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PARTITIONS OF LARGE MULTIPARTITES WITH CONGRUENCE CONDITIONS. I

BY

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ABSTRACT. Let $p(n_1, \ldots, n_j: A_1, \ldots, A_j)$ be the number of partitions of $(n_1, \ldots, n_j)$ where, for $1 \leqslant l \leqslant j$, the lth component of each part belongs to the set $A_l = \bigcup_{h(l)=1}^{q(l)} \{a_{lh(l)} + Mv: v = 0, 1, 2, \ldots\}$ and M, $q(l)$ and the $a_{lh(l)}$ are positive integers such that $0 < a_{l1} < \cdots < a_{lq(l)} \leqslant M$. Asymptotic expansions for $p(n_1, \ldots, n_j: A_1, \ldots, A_j)$ are derived, when the $n_l \to \infty$ subject to the restriction that $n_1 \cdots n_j \leqslant n_l^{j+1-\epsilon}$ for all l, where ϵ is any fixed positive number. The case $M = 1$ and arbitrary j was investigated by Robertson [10] while several authors between 1940 and 1960 investigated the case $j = 1$ for different values of M.

1. **Introduction.** Many authors have evaluated the number of different partitions of a multipartite number. A multipartite number of order j is a j-dimensional vector, the components of which are positive integers, and a partition of $(n_1, \ldots, n_j)$ is a solution of the vector equation

$$\sum_k (n_{1k}, \ldots, n_{jk}) = (n_1, \ldots, n_j) \tag{1.1}$$

in multipartites. Two partitions which differ only in the order of the multipartites on the left-hand side of (1.1) are regarded as identical.

In [10] Robertson, extending results of Wright [13], obtained asymptotic expansions for the number of different partitions of $(n_1, \ldots, n_j)$ when $n_1 \cdots n_j < n_l^{j+1-\epsilon_1}$ for all l where ϵ_1 is any fixed positive number less than 1. In this article, we extend these results and obtain, subject to the same conditions on the n_l, asymptotic expansions for $p(n_1, \ldots, n_j: A_1, \ldots, A_j)$ the number of different partitions of $(n_1, \ldots, n_j)$ where, for $1 \leqslant l \leqslant j$, the lth component of each part belongs to the set

$$A_l = \bigcup_{h(l)=1}^{q(l)} \{a_{lh(l)} + Mv: v = 0, 1, 2, \ldots\}$$

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and M, $q(l)$ and the $a_{lh(l)}$ are positive integers such that, for every l, $0 < a_{l1} < \cdots < a_{lq(l)} \leq M$.

The residue classes in which the different components of each part must lie may, in general, be expressed in terms of different moduli, and also the permissible residue classes for any particular component may be expressed in terms of different moduli. These generalizations are, however, only superficial, as all the residue classes may be expressed in terms of residue classes modulo M, the least common multiple of the different moduli.

In [13] there is a fairly comprehensive list of papers concerned with the asymptotic evaluation of the number of partitions of multipartites. The only investigation, of which the authors are aware, concerning multipartites subject to congruence conditions is that of Passi [8]. This paper generalizes the partition problem to lattices but obtains an asymptotic evaluation not of the number of partitions but only of its logarithm.

The case $j = 1$ with different congruence conditions has been investigated by several authors. The Hardy-Ramanujan circle method as modified by Rademacher [9] has been employed to evaluate $p(n_1: A_1)$, where $q(1) = 2$, $a_{12} = M - a_{11}$, as a convergent series. The case $M = 6$ was obtained by Niven [7], $M = 5$ by Lehner [5] and Livingood [6] solved the case where M is any prime > 3. Later Iseki [3] evaluated $p(n_1: A_1)$ when M is composite > 3 and $(a_{11}, M) = 1$. Hagis [2] evaluated $p(n_1: A_1)$ for all odd primes M, where $q(1) = 2r$ and $a_{1h} + a_{1k} = M$ whenever $h + k = 2r$.

In all the cases mentioned in the preceding paragraph, A_1 is symmetrical in the sense that $a_1 \in A_1$ implies that $M - a_1 \in A_1$, and this ensures that the generating function of the $p(n_1: A_1)$ is a modular form. Rademacher's method then leads to a convergent series representation of $p(n_1: A_1)$. Grosswald [1] considered the case where M is any odd prime and A_1 is an arbitrary asymmetrical set. Then the above method cannot be applied and only asymptotic results are obtained.

2. **Notation and definitions.** Throughout this article, $a, d, \Delta, h, k, K, l, m, M, n, N, \nu, q, r, \rho, s$ represent nonnegative integers and j is used for an integer greater than unity. C is a positive number, not necessarily the same at each occurrence, which may depend upon any j, M, ϵ_l but not upon any n_l, x_l, y_l, θ_l, ξ_l, z_l. The numbers ϵ_l are positive and to be thought of as small. The symbols $\sim$, $o(\ )$ always refer to the passage of the n_l to infinity. The symbol $O(\ )$ sometimes refers to the passage of the n_l to infinity and otherwise is obvious from the context. The total differential operator d/dt is always denoted by D and never by a prime. γ, $\zeta(\ )$ represent respectively the Euler constant and the Riemann zeta function.

We write

$$f(x_1, \ldots, x_j) = f(x_1, \ldots, x_j\colon \mathrm{A}_1, \ldots, \mathrm{A}_j)$$

$$(2.1) \qquad = \prod_{\nu_1=0}^{\infty} \cdots \prod_{\nu_j=0}^{\infty} \prod_{h(1)=1}^{q(1)} \cdots \prod_{h(j)=1}^{q(j)} \left(1 - \exp\left(-\sum_{l=1}^{j} (M\nu_l + a_{lh(l)})x_l\right)\right)^{-1}$$

where $\mathrm{Re}(x_l) > 0$ for $1 \leqslant l \leqslant j$. Writing $p(0, \ldots, 0) = 1$ and $p(n_1, \ldots, n_j) = p(n_1, \ldots, n_j\colon \mathrm{A}_1, \ldots, \mathrm{A}_j)$, we can easily verify that

$$(2.2) \qquad f(x_1, \ldots, x_j) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_j=0}^{\infty} p(n_1, \ldots, n_j) e^{-n_1x_1-\cdots-n_jx_j}.$$

We assume for convenience throughout this note that $n_1 \leqslant \cdots \leqslant n_j$. The definition of $p(n_1, \ldots, n_j)$ is such that this assumption involves no loss of generality in the asymptotic results obtained.

When $|t| < 2\pi$, we have

$$(2.3) \qquad te^{ct}(e^t - 1)^{-1} = \sum_{\nu=0}^{\infty} B_\nu(c)t^\nu/\nu!$$

for all c, where the $B_\nu(c)/\nu!$ are the Bernouilli polynomials in c. From pp. 521–523 of Knopp [4], we see that, if we write $P_1(t) = t - [t] - \frac{1}{2}$, then

$$P_1(t) = B_1(t) = -\sum_{r=1}^{\infty} (r\pi)^{-1} \sin 2r\pi t$$

for $0 < t < 1$, and if we write

$$P_{2\nu}(t) = 2(-1)^{\nu-1} \sum_{r=1}^{\infty} (2r\pi)^{-2\nu} \cos 2r\pi t,$$

$$P_{2\nu+1}(t) = 2(-1)^{\nu-1} \sum_{r=1}^{\infty} (2r\pi)^{-2\nu-1} \sin 2r\pi t$$

for all $\nu \geqslant 1$, then $P_\nu(t) = B_\nu(t)/\nu!$ for $0 < t < 1$. Clearly every $P_\nu(t)$ is bounded, has period 1 and, for all $\nu > 1$, $DP_\nu(t) = P_{\nu-1}(t)$. $B_{2\nu}(0)$ for $\nu \geqslant 1$ are the Bernouilli numbers. For $\nu \geqslant 1$, $B_{2\nu+1}(0) = 0$ and $(2\pi)^{2\nu}B_{2\nu}(0) = 2(2\nu)!\zeta(2\nu)$.

