

EXISTENCE OF PERIODIC SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS⁽¹⁾

BY

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ABSTRACT. The nonlinear differential equation $x'' = f(t, x(t))$, f being 2π -periodic in t , is considered for the existence of 2π -periodic solutions. The equation is reduced to an equivalent system of two Hammerstein equations. The case of nonlinear perturbation at resonance is also discussed.

1. Introduction. In this paper we study the existence of 2π -periodic solutions of the nonlinear differential equation

$$(1) \quad x'' = f(t, x(t))$$

where $x(t) = (x_1(t), x_2(t), \dots, x_n(t))$ and f is 2π -periodic (without loss of generality) in t . This paper has been motivated by the recent papers of Loud [10], Lazer and Sanchez [8], Leach [9] and others in nonlinear ordinary differential equations and by the works of Cesari [2] and Hale [4] in alternative problems for nonlinear differential systems.

Equations of the type

$$(2) \quad Ex = Nx$$

over a Hilbert space S , where E is a linear differential operator with preassigned homogeneous boundary conditions such that the null space of E is nontrivial and $N: S \rightarrow S$ is a nonlinear monotone operator, i.e., $\langle Nx - Ny, x - y \rangle \geq 0$ for all $x, y \in S$, were considered in [3]. Equation (2) was reduced to an equivalent system of two equations. One of the equations was over the complement of the null space of E and the other was over the null space of E . This Lyapunov-Schmidt technique of splitting a nonlinear differential equation into an equivalent system of two equations was extensively studied in terms of functional analysis by Cesari, and this work was continued by Hale and others. It has found a wide variety of applications to both ordinary and partial differential equations. For a detailed survey of this technique and its applications one is referred to Cesari [2],

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Hale [4], [5]. Our aim in this paper is to continue the technique developed in [3] and apply it to equations of type (1) where the nonlinear Nemytskii operator $(Nx)(t) = f(t, x(t))$ is not monotone.

In this paper we point out how the reduced equivalent system of two equations can be considered as equations of Hammerstein type. We then draw upon the results on existence of solutions of nonlinear operator equations of Hammerstein type. We would like to point out, however, that throughout this paper we are making the hypothesis that the operator N defined by $Nx(t) = f(t, x(t))$ is defined over the whole Hilbert space S . In the case when the domain of definition of N is not the whole of S the technique as presented in this paper needs to be modified to include a wider class of nonlinearities. These details will be discussed in a subsequent paper.

In §2, we outline the general theory for an abstract equation of type (2) thereby obtaining the equivalent system of two equations. These equations are called the auxiliary and bifurcation equations. §3 of this paper deals with the solvability of the auxiliary equation. We apply here a variant of the Schauder principle of invariance of domain. The continuous dependence on the parameter is the subject of study in §4. We show in §5 that the bifurcation equation can be solved by using results from the theory of monotone operators. In particular we consider nonlinear differential equations of the type $Eu + \mu u = Nu$ where $\mu \in (\lambda_m, \lambda_{m+1})$, λ_m and λ_{m+1} being two consecutive eigenvalues of the associated boundary value problem. This leads to the case of nonlinear perturbation at resonance. Hence, after summarizing the results of the earlier section in §6, we consider in §7 a nonlinear differential equation of the type $Eu + \lambda u = Nu$ where N is a nonlinear operator and λ is an eigenvalue of the associated boundary value problem. We conclude the paper with some examples in §8.

2. Outline of the general theory. We consider the problem of existence of 2π -periodic solutions of period 2π in t of the system of equations

$$(1) \quad x'' = f(t, x(t)),$$

$x(t) = (x_1(t), \dots, x_n(t))$, $f: R \times R^n \rightarrow R^n$ is a function with $f(t + 2\pi, x) = f(t, x)$.

Let S be the real Hilbert space of all n -vector functions $x(t) = (x_1, \dots, x_n)$, $0 \leq t \leq 2\pi$, $x \in (L_2[0, 2\pi])^n$. Let S_E denote the linear subspace of S of all $x \in S$ which are absolutely continuous together with the derivative x' and with $x'' \in (L_2[0, 2\pi])^n$, $x(0) = x(2\pi)$, $x'(0) = x'(2\pi)$. Let $E: S_E \rightarrow S$ be defined by $Ex = x''$. And let $N: S \rightarrow S$ be the Nemytskii operator defined by $(Nx)(t) = f(t, x(t))$. In this paper we will assume that the hypotheses on f are such that $N: S \rightarrow S$.

