h_2'|I^3 - \gamma_1(U)$ is the identity;

(8) $\operatorname{diam}_z(h_2' \circ \gamma_1(g)) < \epsilon_2$ for each $g \in G$;

(9) if $h_2'(\gamma_1(g)) \neq \gamma_1(g), g \in G$, then there exists $g' \in G$ such that $\gamma_1(g) \cup h_2'(\gamma_1(g)) \subset N(\gamma_1(g'), \epsilon_2)$.

Put $h_2 = \gamma_1 \circ h_2' \circ \gamma_1$. Since γ_1 is an isometry and h_1 is an $\epsilon_1/2$-compression, it follows from (8) that

(10) $\operatorname{diam}_y(h_1 \circ h_2(g)) < \epsilon_2$ for each $g \in G$.

If $h_2'(\gamma_1(g)) = \gamma_1(g)$ for some $g \in G$, then $h_2(g) = g$ and thus $\operatorname{diam}_z(h_1 \circ h_2(g)) < \epsilon_1$ by condition (5). Suppose that $h_2'(\gamma_1(g)) \neq \gamma_1(g)$ for some $g \in G$. Then by property (9), $h_2(g) \subset N(g', \epsilon_2)$ for some $g' \in G$ and, by the choice of ϵ_2, $h_1(h_2(g)) \subset N(h_1(g'), \epsilon_1)$. Thus, by (5) we have

(11) $\operatorname{diam}_z(h_1 \circ h_2(g)) < 3\epsilon_1$.

Another key property of the composition $h_1 \circ h_2$ is:

(12) if $h_1(h_2(g)) \neq g$, then there exists $g' \in G$ such that $g \cup h_1(h_2(g)) \subset N(g', 2\epsilon_1)$.

If $h_2(g) = g$ this is an immediate consequence of (6). If $h_2(g) \neq g$ then $h_2'(\gamma_1(g)) \neq \gamma_1(g)$ and condition (9) implies that there is some $g'' \in G$ such that

$g \cup h_2(g) \subset N(g'', \epsilon_2)$. If $h_1(g'') = g''$ then $h_1(h_2(g)) \subset N(g'', \epsilon_1)$ by the choice of ϵ_2, and thus g'' is the required g' in this case. If $h_1(g'') \neq g''$, then (6) implies that there is some g' such that $h_1(g'') \subset N(g', \epsilon_1)$ and $g'' \subset N(g', \epsilon_1)$. Thus, by the way in which g'' was chosen, we have

$$g \subset N(g', \epsilon_2 + \epsilon_1) \quad \text{and} \quad h_1(h_2(g)) \subset N(g', 2\epsilon_1),$$

and (12) is established.

Choose ϵ_3 such that $0 < \epsilon_3 \leqslant \epsilon_1$ and for $p, q \in I^3$,

$$d(h_2(p), h_2(q)) < \epsilon_2 \quad \text{whenever } d(p, q) < \epsilon_3.$$

By applying Lemma 3.3 in much the same way as before we obtain an $\epsilon_3/2$-compression h_3' of the decomposition whose elements are of the form $\gamma_2(g), g \in G$, such that

(13) $h_3'|I^3 - \gamma_2(U)$ is the identity;

(14) $\operatorname{diam}_z(h_3'(\gamma_2(g))) < \epsilon_3$ for each $g \in G$; and

(15) if $h_3'(\gamma_2(g)) \neq \gamma_2(g)$, then there exists $g' \in G$ such that $\gamma_2(g) \cup h_3'(\gamma_2(g)) \subset N(\gamma_2(g'), \epsilon_3)$.

We put $h_3 = \gamma_2 \circ h_3' \circ \gamma_2$. It follows from (14) and the properties of h_1 and h_2 that

(16) $\operatorname{diam}_x(h_1 \circ h_2 \circ h_3(g)) < \epsilon_3$ for each $g \in G$.

From (8), (15), the choice of ϵ_3, and the nature of h_1 it follows that

(17) $\operatorname{diam}_y(h_1 \circ h_2 \circ h_3(g)) < 3\epsilon_2$ for each $g \in G$.

From (11), (15), and the choice of ϵ_2 and ϵ_3 it follows that

(18) $\operatorname{diam}_z(h_1 \circ h_2 \circ h_3(g)) < 5\epsilon_1$ for each g.

Using (12) and an argument similar to the one used to verify that property, we can establish:

(19) if $h_1(h_2(h_3(g))) \neq g$, there exists $g' \in G$ such that $g \cup h_1(h_2(h_3(g))) \subset N(g', 3\epsilon_1)$.