We define

$$\Lambda(t) = \Lambda_{h(1)\ldots h(j)}(t\colon x_1, \ldots, x_j) = t^{-1} \prod_{l=1}^{j} (e^{Mx_lt} - 1)^{-1} e^{(M-a_{lh(l)})x_lt}$$

$$= t^{-1} \sum_{\nu_1=0}^{\infty} \cdots \sum_{\nu_j=0}^{\infty} \exp\left(-t\sum_{l=1}^{j} (M\nu_l + a_{lh(l)})x_l\right),$$

$$F(x_1, \ldots, x_j) = \sum_{l=1}^{j} n_lx_l + \sum_{r=1}^{\infty} \sum_{h(1)=1}^{q(1)} \cdots \sum_{h(j)=1}^{q(j)} \Lambda_{h(1)\ldots h(j)}(r\colon x_1, \ldots, x_j).$$

We put $X = x_1 \cdots x_j$ and x^* for the x_l with maximum modulus, i.e. $|x^*| = \max|x_l|$. Hence, for $|t| < 2\pi/M|x^*|$, we obtain from (2.3)

$$\begin{aligned} XM^j t^{j+1}\Lambda(t) &= \prod_{l=1}^{j} Mx_l t(e^{Mx_l t} - 1)^{-1} e^{(M-a_{lh(l)})x_l t} \\ &= \prod_{l=1}^{j} \sum_{\nu=0}^{\infty} B_\nu\left(\frac{1-a_{lh(l)}}{M}\right)\frac{(Mx_l t)^\nu}{\nu!} = \sum_{m=0}^{\infty} Q_m t^m, \end{aligned} \tag{2.4}$$

where $Q_m = Q_m(x_1, \ldots, x_j)$ are homogeneous polynomials of degree m in $x_1, \ldots, x_j$. We write

$$G(t) = XM^j\Lambda(t) - \sum_{m=0}^{j} Q_m t^{m-j-1}$$

so that $G(t) = \Sigma_{m=j+1}^{\infty} Q_m t^{m-j-1}$ whenever $|t| < 2\pi/M|x^*|$, and we put

$$\begin{aligned} H &= H(x_1, \ldots, x_j) \\ &= \int_0^\infty \left\{XM^j\Lambda(t) - \sum_{m=0}^{j-1} Q_m t^{m-j-1} - Q_j(e^t-1)^{-1}\right\} dt. \end{aligned} \tag{2.5}$$

We write $z_l = x_l/x_1$ for $1 \leqslant l \leqslant j$, $Z = z_2 \cdots z_j$ and

$$U_m = U_m(z_1, \ldots, z_j) = M^{-m}Q_m(z_1, \ldots, z_j)$$

for all $m \geqslant 0$. If we write

$$\Omega(u) = \Omega(u\colon z_2, \ldots, z_j) = u^{-1}\prod_{l=1}^{j}(e^{z_l u} - 1)^{-1} e^{(1-a_{lh(l)}/M)z_l u},$$

then it follows from (2.4) that

$$Zu^{j+1}\Omega(u) = \sum_{m=0}^{\infty} U_m u^m$$

for $|u| < 2\pi/|z^*|$, where z^* is the z_l with maximum modulus. Observing that

$$\begin{aligned} &\int_0^\infty \{(e^t-1)^{-1} - Mx_1(e^{Mx_1 t} - 1)^{-1}\}\, dt \\ &\qquad = [\log\{(1-e^{-Mx_1 t})^{-1}(1-e^{-t})\}]_0^\infty = \log Mx_1, \end{aligned}$$

we substitute $u = Mx_1 t$ in (2.5) and an easy calculation gives

$$H = XM^jI - Q_j \log Mx_1, \tag{2.6}$$

where

$$I = I(z_2, \ldots, z_j) = \int_0^\infty \beta(u: z_2, \ldots, z_j)\, du,$$

$$\beta(u: z_2, \ldots, z_j) = \Omega(u) - Z^{-1}\left\{\sum_{m=0}^{j-1} U_m u^{m-j-1} + U_j(e^u - 1)^{-1}\right\}.$$

For $s \geqslant 1$, let us write $Z' = z_2 \cdots z_s$, $Z'' = z_{s+1} \cdots z_j$ and, for all $m \geqslant 0$,

$$V_m = V_m(z_1, \ldots, z_s) = M^{-m} Q_m(z_1, \ldots, z_s),$$
$$W_m = W_m(z_{s+1}, \ldots, z_j) = M^{-m} Q_m(z_{s+1}, \ldots, z_j).$$

We define

$$\Omega'(u) = \Omega'(u: z_2, \ldots, z_s) = u^{-1} \prod_{l=1}^{s} (e^{z_l u} - 1)^{-1} e^{(1-a_{lh(l)}/M)z_l u},$$

$$\Omega''(u) = \Omega''(u: z_{s+1}, \ldots, z_j) = \prod_{l=s+1}^{j} (e^{z_l u} - 1)^{-1} e^{(1-a_{lh(l)}/M)z_l u},$$

and so, for $|u| < 2\pi/|z^*|$,

$$Z' u^{s+1} \Omega'(u) = \sum_{m=0}^{\infty} V_m u^m, \qquad Z'' u^{j-s} \Omega''(u) = \sum_{m=0}^{\infty} W_m u^m.$$

It follows that, for all $m \geqslant 0$,

$$U_m = \sum_{r=0}^{m} V_r W_{m-r}. \tag{2.7}$$

We define a generalization of the integral I by writing

$$I'_{sr} = I'_{sr}(z_2, \ldots, z_s) = \int_0^\infty \beta'_{sr}(u: z_2, \ldots, z_s)\, du,$$

where, for $r \geqslant j + 1$,

$$\beta'_{sr}(u: z_2, \ldots, z_s) = u^{r+s-j} \Omega'(u)$$

and, for $0 \leqslant r \leqslant j$,

$$\beta'_{sr}(u: z_2, \ldots, z_s) = u^{r+s-j} \Omega'(u) - Z'^{-1}\left\{\sum_{m=0}^{j-r-1} V_m u^{m+r-j-1} + V_{j-r}(e^u - 1)^{-1}\right\}.$$

For $1 \leqslant l \leqslant j$, we write

$$x_l = y_l + i\theta_l = y_l(1 + i\xi_l),$$

where $y_l > 0$. We put

$$Y = y_1 \cdots y_j,$$
$$R_m = R_m(y_1, \ldots, y_j) = Q_m(y_1, \ldots, y_j),$$
$$\overline{Q}_m = \overline{Q}_m(x_1, \ldots, x_j) = \sum_{h(1)=1}^{q(1)} \cdots \sum_{h(j)=1}^{q(j)} Q_m(x_1, \ldots, x_j),$$
$$\overline{R}_m = \overline{R}_m(y_1, \ldots, y_j) = \sum_{h(1)=1}^{q(1)} \cdots \sum_{h(j)=1}^{q(j)} R_m(y_1, \ldots, y_j),$$
$$\overline{H} = \overline{H}(x_1, \ldots, x_j) = \sum_{h(1)=1}^{q(1)} \cdots \sum_{h(j)=1}^{q(j)} H(x_1, \ldots, x_j).$$

We let $y_l \sim \mu_l$, where the μ_l are defined by

$$\mu_l = n_l^{-1}\{M^{-j}\zeta(j+1)\overline{R}_0 n_1 \cdots n_j\}^{1/(j+1)}. \tag{2.8}$$

Since $n_1 \leqslant \cdots \leqslant n_j$, clearly $\mu_1 \geqslant \cdots \geqslant \mu_j$.

From (6.2), which we prove later, we can easily deduce that a positive integer Δ can be found such that Δ is the smallest integer for which $\mu_1^{2\Delta+j-1} = o(\mu_1 \cdots \mu_j)$. With this value of Δ, we define

$$\begin{aligned} F^*(x_1, \ldots, x_j) &= \sum_{l=1}^{j} n_l x_l \\ &\quad + X^{-1}M^{-j}\left\{\sum_{m=0;m\neq j}^{2\Delta+j-1} \zeta(j+1-m)\overline{Q}_m + \gamma\overline{Q}_j + \overline{H}\right\}. \end{aligned} \tag{2.9}$$

For $1 \leqslant l \leqslant j$, we write d_l for the greatest common divisor $(M, a_{l1} - a_{l2}, \ldots, a_{l1} - a_{lq(l)})$. Then we define $K = K(n_1, \ldots, n_j\colon \mathrm{A}_1, \ldots, \mathrm{A}_j)$ as the number of ν in $0 \leqslant \nu \leqslant M-1$ which satisfy the simultaneous congruences $n_l \equiv \nu a_{l1} \pmod{d_l}$, $1 \leqslant l \leqslant j$. Clearly, since d_l divides $a_{l1} - a_{lh(l)}$ for $2 \leqslant h(l) \leqslant q(l)$, $1 \leqslant l \leqslant j$, these congruences are equivalent to $n_l \equiv \nu a_{lh(l)} \pmod{d_l}$ for any set of $h(1), \ldots, h(j)$. Finally, we write $q = \overline{Q}_0 = \overline{R}_0 = q(1) \cdots q(j)$.

3. Statement and proof of the main result.

Employing the definitions of the last section, we can now state our principal result.