The linear associated problem $Ex + \lambda x = 0$, with boundary conditions $x(0) = x(2\pi)$, $x'(0) = x'(2\pi)$ has a countable system of eigenvalues λ_i with corresponding eigenfunctions $\{\varphi_i\}$, which form a complete orthonormal system in S . Also we have $\lambda_i \geq 0$ and $\lambda_i \rightarrow \infty$. Let $P: S \rightarrow S_0$ be the projection operator with $S_0 = \{\varphi_1, \dots, \varphi_m\}$, so that if $x = \sum_1^\infty c_i \varphi_i \in S$ then $Px = \sum_1^m c_i \varphi_i$, where m is a number ≥ 1 to be suitably chosen. Let $S_1 = (I - P)S$ and let $H: S_1 \rightarrow S_1$ be the linear operator defined by $Hx = -\sum_{m+1}^\infty c_i \lambda_i^{-1} \varphi_i$ for $x = \sum_{m+1}^\infty c_i \varphi_i \in S_1$, m being such that $\lambda_{m+1} > 0$.

With these notations we see that

$$(3) \quad H(I - P)Ex = (I - P)x \quad \text{for } x \in S_E,$$

$$(4) \quad EH(I - P)Nx = (I - P)Nx \quad \text{for } x \in S,$$

$$(5) \quad PEx = EPx \quad \text{for } x \in S_E.$$

Equation (1) can now be written as

$$(2) \quad Ex = Nx.$$

If $x \in S_E$ satisfies (2), then, by applying $H(I - P)$ and by force of (3), we obtain $(I - P)x = H(I - P)Nx$ and thus a solution $x \in S_E$ of (2) is also a solution of

$$(6) \quad x - H(I - P)Nx = Px.$$

And if x is a solution of (6), by applying E to both sides of (6) we have, by virtue of (4) and (5),

$$Ex - (I - P)Nx = EPx = PEx \quad \text{or} \quad Ex - Nx = P(Ex - Nx).$$

Thus a solution of (6) is a solution of (2) if and only if

$$(7) \quad P(Ex - Nx) = 0.$$

Thus we conclude that solving equation (2) is equivalent to solving the system of equations (6) and (7).

Let now x^* be any arbitrary element of S_0 . If the equation

$$(8) \quad x - H(I - P)Nx = x^*$$

is uniquely solvable for each $x^* \in S_0$, then the solution x is such that $Px = x^*$. And $PEx = EPx = Ex^*$. Thus equation (7) reduces to

$$(9) \quad PN[I - H(I - P)N]^{-1}x^* - Ex^* = 0.$$

Hence if equation (8) is uniquely solvable for each $x^* \in S_0$, solving equation (1) is equivalent to solving the system of equations (8) and (9). Equations (8) and (9) are called the auxiliary and bifurcation equations, respectively.

3. **Solvability of the auxiliary equation (8).** We now consider the solvability of the auxiliary equation (8). We first recall that if $x = \sum_1^\infty c_i \varphi_i \in S$ then $-H(I - P)x = \sum_{i=m+1}^\infty c_i \lambda_i^{-1} \varphi_i$ and, as observed in [2] and [3], we have

$$(10) \quad \langle -H(I - P)x, x \rangle \geq \lambda_{m+1} \| -H(I - P)x \|^2$$

and

$$(11) \quad \| -H(I - P)x \| \leq \lambda_{m+1}^{-1} \| x \|.$$

Also $-H(I - P): S \rightarrow S$ is linear and compact. Thus if the operator $N: S \rightarrow S$ is continuous and bounded (i.e., N takes bounded sets into bounded sets), then $-H(I - P)N: S \rightarrow S$ is compact. Hence (8) reduces to an equation of the type $(I + T)x = x^*$ where $T: S \rightarrow S$ is compact. We now state the following lemma which is a variant of the Schauder principle of invariance of domain.

LEMMA 1. *If $T: S \rightarrow S$ is compact, $(I + T)$ is one-to-one and $(I + T)^{-1}$ is bounded (i.e., there exists a continuous function $k: R^+ \rightarrow R^+$ such that if $(I + T)u = v$ then $\|u\| \leq k(\|v\|)$), then for each $v \in S$ there exists exactly one solution $u \in S$ of the equation $(I + T)u = v$.*

The proof of the lemma may be seen in Nagumo [12].