Reviewing the definitions of $\epsilon_1, \epsilon_2, \epsilon_3$ and conditions (16), (17), (18), (19), (13), (7), and (4); we may observe that $h_1 \circ h_2 \circ h_3$ is the promised homeomorphism h satisfying (1), (2), and (3). The proof is therefore complete.

It is helpful to observe at this point that the conclusion of Lemma 4.1 remains valid if the unit cube I^3 in the hypothesis is replaced by a cube C, each of whose edges is parallel to one of the coordinate axes. We can also observe that the conclusion of Lemma 4.1 is equivalent in the present situation to the criterion of shrinkability stated by McAuley in [7]. To show this is a technical exercise which we leave to the interested reader. Lemma 4.1 combined with Theorem 2 of [7] yields the following result.

THEOREM 4.2. *Let C be a cube in E^3, each of whose edges is parallel to one of the coordinate axes. Let G be a u.s.c. decomposition of C into universally* *monotone sets such that the union of the nondegenerate elements of G is contained in the interior of C. Then C/G is homeomorphic to* I^3.

From this theorem one can easily prove that the space associated with a u.s.c. decomposition G of E^3 into universally monotone sets is a 3-manifold and thus, by a result of Armentrout [1], homeomorphic to E^3. To this end let g be an arbitrary element of G. Choose geometric cubes C_1 and C_2, each of whose faces is parallel to one of the coordinate planes, such that $g \subset \text{Int } C_1$ and, for each $g' \in G$, $g' \subset \text{Int } C_2$ whenever $g' \cap C_1 \neq \emptyset$. Put

$$U = \{g' \in G | g' \subset \text{Int } C_1\},$$

$$K = \{g' \in G | g' \cap C_1 \neq \emptyset\}, \text{ and}$$

$$G' = K \cup (C_2 - K^*).$$

Theorem 4.2, applied to C_2 and G', gives a homeomorphism of C_2/G' onto I^3. Since $h(U)$ is an open subset of Int I^3 containing $h(g)$, we conclude that U contains a neighborhood of g which is homeomorphic to an open ball; hence, that E^3/G is a 3-manifold and homeomorphic to E^3.

An obvious generalization of Theorem 1 to a class of decompositions of E^n $(n > 3)$ can be proved in much the same way as we have proceeded here. A universally monotone set in E^n would be defined as in §1, with the exception that the partial ordering would be based upon n inequalities involving n coordinates. The answer to each question below is presently unknown to the author.

Question 1. Suppose that $\{\gamma_i\}$ is a sequence of isometries of E^3 onto E^3 and G is a monotone decomposition of E^3 into compact sets such that, for each $g \in G$, there is some integer n such that $\gamma_n(g)$ is a universally monotone set. Is E^3/G homeomorphic to E^3?

Question 2. One might say that a compact set X in E^3 is collapsible onto a plane P if for each point $q \in X$, either $q \in P$ or the line segment perpendicular to P and joining q to P is contained in X. Suppose that E^3 is decomposed into pointlike sets each of which is collapsible onto a horizontal plane (or perhaps one of a countable family of planes). Is the resulting decomposition space homeomorphic to E^3?

The latter question has an affirmative answer, given by Dyer and Hamstrom [4], in the special case that each element is a subset of some horizontal plane. An affirmative answer to this question would yield the main theorem of this paper as a corollary.

REFERENCES

1. S. Armentrout, *Cellular decompositions of* 3-*manifolds that yield* 3-*manifolds*, Mem. Amer. Math. Soc. No. 107 (1971).

2. R. H. Bing, *Upper semicontinuous decompositions of E^3*, Ann. of Math. (2) 65 (1957), 363–374. MR 19, 1187.

3. ———, *Decompositions of E^3*, Topology of 3-manifolds and related topics (Proc. the Univ. of Georgia Institute, 1961), Prentice-Hall, Englewood Cliffs, N. J., 1962, pp. 5–21. MR 25 #4501.

4. E. Dyer and M.-E. Hamstrom, *Completely regular mappings*, Fund. Math. 45 (1958), 103–118. MR 19, 1187.

5. W. T. Eaton, *Applications of a mis-match theorem to decomposition spaces*, Fund. Math. (to appear).

6. L. F. McAuley, *Some upper semi-continuous decompositions of E^3 into E^3*, Ann. of Math. (2) 73 (1961), 437–457. MR 23 #A3554.

7. ———, *Upper semicontinuous decompositions of E^3 into E^3 and generalizations to metric spaces*, Topology of 3-manifolds and related topics (Proc. the Univ. of Georgia Institute, 1961), Prentice-Hall, Englewood Cliffs, N. J., 1962, pp. 21–26. MR 25 #4502.

DEPARTMENT OF MATHEMATICS, PEMBROKE STATE UNIVERSITY, PEMBROKE, NORTH CAROLINA 28372