THEOREM 1. *If, for* $1 \leqslant l \leqslant j$ $(j > 1)$, *every* n_l *tends to infinity subject to the condition that*

$$n_1 \cdots n_j < n_l^{j+1-\epsilon_1} \tag{3.1}$$

for any fixed positive number ϵ_1, *then*

$$p(n_1, \ldots, n_j) \sim Kd_1 \cdots d_j M^{-1}(j+1)^{-\frac{1}{2}}\{M^j/2\pi\zeta(j+1)q\}^{\frac{1}{2}j} \times Y^{\frac{1}{2}(j+2)} e^{F^*(y_1,\ldots,y_j)},$$

where the y_l *are functions of* $\mu_1, \ldots, \mu_j$ *such that* $y_l \sim \mu_l$ *and*

$$y_l \partial F^*(y_1, \ldots, y_j)/\partial y_l = o\{(\mu_1 \cdots \mu_j)^{-\frac{1}{2}+\epsilon_2}\} \tag{3.2}$$

for $1 \leqslant l \leqslant j$ *and any fixed positive number* ϵ_2.

PROOF. First we choose ϵ_2 to satisfy

$$0 < 2(j+1)\epsilon_2 < \epsilon_1, \tag{3.3}$$

an inequality which will be used in the proof of Lemma 4. Next, we define η and χ by

$$\eta = (\mu_1 \cdots \mu_j)^{\frac{1}{2}-\epsilon_2}, \quad \chi = \eta\{Y^{-1}M^{-j}\zeta(j+1)\overline{R}_0\}^{\frac{1}{2}} \sim C(n_1 \cdots n_j)^{\epsilon_2/(j+1)}.$$

From (2.2), we obtain

$$p(n_1, \ldots, n_j) = (2\pi)^{-j}\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} e^{n_1x_1+\ldots+n_jx_j+\log f(x_1,\ldots,x_j)}\, d\theta_1 \cdots d\theta_j.$$

Now, from (2.1),

$$\begin{aligned}\log f(x_1, \ldots, x_j) &= \sum_{\nu_1=0}^{\infty} \cdots \sum_{\nu_j=0}^{\infty} \sum_{h(1)=1}^{q(1)} \cdots \sum_{h(j)=1}^{q(j)} \sum_{r=1}^{\infty} r^{-1} \exp\left(-r\sum_{l=1}^{j}(M\nu_l + a_{lh(l)})x_l\right) \\ &= \sum_{r=1}^{\infty} \sum_{h(1)=1}^{q(1)} \cdots \sum_{h(j)=1}^{q(j)} \Lambda_{h(1)\ldots h(j)}(r: x_1, \ldots, x_j)\end{aligned} \tag{3.4}$$

and so,

$$p(n_1, \ldots, n_j) = (2\pi)^{-j}\int_{-\pi/M}^{\pi/M} \cdots \int_{-\pi/M}^{\pi/M} g(x_1, \ldots, x_j)\, d\theta_1 \cdots d\theta_j, \tag{3.5}$$

where, for $\omega = e^{2\pi i/M}$,

$$g(x_1, \ldots, x_j) = e^{n_1x_1+\ldots+n_jx_j} \sum_{s_1=0}^{M-1} \cdots \sum_{s_j=0}^{M-1} \omega^{n_1s_1+\ldots+n_js_j}$$

$$(3.6) \qquad \times \exp\left\{\sum_{r=1}^{\infty} \sum_{h(1)=1}^{q(1)} \cdots \sum_{h(j)=1}^{q(j)} \Lambda_{h(1)\ldots h(j)}(r: x_1, \ldots, x_j) \times \omega^{-r\Sigma_{l=1}^{j} a_{lh(l)}s_l}\right\}.$$

In order to prove Theorem 1, we require an asymptotic expansion for the logarithm of the generating function $f(x_1, \ldots, x_j)$. The following result, which is of interest in itself, will be proved in §5.

THEOREM 2. *When* $|\arg x_l| < \frac{1}{2}\pi - \epsilon_3$ *for* $1 \leq l \leq j$,

$$XM^j \log f(x_1, \ldots, x_j) = \sum_{m=0;\, m\neq j}^{2k+j-1} \zeta(j+1-m)\overline{Q}_m + \gamma\overline{Q}_j + \overline{H} + O(x^{*2k+j})$$

as $x^* \longrightarrow 0$.

We also require the following lemmas.

LEMMA 1. *An equivalence relation is defined on* $\{(s_1, \ldots, s_j)\colon 0 \leq s_l \leq M-1$ *for* $1 \leq l \leq j\}$ *by setting* $(s_1, \ldots, s_j)$ *equivalent to* $(s'_1, \ldots, s'_j)$ *whenever* $\Sigma_1^j a_{lh(l)}s_l \equiv \Sigma_1^j a_{lh(l)}s'_l \pmod M$ *for all* $h(1), \ldots, h(j)$. *Then*

$$g(x_1, \ldots, x_j) = e^{n_1x_1+\ldots+n_jx_j} Kd_1 \cdots d_j M^{-1} \sum\nolimits^* \omega^{n_1s_1^*+\ldots+n_js_j^*}$$

$$\times \exp\left\{\sum_{r=1}^{\infty} \sum_{h(1)=1}^{q(1)} \cdots \sum_{h(j)=1}^{q(j)} \Lambda_{h(1)\ldots h(j)}(r: x_1, \ldots, x_j) \times \omega^{-r\Sigma_{l=1}^{j} a_{lh(l)}s_l^*}\right\},$$

where the sum Σ^* *is taken over a complete set of representatives of the equivalence classes.*

LEMMA 2. *If* $\Sigma_1^j |\xi_l| \geq \eta$ *and every* $|\theta_l| \leq \pi/M$, *then*

$$g(x_1, \ldots, x_j) = O\{\exp(F(y_1, \ldots, y_j) - C\chi^2)\}.$$

LEMMA 3. *If* $\Sigma_1^j |\xi_l| \leq \eta$, *then*

$$\sum{}^{(*)}\omega^{n_1s_1+\cdots+n_js_j}\exp\left\{\sum_{r=1}^{\infty}\sum_{h(1)=1}^{q(1)}\cdots\sum_{h(j)=1}^{q(j)}\Lambda_{h(1)\ldots h(j)}(r\colon x_1,\ldots,x_j)\right.$$
$$\left.\times\,\omega^{-r\sum_{l=1}^{j}a_{lh(l)}s_l^*}\right\}$$
$$=O\left\{\exp\left(F(y_1,\ldots,y_j)-\sum_{l=1}^{j}n_ly_l-C(\mu_1\cdots\mu_j)^{-1}\right)\right\},$$

where the sum $\Sigma^{(*)}$ *is taken over the same set as* Σ^* *except that the term corresponding to the equivalence class where every* $\Sigma_1^j a_{lh(l)}s_l \equiv 0 \pmod M$ *is omitted.*

LEMMA 4. *If* $\Sigma_1^j|\xi_l|\leqslant\eta$, *then*

$$F(x_1,\ldots,x_j)=F^*(y_1,\ldots,y_j)$$
$$-\,Y^{-1}M^{-j}\zeta(j+1)\bar{R}_0\left\{\sum_{l=1}^{j}\xi_l^2+\sum_{l=1}^{j-1}\sum_{m=l+1}^{j}\xi_l\xi_m\right\}+o(1).$$

LEMMA 5. *It is always possible to choose* $y_1,\ldots,y_j$ *so that* $y_l\sim\mu_l$ *and* (3.2) *holds for* $1\leqslant l\leqslant j$.

From (3.5), (3.6) and Lemma 1, it follows that

$$M(2\pi)^j\frac{p(n_1,\ldots,n_j)}{Kd_1\cdots d_j}$$
$$=\int_{-\pi/M}^{\pi/M}\cdots\int_{-\pi/M}^{\pi/M}e^{n_1x_1+\cdots+n_jx_j}\sum{}^{*}\omega^{n_1s_1^*+\cdots+n_js_j^*}$$
$$\times\exp\left\{\sum_{r=1}^{\infty}\sum_{h(1)=1}^{q(1)}\cdots\sum_{h(j)=1}^{q(j)}\Lambda_{h(1)\ldots h(j)}(r\colon x_1,\ldots,x_j)\right.$$
$$\left.\times\,\omega^{-r\sum_{l=1}^{j}a_{lh(l)}s_l^*}\right\}d\theta_1\cdots d\theta_j$$

and, by Lemmas 2 and 3,this is equal to

$$\int_{\Sigma|\xi_l|<\eta}\!\!\cdots\int e^{F(x_1,\ldots,x_j)}\,d\theta_1\cdots d\theta_j+O\{e^{F(y_1,\ldots,y_j)-C\chi^2}\}$$
$$+\,O\{\eta^je^{F(y_1,\ldots,y_j)-C/\mu_1\cdots\mu_j}\}.$$

By Lemma 4, the latter integral is asymptotic to

$$Ye^{F^*(y_1,\dots,y_j)} \int\cdots\int_{\Sigma|\xi_l|\leq\eta} \exp\left\{-Y^{-1}M^{-j}\zeta(j+1)\bar{R}_0 \times\left(\sum_{l=1}^{j}\xi_l^2+\sum_{l=1}^{j-1}\sum_{m=l+1}^{j}\xi_l\xi_m\right)\right\}d\xi_1\cdots d\xi_j.$$