PROPOSITION 1. *If the Nemytskii operator N defined by $(Nx)(t) = f(t, x(t))$ is such that $N: S \rightarrow S$ and*

- (i) *N is continuous and bounded,*
- (ii) *there exist $p, m > 0$ such that for all $u, v \in S$*

$$\langle Nu - Nv, u - v \rangle \geq -p \|u - v\|^2, \quad p < \lambda_{m+1},$$

then the operator $[I - H(I - P)N]^{-1}$ is bounded and the auxiliary equation (8) has a unique 2π -periodic solution in S for each $x^ \in S_0$.*

PROOF. By virtue of the above lemma, it suffices to prove that $I - H(I - P)N$ is one-to-one and $[I - H(I - P)N]^{-1}$ is bounded. If possible let

$$u - H(I - P)Nu = v - H(I - P)Nv, \quad u, v \in S.$$

Then, by (10),

$$\begin{aligned} \langle Nu - Nv, u - v \rangle &= -\langle Nu - Nv, -H(I - P)Nu + H(I - P)Nv \rangle \\ &\leq -\lambda_{m+1} \| -H(I - P)Nu + H(I - P)Nv \|^2 \\ &= -\lambda_{m+1} \| u - v \|^2. \end{aligned}$$

But this contradicts hypothesis (ii) of the proposition. Thus $[I - H(I - P)N]$ is one-to-one.

Now let $[I - H(I - P)N]^{-1}w = u$, $\|w\| \leq R$. Then, by (10),

$$\begin{aligned}
\lambda_{m+1} \|w - u\|^2 &= \lambda_{m+1} \| -H(I - P)Nu \|^2 \\
&\leq \langle Nu, -H(I - P)Nu \rangle \\
&= \langle Nu, w - u \rangle \\
&= -\langle Nw - Nu, w - u \rangle + \langle Nw, w - u \rangle.
\end{aligned}$$

By hypothesis (ii) we have now

$$\lambda_{m+1} \|w - u\|^2 \leq p \|w - u\|^2 + \langle Nw, w - u \rangle.$$

Thus $(\lambda_{m+1} - p) \|w - u\|^2 \leq \|Nw\| \|w - u\|$, and hence $\|u\| \leq \|w\| + (\lambda_{m+1} - p)^{-1} \|Nw\|$. Since N is bounded and $p < \lambda_{m+1}$, it follows that $\|u\| \leq A(R)$ if $\|w\| \leq R$, i.e., $[I - H(I - P)N]^{-1}$ is bounded.

We now apply the lemma and conclude that the auxiliary equation (8) has a unique solution for each $x^* \in S_0$. This proves Proposition 1.

4. Continuity of $[I - H(I - P)N]^{-1}$. The two lemmas of this section show the continuity of $[I - H(I - P)N]^{-1}$ under two distinct hypotheses.

LEMMA 2. Let $0 < t < \lambda_{m+1}$ be such that $\|Nu - Nv\| \leq t \|u - v\|$, $u, v \in S$. Then $[I - H(I - P)N]^{-1}$ is continuous.

PROOF. Let $[I - H(I - P)N]^{-1}x^* = u$ and $[I - H(I - P)N]^{-1}y^* = v$. Then, by (11) and the hypothesis of the lemma, we have

$$\begin{aligned}
\|u - v\| &= \|(x^* - y^*) - (-H(I - P)Nu + H(I - P)Nv)\| \\
&\leq \|x^* - y^*\| + \| -H(I - P)(Nu - Nv) \| \\
&\leq \|x^* - y^*\| + t\lambda_{m+1}^{-1} \|u - v\|.
\end{aligned}$$

Then, $\|u - v\| \leq (1 - t\lambda_{m+1}^{-1}) \|x^* - y^*\|$ and this proves the lemma.

LEMMA 3. Let $N: S \rightarrow S$ be such that

- (i) there exists $p < \lambda_{m+1}$ such that, for all $u, v \in S$, $\langle Nu - Nv, u - v \rangle \geq -p \|u - v\|^2$;
- (ii) there exists $q \geq 0$ such that $p \geq q > \lambda_m$ and $\langle Nu - Nv, x^* - y^* \rangle \leq -q \|x^* - y^*\|^2$, for all $x^*, y^* \in S_0$ and u, v the solutions of the auxiliary equation (8) corresponding to x^* and y^* .

Then $[I - H(I - P)N]^{-1}$ is continuous.