If we transform the variables in this integral by writing

$$\xi_l = \{YM^j/\zeta(j+1)\bar{R}_0\}^{1/2}u_l \quad \text{for } 1 \leq l \leq j,$$

we obtain

$$M(2\pi)^j p(n_1, \dots, n_j)/Kd_1\cdots d_j \sim Y^{1/2(j+2)}\{M^j/\zeta(j+1)\bar{R}_0\}^{1/2 j} \times e^{F^*(y_1,\dots,y_j)}\int\cdots\int_{\Sigma|u_l|\leq\chi}\exp\left(-\sum u_l^2-\sum\sum u_lu_m\right)du_1\cdots du_j.$$

Since the integrand is positive everywhere,

$$\int_{-\chi/j}^{\chi/j}\cdots\int_{-\chi/j}^{\chi/j} < \int\cdots\int_{\Sigma|u_l|\leq\chi} < \int_{-\chi}^{\chi}\cdots\int_{-\chi}^{\chi}\exp\left(-\sum u_l^2-\sum u_lu_m\right)du_1\cdots du_j$$

and so, by Lemma 4 of [10],

$$p(n_1, \dots, n_j) \sim Kd_1\cdots d_jM^{-1}(j+1)^{-1/2}\{M^j/2\pi\zeta(j+1)\bar{R}_0\}^{1/2 j} \times Y^{1/2(j+2)}e^{F^*(y_1,\dots,y_j)}.$$

Theorem 1 follows from this and Lemma 5.

In order to examine $F^*(y_1, \dots, y_j)$ more precisely, it is necessary to investigate I and, for this purpose, we require two further lemmas.

LEMMA 6. *If $g(t)$ and all its derivatives are continuous in $0 \leq t \leq 1$, then, for any τ satisfying $0 < \tau < 1$,*

$$g(\tau) = \int_0^1 g(t)\,dt + \sum_{r=1}^{k}P_r(\tau)\{D^{r-1}g(1) - D^{r-1}g(0)\} - \sigma_k$$

for all $k \geq 1$, where

$$\sigma_k = \int_0^1 P_k(\tau - t)D^k g(t)\,dt.$$

PROOF. For $r > 1$, $P_r(t)$ is continuous, $DP_r(\tau - t) = -P_{r-1}(\tau - t)$ and $P_r(\tau - 1) = P_r(\tau)$. Therefore, by integration by parts,

$$\sigma_r = P_r(\tau)\{D^{r-1}g(1) - D^{r-1}g(0)\} + \sigma_{r-1} \tag{3.7}$$

for all $r > 1$. Also, $P_1(-0) - P_1(+0) = 1$, $DP_1(\tau - t) = -1$ for all $t \neq \tau$ and $P_1(\tau - 1) = P_1(\tau)$.

Now,

$$\int_0^\tau DP_1(\tau - t)g(t)\,dt = P_1(+0)g(\tau) - P_1(\tau)g(0) - \int_0^\tau P_1(\tau - t)Dg(t)\,dt,$$

$$\int_\tau^1 DP_1(\tau - t)g(t)\,dt = P_1(\tau)g(1) - P_1(-0)g(\tau) - \int_\tau^1 P_1(\tau - t)Dg(t)\,dt$$

and so,

$$\begin{aligned} g(\tau) - \int_0^1 g(t)\,dt &= \{P_1(-0) - P_1(+0)\}g(\tau) + \int_0^1 DP_1(\tau - t)g(t)\,dt \\ &= P_1(\tau)\{g(1) - g(0)\} - \int_0^1 P_1(\tau - t)Dg(t)\,dt. \end{aligned}$$

The lemma follows easily from (3.7).

COROLLARY. *For* $|\arg v| < \frac{1}{2}\pi - \epsilon_4$, $0 \leqslant \tau < 1$, *and any fixed positive integer* $k \geqslant 2$,

$$ve^{v\tau}(e^v - 1)^{-1} = 1 + \sum_{r=1}^{k-1} P_r(\tau)v^r + O(v^k)$$

where the constant implied in the order term depends upon τ, k *but is independent of* v.

PROOF. For $0 < \tau < 1$, putting $g(t) = e^{vt}$ in Lemma 6, we obtain

$$e^{v\tau} = v^{-1}(e^v - 1) + \sum_{r=1}^{k} P_r(\tau)v^{r-1}(e^v - 1) - v^k \int_0^1 P_k(\tau - t)e^{vt}\,dt.$$

Since $v(e^{\mathrm{Re}(v)} - 1)/(e^v - 1)\mathrm{Re}(v)$ is bounded, it follows that

$$ve^{v\tau}(e^v - 1)^{-1} = 1 + \sum_{r=1}^{k-1} P_r(\tau)v^r + O(v^k).$$

It is well known that this formula also holds for $\tau = 0$. (See, for example, [4, pp. 534–535].)

LEMMA 7. *If* $|\arg z_l| < \frac{1}{2}\pi - \epsilon_5$ *for* $2 \leqslant l \leqslant j$, *then*

$$Z''I(z_2, \ldots, z_j) = \sum_{r=0}^{k-1} I'_{sr} W_r + \sum_{r=k}^{(j-s)k} O(I'_{sr} z''^r)$$

for all $k > j$.

PROOF. From (2.7) and the preceding corollary,

$$\beta(u\colon z_2, \ldots, z_j) = Z''^{-1} u^{s-j-1} \prod_{l=1}^{s} (e^{z_l u} - 1)^{-1} e^{(1-a_{lh(l)}/M) z_l u}$$

$$\times \left\{ \sum_{r=0}^{k-1} W_r u^r + \sum_{r=k}^{(j-s)k} O(z''^r u^r) \right\}$$

$$- Z^{-1} \left\{ \sum_{m=0}^{j-1} \sum_{r=0}^{m} V_r W_{m-r} u^{m-j-1} + \sum_{r=0}^{j} V_r W_{j-r} (e^u - 1)^{-1} \right\}$$

$$= Z''^{-1} \left\{ \sum_{r=0}^{k-1} \beta'_{sr}(u\colon z_2, \ldots, z_s) W_r \right.$$

$$\left. + \sum_{r=k}^{(j-s)k} O\{\beta'_{sr}(u\colon z_2, \ldots, z_s) z''^r\} \right\}.$$

The lemma follows by integration over u from 0 to ∞.

Since $e^t - 1 > t^m/m!$ for all positive t and all positive integers m, we have, for all $r \geqslant j + 1$,

$$\beta'_{sr}(u\colon y_2/y_1, \ldots, y_s/y_1) = O\{y_1^s (y_1 \cdots y_s)^{-1}\}$$

for $0 \leqslant u \leqslant \frac{1}{2}\pi$ and

$$\beta'_{sr}(u\colon y_2/y_1, \ldots, y_s/y_1) = O\{y_1^s (y_1 \cdots y_s)^{-1} u^{-2}\}$$

for $u \geqslant \frac{1}{2}\pi$. It follows that, for all $r \geqslant j + 1$,

$$I'_{sr}(y_2/y_1, \ldots, y_s/y_1) = O\{\mu_1^s (\mu_1 \cdots \mu_s)^{-1}\}.$$

From (2.4),

$$\sum_{m=0}^{\infty} V_m \left(1, \frac{y_2}{y_1}, \ldots, \frac{y_s}{y_1}\right) u^m$$

$$= \prod_{l=1}^{s} (e^{u y_l / y_1} - 1)^{-1} \, e^{(1-a_{lh(l)}/M) u y_l / y_1} \, \frac{u y_l}{y_1}$$

and so,

$$|V_m(1, y_2/y_1, \ldots, y_s/y_1)| \pi^m \leqslant e^{s\pi} \left\{ 1 + \frac{\pi}{2} + \sum_{\nu=1}^{\infty} B_{2\nu}(0) \pi^{2\nu}/(2\nu)! \right\} = C.$$

Therefore, for $0 \leqslant r \leqslant j$ and $0 \leqslant u \leqslant \frac{1}{2}\pi$,

$$\beta'_{sr}(u:y_2/y_1, \ldots, y_s/y_1)$$
$$= y_1^s(y_1 \cdots y_s)^{-1}\left\{V_{j-r}\left(1, \frac{y_2}{y_1}, \ldots, \frac{y_s}{y_1}\right)\{u^{-1} - (e^u - 1)^{-1}\}\right.$$
$$\left. + \sum_{m=j-r+1}^{\infty} V_m\left(1, \frac{y_2}{y_1}, \ldots, \frac{y_s}{y_1}\right)u^{m+r-j-1}\right\}$$

and so,

$$\int_0^{\frac{1}{2}\pi} \beta'_{sr}\left(u: \frac{y_2}{y_1}, \ldots, \frac{y_s}{y_1}\right) du = O\left\{y_1^s(y_1 \cdots y_s)^{-1} \sum_{m=0}^{\infty} 2^{-m}\right\}$$
$$= O\{\mu_1^s(\mu_1 \cdots \mu_s)^{-1}\}.$$

Also, for $0 \leqslant r \leqslant j$ and $u \geqslant \frac{1}{2}\pi$,

$$\beta'_{sr}(u: y_2/y_1, \ldots, y_s/y_1) = O\{y_1^s(y_1 \cdots y_s)^{-1}u^{-2}\}$$

and so,

$$\int_{\frac{1}{2}\pi}^{\infty} \beta'_{sr}(u: y_2/y_1, \ldots, y_s/y_1)du = O\{\mu_1^s(\mu_1 \cdots \mu_s)^{-1}\}.$$

Hence, for $0 \leqslant r \leqslant j$,

$$I'_{sr}(y_2/y_1, \ldots, y_s/y_1) = O\{\mu_1^s(\mu_1 \cdots \mu_s)^{-1}\}.$$

Now, if $\mu_{s+1}, \ldots, \mu_j$ are each $O(\mu_1^{1+\epsilon_6})$ for some fixed positive number ϵ_6 but none of $\mu_2, \ldots, \mu_s$ is $O(\mu_1^{1+\epsilon_7})$ for any fixed positive number ϵ_7, then it follows from Lemma 7 that

$$I\left(\frac{y_2}{y_1}, \ldots, \frac{y_j}{y_1}\right) = y_1^{j-s}(y_{s+1} \cdots y_j)^{-1} \sum_{r=0}^{k-1} I'_{sr}\left(\frac{y_2}{y_1}, \ldots, \frac{y_s}{y_1}\right)$$
$$\times (My_1)^{-r}Q_r(y_{s+1}, \ldots, y_j) + O\{\mu_1^{j-k}(\mu_1 \cdots \mu_j)^{-1}\mu_{s+1}^k\}. \tag{3.8}$$

If k is chosen so that $\mu_1^{j-k}(\mu_1 \cdots \mu_j)^{-1}\mu_{s+1}^k = o(1)$, then the expression for I given in (3.8) may be used to calculate $F^*(y_1, \ldots, y_j)$ in Theorem 1.

4. Proofs of the first three lemmas.

PROOF OF LEMMA 1. It is easily verified that the relation is an equivalence relation. We observe that the final exponential expression in (3.6) is the same for all $(s_1, \ldots, s_j)$ for which the sums $\Sigma a_{lh(l)}s_l$ are congruent modulo M for all $h(1), \ldots, h(j)$. To evaluate $g(x_1, \ldots, x_j)$ therefore, we first compute the sums $\Sigma'\omega^{n_1s_1+\cdots+n_js_j}$, where each sum Σ' is taken over all the $(s_1, \ldots, s_j)$ for which $\Sigma a_{lh(l)}s_l \equiv \rho_{h(1)\ldots h(j)} \pmod M$ for all $h(1), \ldots, h(j)$ and the $\rho_{h(1)\ldots h(j)}$ are nonnegative integers less than M.

For any particular set of $\rho_{h(1)\ldots h(j)}$, for which there is no $(s_1, \ldots, s_j)$ such that $\Sigma a_{lh(l)}s_l \equiv \rho_{h(1)\ldots h(j)} \pmod M$ for all $h(1), \ldots, h(j)$, then clearly the sum is zero. Otherwise, there is at least one $(s_1, \ldots, s_j)$, say $(s_1^*, \ldots, s_j^*)$, such that $\Sigma a_{lh(l)}s_l^* \equiv \rho_{h(1)\ldots h(j)} \pmod M$ for all $h(1), \ldots, h(j)$. It follows that, for any $(s_1, \ldots, s_j)$ for which $\Sigma a_{lh(l)}s_l \equiv \rho_{h(1)\ldots h(j)} \pmod M$ for all $h(1), \ldots, h(j)$, then $\Sigma a_{lh(l)}(s_l - s_l^*) \equiv 0 \pmod M$ for all $h(1), \ldots, h(j)$. These congruences are equivalent to the $j + 1$ congruences $\Sigma a_{l1}(s_l - s_l^*) \equiv 0 \pmod M$ and $d_h(s_h - s_h^*) \equiv 0 \pmod M$ for $1 \leqslant h \leqslant j$. Hence, in this case,

$$\sum{}' \omega^{n_1s_1+\cdots+n_js_j} = \omega^{n_1s_1^*+\cdots+n_js_j^*} \underset{\Sigma Ma_{l1}s_l/d_l\equiv 0 \pmod M}{\sum_{s_1=0}^{d_1-1} \cdots \sum_{s_j=0}^{d_j-1}} \omega^{M(n_1s_1/d_1+\cdots+n_js_j/d_j)}$$

$$= \omega^{n_1s_1^*+\cdots+n_js_j^*} \sum_{s_1=0}^{d_1-1} \cdots \sum_{s_j=0}^{d_j-1} \omega^{M(n_1s_1/d_1+\cdots+n_js_j/d_j)}$$

$$\times M^{-1} \sum_{\nu=0}^{M-1} \omega^{-\nu\Sigma Ma_{l1}s_l/d_l}$$

$$= \omega^{n_1s_1^*+\cdots+n_js_j^*} M^{-1} \sum_{\nu=0}^{M-1} \sum_{s_1=0}^{d_1-1} \omega^{Ms_1(n_1-\nu a_1)/d_1}$$

$$\cdots \sum_{s_j=0}^{d_j-1} \omega^{Ms_j(n_j-\nu a_{j1})/d_j}$$

$$= \omega^{n_1s_1^*+\cdots+n_js_j^*} M^{-1} d_1 \cdots d_j K.$$

Lemma 1 follows immediately.

Proof of Lemma 2. At least one ξ_l satisfies $|\xi_l| > \eta/j$. Then

$$|e^{Mx_l} - 1|^2 = e^{2My_l} + 1 - 2e^{My_l}\cos My_l\xi_l$$

$$= (e^{My_l} - 1)^2 + 4e^{My_l}\sin^2 \tfrac{1}{2}M\theta_l \geqslant (e^{My_l} - 1)^2(1 + C\eta^2)$$

for $|\theta_l| \leqslant \pi/M$. Hence, for any $h(1), \ldots, h(j)$,

$$|\Lambda_{h(1)\ldots h(j)}(1{:}\, x_1, \ldots, x_j)| \leqslant (1 - C\eta^2)\Lambda_{h(1)\ldots h(j)}(1{:}\, y_1, \ldots, y_j).$$

Also, for all $r > 1$ and any $h(1), \ldots, h(j)$,

$$|\Lambda_{h(1)\ldots h(j)}(r: x_1, \ldots, x_j)| \leqslant \Lambda_{h(1)\ldots h(j)}(r: y_1, \ldots, y_j)$$

and so, from (3.6),

$$|g(x_1, \ldots, x_j)| \leqslant C \exp\{F(y_1, \ldots, y_j) - C\chi^2\}.$$

PROOF OF LEMMA 3. Each of the terms in $\Sigma^{(*)}$ has at least one of the sums $\Sigma a_{lh(l)} s_l^* \not\equiv 0 \pmod M$. Then, for some $h(1), \ldots, h(j)$, $\Sigma a_{lh(l)} s_l^* \equiv \rho \not\equiv 0 \pmod M$. Therefore,

$$\Lambda_{h(1)\ldots h(j)}(1: x_1, \ldots, x_j)\omega^{-\Sigma a_{lh(l)} s_l^*}$$

$$= \Lambda_{h(1)\ldots h(j)}(1: y_1, \ldots, y_j)\,\{1 + O(\eta)\}(\cos 2\pi\rho/M - i \sin 2\pi\rho/M).$$

It follows that

$$\left|\exp\left\{\sum_{r=1}^{\infty} \sum_{h(1)=1}^{q(1)} \cdots \sum_{h(j)=1}^{q(j)} \Lambda_{h(1)\ldots h(j)}(r: x_1, \ldots, x_j)\omega^{-r\Sigma a_{lh(l)} s_l^*}\right\}\right|$$

$$= \exp \operatorname{Re}\left\{\sum_{r=1}^{\infty} \sum_{h(1)=1}^{q(1)} \cdots \sum_{h(j)=1}^{q(j)} \Lambda_{h(1)\ldots h(j)}(r: x_1, \ldots, x_j)\omega^{-r\Sigma a_{lh(l)} s_l^*}\right\}$$

$$\leqslant \exp\left\{\sum_{r=1}^{\infty} \sum_{h(1)=1}^{q(1)} \cdots \sum_{h(j)=1}^{q(j)} \Lambda_{h(1)\ldots h(j)}(r: y_1, \ldots, y_j) - Y^{-1}(1 - \cos 2\pi\rho/M)\{1 + O(\eta) + O(\mu_1)\}\right\}$$

$$\leqslant \exp\left\{\sum_{r=1}^{\infty} \sum_{h(1)=1}^{q(1)} \cdots \sum_{h(j)=1}^{q(j)} \Lambda_{h(1)\ldots h(j)}(r: y_1, \ldots, y_j) - CY^{-1}\right\}.$$

Lemma 3 follows immediately.