PROOF. Let $u - H(I - P)Nu = x^*$, and $v - H(I - P)Nv = y^*$. Then

$$\langle Nu - Nv, u - v \rangle + \langle Nu - Nv, -H(I - P)Nu + H(I - P)Nv \rangle = \langle Nu - Nv, x^* - y^* \rangle.$$

By hypothesis (i) and (ii) of Lemma 3, and by (10) we have

$$-p \|u - v\|^2 + \lambda_{m+1} \| -H(I - P)[Nu - Nv] \|^2 \leq -q \|x^* - y^*\|^2.$$

Thus

$$\begin{aligned}\lambda_{m+1} \| -H(I-P)[Nu - Nv] \|^2 &\leq p \|u - v\|^2 - q \|x^* - y^*\|^2 \\ &= p \|(x^* - y^*) - (-H(I-P)[Nu - Nv])\|^2 - q \|x^* - y^*\|^2.\end{aligned}$$

Since $x^* - y^*$ and $-H(I-P)[Nu - Nv]$ are orthogonal we have,

$$\lambda_{m+1} \| -H(I-P)[Nu - Nv] \|^2 \leq (p - q) \|x^* - y^*\|^2 + p \| -H(I-P)[Nu - Nv] \|^2.$$

Hence, $(\lambda_{m+1} - p) \| -H(I-P)[Nu - Nv] \|^2 \leq (p - q) \|x^* - y^*\|^2$. Also $\|u - v\| \leq \|x^* - y^*\| + \| -H(I-P)[Nu - Nv] \|$. The above two inequalities together imply that $[I - H(I-P)N]^{-1}$ is continuous.

5. Solvability of the bifurcation equation (9). We now consider the solvability of the bifurcation equation

$$(9) \quad PN[I - H(I-P)N]^{-1}x^* - Ex^* = 0.$$

This equation, by virtue of the continuity and boundedness of both N and $[I - H(I-P)N]^{-1}$, could be discussed by using known results from the Leray-Schauder theory. However, we shall apply instead a well-known result from the theory of nonlinear monotone operators.

PROPOSITION 2. *Let the Nemytskii operator defined by $(Nx)(t) = f(t, x(t))$ be such that $N: S \rightarrow S$, and*

- (i) *N is continuous and bounded;*
- (ii) *there exists $\lambda_m < p < \lambda_{m+1}$ such that, for all $u, v \in S$, $\langle Nu - Nv, u - v \rangle \geq -p \|u - v\|^2$;*
- (iii) *there exists q such that $\lambda_m < q \leq p$ and, for all $x^*, y^* \in S_0$, $\langle Nu - Nv, x^* - y^* \rangle \leq -q \|x^* - y^*\|^2$ where u, v are the solutions of the auxiliary equation corresponding to x^*, y^* .*

Then the bifurcation equation (9) has a unique solution.

PROOF. *Uniqueness of the solution of (9).* We note that (i) and (ii) guarantee the solvability of the auxiliary equation and the uniqueness of the solution.

Let $PN[I - H(I-P)N]^{-1}x^* - Ex^* = PN[I - H(I-P)N]^{-1}y^* - Ey^*$. Thus,

$$Ex^* - Ey^* = PN[I - H(I-P)N]^{-1}x^* - PN[I - H(I-P)N]^{-1}y^*,$$

or

$$\langle Ex^* - Ey^*, x^* - y^* \rangle = \langle PNu - PNv, x^* - y^* \rangle,$$

where $[I - H(I-P)N]^{-1}x^* = u$, $[I - H(I-P)N]^{-1}y^* = v$. Thus, by (iii)

$$\langle Ex^* - Ey^*, x^* - y^* \rangle = \langle Nu - Nv, x^* - y^* \rangle \leq -q \|x^* - y^*\|^2.$$

But if $x^* = \sum_1^m c_i \varphi_i$, then $\langle Ex^*, x^* \rangle \geq -\lambda_m \|x^*\|^2$. Hence,

$$-\lambda_m \|x^* - y^*\|^2 \leq -q \|x^* - y^*\|^2,$$

which contradicts the hypothesis that $q > \lambda_m$. Thus the uniqueness of the solution to the bifurcation equation is proved.

Existence of a solution of (9). Let $T_1: S_0 \rightarrow S_0$ be defined by

$$T_1 x^* = Ex^* - PN[I - H(I - P)N]^{-1} x^*.$$

The bifurcation equation reduces to $T_1 x^* = 0$. Let x^*, y^* be any two elements of S_0 and let $u, v \in S$ be such that

$$[I - H(I - P)N]^{-1} x^* = u, \quad [I - H(I - P)N]^{-1} y^* = v.$$

The existence of u, v is guaranteed by hypotheses (i), (ii) and Proposition 1.

Now

$$\begin{aligned} \langle T_1 x^* - T_1 y^*, x^* - y^* \rangle &= \langle Ex^* - Ey^*, x^* - y^* \rangle - \langle PNu - PNv, x^* - y^* \rangle \\ &\geq -\lambda_m \|x^* - y^*\|^2 - \langle Nu - Nv, x^* - y^* \rangle \\ (12) \quad &= -\lambda_m \|x^* - y^*\|^2 + q \|x^* - y^*\|^2 \\ &= (q - \lambda_m) \|x^* - y^*\|^2. \end{aligned}$$

Since $q > \lambda_m$, $T_1: S_0 \rightarrow S_0$ is strictly monotone. Also, T_1 is continuous since N and $[I - H(I - P)N]^{-1}$ are so and E is a continuous operator on the finite-dimensional space S_0 . Finally, it can be derived from (12) that T_1 is coercive, i.e., $\langle T_1 x^*, x^* \rangle / \|x^*\| \rightarrow \infty$ as $\|x^*\| \rightarrow \infty$. Hence, by the well-known theorem of Minty [11] on strictly monotone operators, the bifurcation equation is uniquely solvable.

6. THEOREM 1. *Let the Nemytskii operator defined by $(Nx)(t) = f(t, x(t))$ be such that $N: S \rightarrow S$, and*

- (i) f is 2π -periodic in t ;
- (ii) $N: S \rightarrow S$ is continuous and bounded;
- (iii) there exist p, q, m such that $m^2 < q \leq p < (m+1)^2$;
- (iv) for all $u, v \in S$, $\langle Nu - Nv, u - v \rangle \geq -p \|u - v\|^2$;
- (v) for all $u, v \in S$, $\langle Nu - Nv, x^* - y^* \rangle \leq -q \|x^* - y^*\|^2$ where u and v are the solutions of the auxiliary equation corresponding to x^*, y^* .

Then the equation $x'' = f(t, x(t))$ has a unique 2π -periodic solution.

Actually it is enough to know that (v) is true for all $x^*, y^* \in S_0$, and u, v corresponding solutions of the auxiliary equation (8).

By virtue of Lemma 2 we have, with the same symbols as above,

COROLLARY 1. Let $N: S \rightarrow S$ be such that

- (i) $\|Nu - Nv\| \leq p\|u - v\|$, $u, v \in S$, $p < (m + 1)^2$;
- (ii) $\langle Nu - Nv, x^* - y^* \rangle \leq -q\|x^* - y^*\|^2$, for all $u, v \in S$, $m^2 < q \leq p$ and $Pu = x^*$, $Pv = y^*$.

Then the equation $x'' = f(t, x(t))$ has a unique 2π -periodic solution.

7. **Case of resonance.** As is seen from the proofs of the existence of a solution to the bifurcation equation (§5), the condition $q > \lambda_m$ is important. When $q = \lambda_m$, we have the case of perturbation at resonance. More generally, we are interested in considering the existence of a solution to the nonlinear differential equation

$$(12') \quad x'' + \lambda_m x = f(t, x(t)),$$

where $f: R \times R^n \rightarrow R^n$ is a function which is 2π -periodic in t and λ_m is an eigenvalue of the associated homogeneous boundary value problem. The question of the existence of a 2π -periodic solution to (12') is the topic of discussion of this section. We give below a class of nonlinear perturbations f for which equation (12') has at least one 2π -periodic solution. As will be obvious from the nature of the proof, the method is powerful to treat various classes of nonlinearities.

THEOREM 2. Let $f: R \times R^n \rightarrow R^n$ be a function which is 2π -periodic in t . Let the Nemytskii operator $(Mx)(t) = f(t, x(t))$ be such that

- (1) $M: S \rightarrow S = (L_2[0, 2\pi])^n$,
- (2) M is continuous and bounded,
- (3) M is monotone, i.e., $\langle Mu - Mv, u - v \rangle \geq 0$, for all $u, v \in S$,
- (4) there exists $R > 0$ such that $\langle Mu, x^* \rangle \leq 0$, for all x^* satisfying $\|x^*\| = R$ and u is the solution of the auxiliary equation corresponding to x^* .