5. **Proof of Theorem 2.** In order to prove Theorem 2, we first establish the following lemma.

LEMMA 8. *As $t \longrightarrow \infty$ through real positive values, $D^k G(t) \longrightarrow 0$ for all $k \geqslant 0$ and, for all $k \geqslant 1$,*

$$\int_0^\infty |D^k G(t)|\, dt = O(x^{*j+k}).$$

PROOF. Obviously $G(t) \longrightarrow 0$ as $t \longrightarrow \infty$. Let $t_0 = \tfrac{1}{2}\pi/M|x^*|$. Now $D^k \Sigma_0^j Q_m t^{m-j-1} \longrightarrow 0$ as $t \longrightarrow \infty$ and

$$\int_{t_0}^{\infty}\left|D^k \sum_{m=0}^{j} Q_m t^{m-j-1}\right| dt$$

$$\leqslant \int_{t_0}^{\infty} \sum_{m=0}^{j} |Q_m|(j+1-m)\cdots(j+k-m)t^{m-j-k-1}\,dt$$

$$= \sum_{m=0}^{j} |Q_m|(j+1-m)\cdots(j+k-1-m)t_0^{m-j-k} = O(x^{*j+k}).$$

Since

$$D\{(e^{Mx_l t}-1)^{-1}\} = -Mx_l\{(e^{Mx_l t}-1)^{-1} + (e^{Mx_l t}-1)^{-2}\},$$

we see that $D^k\Lambda(t)$ is the sum of a finite number of terms of the form

$$Ct^{-a-1}\prod_{l=1}^{j}(Mx_l)^{r_l}\{(M-a_{lh(l)})x_l\}^{s_l}(e^{Mx_l t}-1)^{-\rho_l-1},$$

where $a \geqslant 0$, $r_l \geqslant \rho_l \geqslant 0$, $s_l \geqslant 0$, $a + \Sigma_1^j(r_l+s_l) = k$. Hence, $D^k\Lambda(t) \longrightarrow 0$ as $t \longrightarrow \infty$ and $\int_{t_0}^{\infty}|D^k\Lambda(t)|\,dt$ is dominated by the sum of a finite number of terms of the form

$$C|X|\int_{t_0}^{\infty}\left\{\prod_{l=1}^{j}|x_l|^{r_l+s_l}|e^{Mx_l t}-1|^{-\rho_l-1}\right\}t^{-a-1}\,dt$$

$$\leqslant C|X|\int_{t_0}^{\infty}\left\{\prod_{l=1}^{j}|x_l|^{r_l+s_l}|x_l t|^{-\rho_l-1}\right\}t^{-a-1}\,dt$$

$$\leqslant C\left\{\prod_{l=1}^{j}|x_l|^{r_l+s_l-\rho_l}\right\}t_0^{-j-a-\Sigma\rho_l}$$

$$\leqslant C|x^*|^{\Sigma(r_l+s_l-\rho_l)}|x^*|^{j+a+\Sigma\rho_l} = O(x^{*j+k}).$$

Therefore, $D^kG(t) \longrightarrow 0$ as $t \longrightarrow \infty$ and $\int_{t_0}^{\infty}|D^kG(t)|\,dt = O(x^{*j+k})$.

By substituting $t = \pi/M|x^*|$ in (2.4), we obtain

$$\sum_{m=0}^{\infty}|Q_m|(\pi/M|x^*|)^m \leqslant \prod_{l=1}^{j} e^{\pi}\left\{1+\frac{\pi}{2}+\sum_{\nu=1}^{\infty}B_{2\nu}(0)\pi^{2\nu}/(2\nu)!\right\} = C$$

and so, $|Q_m| \leqslant C(M|x^*|/\pi)^m$. Now, for $0 \leqslant t \leqslant t_0$,

$$|D^kG(t)| \leqslant \sum_{m=j+k+1}^{\infty}(m-j-1)\cdots(m-j-k)|Q_m|t^{m-j-k-1}$$

and hence,

$$\int_0^{t_0} |D^k G(t)|\,dt \leqslant C \sum_{m=j+k+1}^{\infty} (m-j-1)\cdots(m-j-k+1)2^{j+k-m}|x^*|^{j+k}$$
$$= O(x^{*j+k}).$$

This completes the proof of the lemma.

PROOF OF THEOREM 2. For any positive integer N, we have

$$\int_0^N P_1(t)DG(t)\,dt = \sum_{r=0}^{N-1} \int_0^1 (t-\tfrac{1}{2})DG(t+r)\,dt$$
(5.1)
$$= \tfrac{1}{2}\{G(0)+G(N)\} + \sum_{r=1}^{N-1} G(r) - \int_0^N G(t)\,dt.$$

By repeated integration by parts, Lemma 8 shows that

$$\int_0^\infty P_1(t)DG(t)\,dt = \sum_{s=1}^{2k} (-1)^s P_{s+1}(0)D^s G(0) + \int_0^\infty P_{2k+1}(t)D^{2k+1}G(t)\,dt$$
(5.2)
$$= \sum_{s=1}^{k-1} (-1)^s B_{2s}(0)Q_{2s+j}/2s + O(x^{*2k+j}).$$

Now,

$$\sum_{r=1}^{N-1} G(r) - \int_0^N G(t)\,dt = A_1 + A_2 + A_3,$$

where

$$A_1 = \sum_{r=1}^{N-1} \{G(r) + Q_j/r\},$$

$$A_2 = Q_j\left\{\int_0^N (t^{-1} - (e^t-1)^{-1})\,dt - \sum_{r=1}^{N-1} r^{-1}\right\},$$

$$A_3 = -\int_0^N \{G(t) + Q_j(t^{-1} - (e^t-1)^{-1})\}\,dt.$$

As $N \longrightarrow \infty$,

$$A_1 \longrightarrow XM^j \sum_{r=1}^{\infty} \Lambda(r) - \sum_{m=0}^{j-1} \zeta(j+1-m)Q_m,$$

$$A_2 = Q_j\left\{\log N - \log(1-e^{-N}) - \sum_{r=1}^{N-1} r^{-1}\right\} \longrightarrow -\gamma Q_j,$$

$$A_3 \longrightarrow -\int_0^\infty \{G(t) + Q_j(t^{-1} - (e^t-1)^{-1})\}\,dt = -H.$$

Therefore, letting $N \longrightarrow \infty$ in (5.1), we obtain

$$\int_0^\infty P_1(t)DG(t)\,dt = XM^j\sum_{r=1}^{\infty}\Lambda(r) - \sum_{m=0}^{j-1}\zeta(j+1-m)Q_m - \gamma Q_j + \tfrac{1}{2}Q_{j+1} - H,$$

and Theorem 2 follows from (3.4) and (5.2) since $\zeta(0) = -\frac{1}{2}$ and $\zeta(-2s) = 0$, $\zeta(1-2s) = (-1)^s B_{2s}(0)/2s$ for all positive integers s.

6. **Proof of Lemma 4.** We define the linear operator T by

$$Th(x_1, \ldots, x_j) = h(x_1, \ldots, x_j) - h(y_1, \ldots, y_j) - \sum_{l=1}^{j} i\xi_l y_l \frac{\partial h}{\partial y_l}(y_1, \ldots, y_j),$$

where $h(x_1, \ldots, x_j)$ is any function of $x_1, \ldots, x_j$ with continuous first order partial derivatives. First we prove that

$$TF^*(x_1, \ldots, x_j) = -Y^{-1}M^{-j}\zeta(j+1)\overline{R}_0\left(\sum_{l=1}^{j}\xi_l^2 + \sum_{l=1}^{j-1}\sum_{m=l+1}^{j}\xi_l\xi_m\right) + o(1). \tag{6.1}$$