Then the equation

$$(13) \quad x'' + \lambda_m x = f(t, x(t))$$

has at least one 2π -periodic solution.

PROOF. If we write (13) in the form $Ex = Nx$, we have $(Nx)(t) = -\lambda_m x + f(t, x(t))$, $Ex = x''$ with 2π -periodicity boundary conditions, and we have the corresponding system (8), (9).

Solvability of the auxiliary equation (8). This equation can be rewritten as in [3]:

$$0 \in [-H(I - P)]^{-1}(x - x^*) + Nx.$$

Setting $y = x - x^*$, the equation reduces to

$$0 \in T_2 y = [-H(I-P)]^{-1}(y) + N_{x^*}(y),$$

where $N_{x^*}(y) = N(y + x^*)$. For $y_1 = -H(I-P)z_1$, $y_2 = -H(I-P)z_2$, we have from (10),

$$\begin{aligned} \langle T_2 y_1 - T_2 y_2, y_1 - y_2 \rangle &= \langle z_1 - z_2, -H(I-P)z_1 + H(I-P)z_2 \rangle \\ &\quad + \langle -\lambda_m(y_1 + x^*) + \lambda_m(y_2 + x^*), y_1 - y_2 \rangle \\ &\quad + \langle My_1 - My_2, y_1 - y_2 \rangle \\ &\geq (\lambda_{m+1} - \lambda_m) \|y_1 - y_2\|^2. \end{aligned} \quad (14)$$

Thus T_2 is strongly monotone and hence, coercive. Also, by Brezis, Crandall, and Pazy [1], T_2 can be seen to be maximal monotone. Thus, T_2 is a maximal monotone, coercive operator. Hence the auxiliary equation is solvable for each $x^* \in S_0$ and by virtue of the strong monotonicity of T_2 , the solution is unique. As proved in §§3 and 4, $[I - H(I-P)N]^{-1}$ is continuous and bounded.

Solvability of the bifurcation equation (9). The bifurcation equation (9) can be rewritten as

$$x + P[E - I - PN(I - H(I-P)N)^{-1}]x^* = 0.$$

This equation is of the type $(I + T_3)x^* = 0$, where $T_3: S_0 \rightarrow S_0$ is compact. Let G denote the set $\{x^* \in S_0: \|x^*\| < R\}$ and let $t \in [0, 1)$. We now show that the equation $x^* + tT_3x^* = 0$ cannot have a solution on $\{x^* \in S_0: \|x^*\| = R\}$. The case $t = 0$ is obvious. If $t \in (0, 1)$ and $x^* + tT_3x^* = 0$ where $\|x^*\| = R$, then

$$\langle T_3x^*, x^* \rangle + t\langle T_3x^*, T_3x^* \rangle = 0. \quad (15)$$

But $\langle T_3x^*, x^* \rangle = \langle Ex^*, x^* \rangle - \|x^*\|^2 - \langle PNu, x^* \rangle$ where $u = [I - H(I-P)N]^{-1}x^*$ and therefore $Pu = x^*$. Thus

$$\begin{aligned} \langle T_3x^*, x^* \rangle &\geq -\lambda_m \|x^*\|^2 - \|x^*\|^2 + \langle \lambda_m u, x^* \rangle - \langle Mu, x^* \rangle \\ &= -\|x^*\|^2 - \langle Mu, x^* \rangle \\ &\geq -\|x^*\|^2, \quad \text{by hypothesis (4).} \end{aligned}$$

Thus by (15), $0 \geq -\|x^*\|^2 + t\|T_3x^*\|^2$. Also $x^* + tT_3x^* = 0$ implies $0 \geq -\|x^*\|^2 + \|x^*\|^2/t$, i.e., $x^* = 0$. Thus $x^* \in G$, which is a contradiction. Hence, the equation $x^* + tT_3x^* = 0$, $t \in [0, 1)$ does not have a solution on $\{x^* \in S_0: \|x^*\| = R\}$.

By the Leray-Schauder theory we conclude that $x^* + T_3x^* = 0$ has a solution in G . This completes the proof of the theorem.

COROLLARY 2. *Let f and M be the same as in Theorem 2 and let M be such that*

- (i) $M: S \rightarrow S$,
- (ii) M is continuous and bounded,
- (iii) $\langle Mu - Mv, u - v \rangle \geq -p\|u - v\|^2$, where $p < \lambda_{m+1} - \lambda_m$,
- (iv) there exists $R > 0$ such that $\langle Mu, x^* \rangle \leq 0$ for all x^* satisfying $\|x^*\| = R$ and u is the solution of the auxiliary equation corresponding to x^* .