Trivial calculations show that

$$T\left(\sum_{l=1}^{j} n_l x_l\right) = 0,$$

$$T(X^{-1}) = -Y^{-1}\left(\sum_{l=1}^{j}\xi_l^2 + \sum_{l=1}^{j-1}\sum_{m=l+1}^{j}\xi_l\xi_m\right) + O(\eta^3 Y^{-1}),$$

$$T(X^{-1}Q_j \log x_1) = O(\eta^2 Y^{-1}R_j \log y_1)$$

and, for all $m > 0$,

$$T(X^{-1}Q_m) = O(\eta^2 Y^{-1}R_m).$$

Now

$$\eta^3 Y^{-1} = O\{(\mu_1 \cdots \mu_j)^{\frac{1}{2}-3\epsilon_2}\} = o(1).$$

From (3.1), $n_1^j \leqslant n_1 \cdots n_j < n_1^{j+1-\epsilon_1}$, $n_j < n_1^{2-\epsilon_1}$, and therefore it follows from (2.8) that

$$Cn_1^{-1+j\epsilon_1/(j+1)} < \mu_j \leqslant \mu_1 < Cn_1^{-\epsilon_1/(j+1)}.$$

Hence, we have

$$\mu_j \leqslant \mu_1 < \mu_j^{\epsilon_1/(j+1)}, \quad \log \mu_j = O(\log \mu_1). \tag{6.2}$$

Also, $\eta^2 Y^{-1}|R_j \log y_1|$ and $\eta^2 Y^{-1}|R_m|$ for all $m > 0$ are each less than

$$C\mu_1(\mu_1 \cdots \mu_j)^{-2\epsilon_2} = Cn_1^{-1}(n_1 \cdots n_j)^{(1+2\epsilon_2)/(j+1)}$$
$$< Cn_1^{2\epsilon_2-\epsilon_1(1+2\epsilon_2)/(j+1)} = o(1)$$

by (3.3). We have, therefore,

$$T(X^{-1}) = -Y^{-1}\left(\sum_{l=1}^{j} \xi_l^2 + \sum_{l=1}^{j-1} \sum_{m=l+1}^{j} \xi_l \xi_m\right) + o(1),$$

$$T(X^{-1}Q_j \log x_1) = o(1)$$

and, for all $m > 0$, $T(X^{-1}Q_m) = o(1)$.

If we write $\Gamma(t) = h\{y_1(1 + i\xi_1 t), \ldots, y_j(1 + i\xi_j t)\}$, where $h(x_1, \ldots, x_j)$ has continuous second order partial derivatives, then Taylor's theorem gives

$$\Gamma(1) = \Gamma(0) + D\Gamma(0) + \tfrac{1}{2}D^2\Gamma(\psi)$$

for some ψ which satisfies $0 < \psi < 1$. It follows that

$$Th(x_1, \ldots, x_j)$$
$$= -\frac{1}{2}\sum_{l=1}^{j}\sum_{m=1}^{j} \xi_l \xi_m y_l y_m \frac{\partial^2 h}{\partial x_l \partial x_m} \{y_1(1 + i\xi_1\psi), \ldots, y_j(1 + i\xi_j\psi)\}.$$

For all $l > 1$, routine calculations give

$$\partial^2\Omega(u)/\partial x_l^2 = \Omega(u)\{(a_{lh(l)}u/Mx_1 + u/x_1(e^{z_l u} - 1))^2 + u^2 e^{z_l u}/x_1^2(e^{z_l u} - 1)^2\},$$

with similar expressions for the other second order partial derivatives. Since $e^t - 1 > t^r/r!$ for all positive t and all positive integers r, we have

$$T\Omega(u) = O\{\eta^2 \mu_1^j(\mu_1 \ldots \mu_j)^{-1}u^{-j-1}\}$$

and so,

$$T\left\{\int_{\frac{1}{2}\pi}^{\infty} \Omega(u)\,du\right\} = O(\eta^2\mu_1^j/\mu_1 \cdots \mu_j).$$

If we substitute $t = \pi/My_1$ in (2.4), we obtain

$$\sum_{m=0}^{\infty} R_m^*(\pi/My_1)^m \leqslant e^{j\pi}\left\{1 + \frac{\pi}{2} + \sum_{\nu=1}^{\infty} B_{2\nu}(0)\pi^{2\nu}/(2\nu)!\right\}^j = C,$$

where R_m^* is written for the sum of the moduli of the monomials of which the polynomial $R_m = Q_m(y_1, \ldots, y_j)$ is composed. Therefore, for all $m \geqslant 0$,

$R^*_m y_1^{-m} \leq CM^m \pi^{-m}$. Since $Z^{-1}U_m$ is a sum of monomials of the form $CM^{-m}\Pi_{l=1}^{j}(x_l/x_1)^{r_l-1}$, where $\Sigma r_l = m$ and

$$\sum_{l=1}^{j}\sum_{m=1}^{j} x_l x_m \frac{\partial^2}{\partial x_l \partial x_m}\left\{\prod_{l=1}^{j}\left(\frac{x_l}{x_1}\right)^{r_l-1}\right\} \leq j^2 m^2 \prod_{l=1}^{j}\left(\frac{x_l}{x_1}\right)^{r_l-1},$$

it follows that

$$|T(Z^{-1}U_m)| \leq C\eta^2 m^2 M^{-m} Y^{-1} R^*_m y_1^{j-m} \leq C\eta^2 m^2 \pi^{-m} y_1^j Y^{-1}.$$

We can immediately deduce that

$$T\left(\int_{\frac{1}{2}\pi}^{\infty} Z^{-1}\left\{\sum_{m=0}^{j-1} U_m u^{m-j-1} + U_j(e^u - 1)^{-1}\right\} du\right) = O\left(\frac{\eta^2 \mu_1^j}{\mu_1 \cdots \mu_j}\right).$$

Also, since for $0 \leq u \leq \frac{1}{2}\pi$

$$\beta(u\colon z_2, \ldots, z_j) = Z^{-1}\left\{\sum_{m=j+1}^{\infty} U_m u^{m-j-1} + U_j(u^{-1} - (e^u - 1)^{-1})\right\},$$

we deduce that

$$T\left\{\int_0^{\frac{1}{2}\pi} \beta(u\colon z_2, \ldots, z_j)\, du\right\} = O\left(\eta^2 y_1^j Y^{-1} \sum_{m=0}^{\infty} m^2 2^{-m}\right)$$

$$= O(\eta^2 \mu_1^j/\mu_1 \cdots \mu_j).$$

Combining the results of the last two paragraphs, we have

$$T(I) = O(\eta^2 \mu_1^j/\mu_1 \cdots \mu_j) = O(n_1^{2\epsilon_2 - \epsilon_1 (j+2\epsilon_2)/(j+1)}) = o(1)$$

by (3.3). (6.1) follows from (2.6) and (2.9). From (3.2), we have

$$i\xi_l y_l \partial F^*(y_1, \ldots, y_j)/\partial y_l = o(1)$$

for $1 \leq l \leq j$. Therefore, from (6.1),

$$F^*(x_1, \ldots, x_j) = F^*(y_1, \ldots, y_j)$$
$$- Y^{-1}M^{-j}\zeta(j+1)\overline{R}_0\left(\sum_{l=1}^{j}\xi_l^2 + \sum_{l=1}^{j-1}\sum_{m=l+1}^{j}\xi_l\xi_m\right) + o(1).$$

Again, we see from (2.9), (3.4) and Theorem 2 that

$$F(x_1, \ldots, x_j) = F^*(x_1, \ldots, x_j) + o(1)$$

and Lemma 4 follows immediately.

7. **Proof of Lemma 5.** As the proof of this lemma is very similar to that of Lemma 3 in [11], we provide only an outline. It is easily seen that, for all $l > 1$,

$$x_l \partial\Omega(u)/\partial x_l = \Omega(u)\{(1 - a_{lh(l)}/M)z_l u - z_l u e^{z_l u}(e^{z_l u} - 1)^{-1}\}$$

$$= - z_l \Omega(u)\{a_{lh(l)} u/M + u(e^{z_l u} - 1)^{-1}\}.$$

Because of the uniqueness of power series expansions, this equality implies that, if both sides are expanded in increasing powers of u, the corresponding coefficients are equal. Therefore,

$$x_l \partial I/\partial x_l = - z_l\{(a_{lh(l)}/M)I'_{j1} + I^{(l)}\},$$

where

$$I^{(l)} = I^{(l)}(z_2, \dots, z_j) = I'_{j+1,1}(z_2, \dots, z_j, z_l)$$

and the $a_{lh(l)}$ associated with the second z_l is M. It follows immediately that

$$y_l \partial I(y_2/y_1, \dots, y_j/y_1)/\partial y_l$$

$$= - (y_l/y_1)\{(a_{lh(l)}/M)I'_{j1}(y_2/y_1, \dots, y_j/y_1) + I^{(l)}(y_2/y_1, \dots, y_j/y_1)\}.$$

Similarly,

$$y_1 \partial I(y_2/y_1, \dots, y_j/y_1)/\partial y_1$$

$$= \sum_{k=2}^{j} (y_k/y_1)\{(a_{kh(k)}/M)I'_{j1}(y_2/y_1, \dots, y_j/y_1) + I^{(k)}(y_2/y_1, \dots, y_j/y_1)\}.$$