Then the equation $x'' + \lambda_m x = f(t, x(t))$ has at least one 2π -periodic solution.

PROOF OF COROLLARY 2. The proof will be similar to that of Theorem 2. It should be noted that the weaker hypothesis (iii) of the corollary is adequate to ensure that T_2 , in the proof of the solvability of the auxiliary equation of Theorem 2, is strictly monotone, as is clear from (14).

8. Examples. We conclude this paper by presenting some applications of our theorems. As a first example let us consider the following periodically perturbed conservative system:

$$x'' + \text{grad} G(x) = p(t)$$

where $p(t) \in (L^2[0, 2\pi])^n$ and is 2π -periodic in t and G is $C^2(R^n, R)$. Clearly, with the boundary conditions $x(0) = x(2\pi)$ and $x'(0) = x'(2\pi)$, this problem can be treated as a case of perturbation at resonance with $\lambda_m = 0$. We now obtain:

THEOREM 3. Let $G \in C^2(R^n, R)$ be such that, for all $a \in R^n$,

$$M^2 I < qI \leq (\partial^2 G(a)/\partial x_i \partial x_j) \leq pI < (M + 1)^2 I$$

where M is an integer. Then the nonlinear differential equation

$$(16) \quad x'' + \text{grad} G(x) = p(t)$$

has a unique 2π -periodic solution.

PROOF. With the usual symbols, let $N: S \rightarrow S$ be defined by $(Nx)(t) = -\text{grad} G(x) + p(t)$. Clearly $N: S \rightarrow S$ is continuous and bounded. λ_m may be treated as 0. Thus S_0 is the space of all constant functions. The auxiliary problem $x - H(I - P)Nx = x^*$ can be rewritten as $x + LN_1 x = L(p(t))$ where $N_1 x = Nx + \mu x$, μ being a suitably chosen number between p and q and L is a linear operator. It can be shown, by the Schauder principle of invariance of domain as in Proposition 1, that the auxiliary problem has a unique solution for each $x^* \in S_0$ and further that $\|x - x^*\| \leq A$, A being independent of x^* . In order to show that (16) has at least one solution, it remains to prove as in the earlier theorems that $\langle Nx, x^* \rangle \leq 0$ for all x^* satisfying $\|x^*\| = R$. Now

$$\langle Nx, x^* \rangle = \langle -\text{grad} G(x), x^* \rangle + \langle p(t), x^* \rangle.$$

Also

$$\begin{aligned}\langle \text{grad } G(x), x^* \rangle &= \langle \text{grad } G(x^*), x^* \rangle + \langle \text{grad } G(x) - \text{grad } G(x^*), x^* \rangle \\ &\geq q \|x^*\|^2 - \|\text{grad } G(x) - \text{grad } G(x^*)\| \|x^*\| \\ &\geq q \|x^*\|^2 - pA \|x^*\|.\end{aligned}$$

Thus there exists $R > 0$ such that $\langle Nx, x^* \rangle \leq 0$ for all x^* satisfying $\|x^*\| = R$. Hence (16) has at least one solution. The uniqueness of the solution follows from the fact that if x_1 and x_2 are any two solutions of (16) then $y = x_1 - x_2$ satisfies $y'' + B(t)y = 0$, where $B(t)$ is obtained from $\text{grad } G(x_1) - \text{grad } G(x_2)$ by the mean value theorem. Now using the fact that $qI \leq (\partial^2 G(a)/\partial x_i \partial x_j) \leq pI$, we can prove that $y = 0$.

REMARKS. This problem was earlier considered by Lazer and Sanchez [8] who proved that it has a solution. In particular, the following result of Leach [9], which is obtained by a different technique and generalizes a result of Loud [10], may be seen to be a particular case of the above theorem:

Let $g(x) \in C^1$ with $g(0) = 0$ and let there exist an integer n such that, for all x , $n^2 < g'(x) < (n+1)^2$. Then if $e(t)$ be 2π -periodic, there exists at most one 2π -periodic solution of $x'' + g(x) = e(t)$.