Now, if $\mu_{s+1}, \dots, \mu_j$ are each $O(\mu_1^{1+\epsilon_6})$ for some fixed positive number ϵ_6 but none of $\mu_2, \dots, \mu_s$ is $O(\mu_1^{1+\epsilon_7})$ for any fixed positive number ϵ_7, then both $I'_{j1}(y_2/y_1, \dots, y_j/y_1)$ and $I^{(l)}(y_2/y_1, \dots, y_j/y_1)$ can be expanded in powers of $y_{s+1}, \dots, y_j$ as $I(y_2/y_1, \dots, y_j/y_1)$ was in (3.8). It follows from (2.9) that $Yy_l \partial F^*(y_1, \dots, y_j)/\partial y_l$ can be written in the form

$$\begin{aligned} & Yy_l n_l + \sum_{k(1)\dots k(j)} c^{(l)}_{k(1)\dots k(j)} y_1^{k(1)} \cdots y_j^{k(j)} \\ \text{(7.1)}\qquad & + y_1^j \sum_{k(s+1)\dots k(j)} c'^{(l)}_{k(s+1)\dots k(j)} \left(\frac{y_{s+1}}{y_1}\right)^{k(s+1)} \cdots \left(\frac{y_j}{y_1}\right)^{k(j)} \\ & + o\{(\mu_1 \cdots \mu_j)^{\frac{1}{2}+\epsilon_2}\}, \end{aligned}$$

where the first sum is taken over all nonnegative integers $k(1), \dots, k(j)$ such that $\mu_1^{k(1)} \cdots \mu_j^{k(j)}$ is not $o\{(\mu_1 \cdots \mu_j)^{\frac{1}{2}+\epsilon_2}\}$ and the second sum is taken over all nonnegative integers $k(s+1), \ldots, k(j)$ such that $(\mu_{s+1}/\mu_1)^{k(s+1)} \cdots (\mu_j/\mu_1)^{k(j)}$ is not $o\{\mu_1^{-j}(\mu_1 \cdots \mu_j)^{1/2+\epsilon_2}\}$. The coefficients in the first sum are constants except when $k(1) + \cdots + k(j) = j$ in which case they are linear functions of $\log y_1$. The coefficients in the second sum are $O(1)$ and involve definite integrals of the form $I'_{sr}(y_2/y_1, \ldots, y_s/y_1)$ and $I'_{s+1,r}(y_2/y_1, \ldots, y_s/y_1, y_l/y_1)$ for $2 \leqslant l \leqslant s$. Also, for $1 \leqslant l \leqslant j$, we have

$$c^{(l)}_{0\ldots 0} = -M^{-j}\zeta(j+1)\overline{R}_0. \tag{7.2}$$

We put

$$\begin{aligned} y_l = \mu_l \sum_{k(1)\ldots k(j)k'(s+1)\ldots k'(j)} & \gamma^{(l)}_{k(1)\ldots k(j)k'(s+1)\ldots k'(j)}\mu_1^{k(1)} \cdots \mu_j^{k(j)} \\ & \times (\mu_{s+1}/\mu_1)^{k'(s+1)} \cdots (\mu_j/\mu_1)^{k'(j)} \end{aligned} \tag{7.3}$$

for $1 \leqslant l \leqslant j$, where the sum is taken over all nonnegative integers $k(1), \ldots, k(j), k'(s+1), \ldots, k'(j)$ such that $\mu_1^{k(1)} \cdots \mu_j^{k(j)}(\mu_{s+1}/\mu_1)^{k'(s+1)} \cdots (\mu_j/\mu_1)^{k'(j)}$ is not $o\{(\mu_1 \cdots \mu_j)^{1/2+\epsilon_2}\}$. From (2.8), $n_l\mu_l(\mu_1 \cdots \mu_j) = M^{-j}\zeta(j+1)\overline{R}_0$ and therefore, it is easily seen from (7.2) that, for all l, $\gamma^{(l)}_{0\ldots 0,0\ldots 0} = 1$. Hence, the expressions for y_l given in (7.3) satisfy $y_l \sim \mu_l$.

Now, by substituting in (7.1) for each y_l the finite series given in (7.3) and by equating the coefficients of $\mu_1^{k(1)} \cdots \mu_j^{k(j)}(\mu_{s+1}/\mu_1)^{k'(s+1)} \cdots (\mu_j/\mu_1)^{k'(j)}$ on each side of the resulting equation, we can calculate the other coefficients successively. Thus, for example,

$$2\gamma^{(l)}_{10\ldots 0,0\ldots 0} + \sum_{\nu=1;\nu\neq l}^{j} \gamma^{(\nu)}_{10\ldots 0,0\ldots 0} = -c^{(l)}_{10\ldots 0}M^{-j}\zeta(j+1)\overline{R}_0$$

for $1 \leqslant l \leqslant j$ and these equations can easily be solved for each $\gamma^{(l)}_{10\ldots 0,0\ldots 0}$ in terms of the $c^{(\nu)}_{10\ldots 0}$ for $1 \leqslant \nu \leqslant j$. Also, for any $k(1), \ldots, k(j), k'(s+1), \ldots, k'(j)$,

$$\begin{aligned} 2\gamma^{(l)}_{k(1)\ldots k(j)k'(s+1)\ldots k'(j)} + \sum_{\nu=1;\nu\neq l}^{j} & \gamma^{(\nu)}_{k(1)\ldots k(j)k'(s+1)\ldots k'(j)} \\ & = c^{*(l)}_{k(1)\ldots k(j)k'(s+1)\ldots k'(j)}, \end{aligned}$$

where the $c^{*(l)}_{k(1)\ldots k(j)k'(s+1)\ldots k'(j)}$ involve the coefficients $c^{(l)}_{k(1)\ldots k(j)}$, $c'^{(l)}_{k(s+1)\ldots k(j)}$ and the $\gamma^{(l)}_{\nu(1)\ldots \nu(j)\nu'(s+1)\ldots \nu'(j)}$ for which every $\nu(l) \leqslant k(l)$, every $\nu'(l) \leqslant k'(l)$ and $\Sigma_1^j \nu(l) + \Sigma_{s+1}^j \nu'(l) < \Sigma_1^j k(l) + \Sigma_{s+1}^j k'(l)$. These equations can be solved to give each $\gamma^{(l)}_{k(1)\ldots k(j)k'(s+1)\ldots k'(j)}$ in terms of the $c^{*(\nu)}_{k(1)\ldots k(j)k'(s+1)\ldots k'(j)}$ for $1 \leqslant \nu \leqslant j$. Therefore, the coefficients $\gamma^{(l)}_{k(1)\ldots k(j)k'(s+1)\ldots k'(j)}$ can be calculated successively, provided that we calculate first the $\gamma^{(l)}_{k(1)\ldots k(j)k'(s+1)\ldots k'(j)}$ for which $\Sigma_1^j k(l) + \Sigma_{s+1}^j k'(l) = 1$, then those for which $\Sigma_1^j k(l) + \Sigma_{s+1}^j k'(l) = 2$, and so on. The y_l can thus be expressed completely in terms of the μ_l and this completes the proof of Lemma 5.

8. **Concluding remarks.** The dominant term in the asymptotic expansion for $p(n_1, \ldots, n_j)$ gives

$$\log p(n_1, \ldots, n_j) \sim (j + 1)\{M^{-j}q(1) \cdots q(j)\zeta(j + 1)n_1 \cdots n_j\}^{1/(j+1)},$$

which generalizes the well-known results for $j = 1$,

$$\log p(n_1\colon A_1) \sim \pi(2n_1/3)^{1/2}, \qquad A_1 = \{1, 2, 3, \ldots\},$$

$$\log p(n_1\colon A_1) \sim \pi(n_1/3)^{1/2}, \qquad A_1 = \{1, 3, 5, \ldots\}.$$

When $\mu_2, \ldots, \mu_j$ are each $O(\mu_1^{1+\epsilon_6})$ for some fixed positive number ϵ_6, the asymptotic formula for $p(n_1, \ldots, n_j)$ is expressed entirely in terms of elementary functions. Otherwise, the formula involves the definite integrals I'_{sr}, which we have been unable to express in terms of elementary functions except in very particular cases. However, since $\log f(x_1, \ldots, x_j) - \log f(kx_1, \ldots, kx_j)$ does not involve the integrals I, asymptotic formulae for the number of partitions of a multipartite number in which each part cannot occur more than a fixed number of times can be expressed entirely in terms of elementary functions.

Our references to the integral I and its properties have been somewhat abbreviated in this note. However, in an article in preparation, one of the authors will discuss in detail the properties of both the integral I and its generalization I'_{sr}. The number $K(n_1, \ldots, n_j\colon A_1, \ldots, A_j)$ also appears to be of independent interest and it is our intention to investigate this in the near future.

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