We now consider the variant of problem (15) where $\text{grad } G(x)$ is replaced by a bounded nonlinear function. Lazer and Leach [7] considered the nonlinear differential equation

$$(16) \quad x'' + n^2 x + h(x) = p(t)$$

for the existence of 2π -periodic solutions. By an application of Theorem 2, or more specifically, Corollary 2 we can now conclude (cf. Theorem 3.1 of Lazer and Leach [11]),

THEOREM 4. *Let $h(x)$ be such that*

- (i) $|h(x)| \leq M$ for all real x ,
- (ii) $0 < h'(x) \leq p < \lambda_{m+1} - \lambda_m = (m+1)^2 - m^2 = 2m+1$,
- (iii) *there is some $R > 0$ such that $\langle -h(x) + p(t), x^* \rangle \leq 0$, for all x^* satisfying $\|x^*\| = R$ and x is the solution of the auxiliary equation corresponding to x^* .*

Then the nonlinear differential equation $x'' + m^2 x + h(x) = p(t)$ has a unique 2π -periodic solution.

PROOF. The existence of a 2π -periodic solution may be seen to be a consequence of Corollary 2.

REMARK. If, with the same symbols as in Theorem 4, it is assumed that there exist numbers c, d, C and $D(c < d)$ such that

- (i) $h(x) \leq C$ for $x \leq c$,
- (ii) $h(x) \geq D$ for $x \geq d$,
- (iii) $(A^2 + B^2)^{1/2} < 2(D - C)$ where $A = \int_0^{2\pi} p(s) \cos ns ds$ and $B = \int_0^{2\pi} p(s) \sin ns ds$,

then we can show that hypothesis (iii) of Theorem 4 is true, thereby obtaining the above-mentioned theorem of Lazer and Leach [7] as a particular case.

In fact, using the same symbols as in Theorem 4, we have

$$\langle p(t) - h(x), x^* \rangle = \int_0^{2\pi} [p(s) - h(x)] [a \cos ns + b \sin ns] ds$$

where $x^* = a \cos ns + b \sin ns \in S_0$ and $Px = x^*$. We can then write as in Lazer and Leach [7], $x = x^* + (I - P)x = a \cos ns + b \sin ns + \alpha(t) \cos ns + \beta(t) \sin ns$ where $\alpha(t)$ and $\beta(t)$ are continuous 2π -periodic and bounded functions. By following the arguments of Lazer and Leach it can be shown that there exists $R > 0$ such that for $\|x^*\| \geq R$, the inequality (iii) given by $(A^2 + B^2)^{1/2} < 2(D - C)$ implies that $\langle p(t) - h(x), x^* \rangle \leq 0$ and thus Theorem 4 can be applied to the nonlinear differential equation (16).

REFERENCES

1. H. Brezis, M. G. Crandall and A. Pazy, *Perturbations of nonlinear maximal monotone sets in Banach spaces*, Comm. Pure Appl. Math. **23** (1970), 123–144. MR **41** #2454.
2. L. Cesari, *Nonlinear functional analysis*, Lecture notes, C.I.M.E., 1972.
3. L. Cesari and R. Kannan, *Functional analysis and nonlinear differential equations*, Bull. Amer. Math. Soc. **79** (1973), 1216–1219. MR **48** #12183.
4. J. K. Hale, *Applications of alternative problems*, Brown University Lecture Notes, 1971.
5. ———, *Ordinary differential equations*, Wiley, New York, 1969.
6. R. Kannan, *Periodically perturbed conservative systems*, J. Differential Equations, **16** (1974), 506–514.
7. A. C. Lazer and D. E. Leach, *Bounded perturbations of forced harmonic oscillators at resonance*, Ann. Mat. Pura Appl. (4) **82** (1969), 49–68. MR **40** #2972.
8. A. C. Lazer and D. A. Sanchez, *On periodically perturbed conservative systems*, Michigan Math. J. **16** (1969), 193–200. MR **39** #7212.
9. D. E. Leach, *On Poincaré's perturbation theorem and a theorem of W. S. Loud*, J. Differential Equations **7** (1970), 34–53. MR **40** #4539.
10. W. S. Loud, *Periodic solutions of nonlinear differential equations of Duffing type*, Proc. U. S.-Japan Seminar on Differential and Functional Equations, Benjamin, New York, 1967, pp. 199–224. MR **36** #6704.
11. G. J. Minty, *On a "monotonicity" method for the solution of nonlinear equations in Banach spaces*, Proc. Nat. Acad. Sci. U. S. A. **50** (1963), 1038–1041. MR **28** #5358.
12. M. Nagumo, *Degree of mapping in convex linear topological spaces*, Amer. J. Math. **73** (1951), 497–511. MR **13**, 150.

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