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# EXACT COLIMITS AND FIXED POINTS

BY

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ABSTRACT. In this paper we shall give details of some work sketched in [6] on the exactness of the functor colim: $\mathrm{Ab}^{\mathcal{C}} \longrightarrow \mathrm{Ab}$. We shall also investigate the connection between this work and a paper of J. Adámek and J. Reiterman [1] characterizing those categories $\mathcal{C}$ with the property that every endomorphism of an indecomposable functor $\mathcal{C} \longrightarrow$ Sets has a fixed point. Exactness of colim implies the fixed point property, and in some cases (such as when $\mathcal{C}$ has only finitely many objects) both conditions turn out to be equivalent to the components of $\mathcal{C}$ being filtered. We do not expect that the two conditions are equivalent in general, although we have no example. However the category of finite ordinals and order preserving injections is an example of a connected, nonfiltered category relative to which colim is exact. This was conjectured by Mitchell, and is proved by Isbell in [5].

**1. Cohn's criterion.** We recall briefly how one generalizes (noncommutative) homological ring theory to ringoids. The reader can consult the early sections of [8] for more details.

A *ringoid* is a small, preadditive category $\mathcal{C}$. A *left* $\mathcal{C}$*-module* is a covariant additive functor $M$: $\mathcal{C} \longrightarrow \mathrm{Ab}$. The category of left $\mathcal{C}$-modules, with natural transformations as morphisms, is denoted by mod $\mathcal{C}$. A module is *free* if it is (isomorphic to) a coproduct of representables. It is *finitely generated* if it is an epimorphic image of a finite coproduct of representables, and is *finitely presented* if it is a cokernel of a map between finitely generated frees. Such a map is given by a matrix whose entries, being maps between representables, may be identified with morphisms of $\mathcal{C}$. Any module is a direct limit of finitely presented modules [2, Chapter I, §2, Exercise 10].

If $M$ is a left $\mathcal{C}$-module, $\alpha \in \mathcal{C}(A, B)$, and $x \in MA$, then we denote $\alpha x = M(\alpha)(x) \in MB$. Similarly if $E$ is a right $\mathcal{C}$-module (that is, a left $\mathcal{C}^{\mathrm{op}}$-module), then we write $y\alpha = E(\alpha)(y) \in EA$ where $y \in EB$ and $\alpha \in \mathcal{C}(A, B)$. Given a right $\mathcal{C}$-module $E$ and a left $\mathcal{C}$-module $M$, we define

$$E \otimes_{\mathcal{C}} M = \bigoplus_{A \in |\mathcal{C}|} EA \otimes_{Z} MA/X$$

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where $X$ is the subgroup of the coproduct generated by elements of the form $y\alpha \otimes x - y \otimes \alpha x$. Then $\otimes_{\mathcal{C}}$ is an additive, abelian group valued bifunctor, and for fixed $E$, $E \otimes_{\mathcal{C}}$ is left adjoint to the functor which assigns to the abelian group $G$ the left $\mathcal{C}$-module whose value at $A$ is $\mathrm{Hom}_Z(EA, G)$. It is easy to establish an isomorphism

$$E \otimes_{\mathcal{C}} \mathcal{C}(A, \ ) \simeq EA \tag{1}$$

which is natural in $E$ and $A$.

The module $E$ is *flat* if $E \otimes_{\mathcal{C}}$ is exact. A monomorphism $E' \longrightarrow E$ is *pure* if $E' \otimes_{\mathcal{C}} M \longrightarrow E \otimes_{\mathcal{C}} M$ is a monomorphism for all left modules $M$. Since every $M$ is a direct limit of finitely presented modules, it suffices to test with $M$ finitely presented. If $0 \longrightarrow E' \xrightarrow{u} E \longrightarrow E'' \longrightarrow 0$ is a short exact sequence with $E''$ flat, then $u$ is pure. On the other hand if $u$ is pure, and if further $E$ is flat, then $E''$ is flat. One can either use the exact Tor sequence, or can proceed directly as in [2, §2].

The general notion of purity for modules over a ring is due to P. M. Cohn [3] who proved the following criterion.

PROPOSITION 1.1. *Let $E'$ be a submodule of $E$. In order that the inclusion be pure, it is necessary and sufficient that for each family of equations of the form*

$$e_i' = \sum_{j \in J} f_j \alpha_{ij}, \qquad i \in I,$$

*where $I$ and $J$ are finite, $e_i' \in E'A_i$, $f_j \in EB_j$, and $\alpha_{ij} \in \mathcal{C}(A_i, B_j)$, there is a family of elements $f_j' \in E'B_j$ such that*

$$e_i' = \sum_{j \in J} f_j' \alpha_{ij}, \qquad i \in I.$$

PROOF. A finitely presented module $M$ is given by an exact sequence

$$\bigoplus_{j \in J} \mathcal{C}(B_j, \ ) \xrightarrow{[\alpha_{ij}]} \bigoplus_{i \in I} \mathcal{C}(A_i, \ ) \longrightarrow M \longrightarrow 0$$

where $I$ and $J$ are finite. Tensoring with $E' \longrightarrow E$ and using the identification (1) and the fact that tensor commutes with coproducts and cokernels, we obtain an exact, commutative diagram of abelian groups

$$\begin{array}{ccccccc} \bigoplus_J E'B_j & \longrightarrow & \bigoplus_I E'A_i & \longrightarrow & E' \otimes_{\mathcal{C}} M & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \bigoplus_J EB_j & \longrightarrow & \bigoplus_I EA_i & \longrightarrow & E \otimes_{\mathcal{C}} M & \longrightarrow & 0. \end{array}$$

A couple of easy diagram chases yield the result.

2. $\mathcal{C}$-sets. If $A$ and $B$ are objects of a small category $\mathcal{C}$, then $A$ *maps to* $B$ if $\mathcal{C}(A, B)$ is not empty. This generates an equivalence relation on $|\mathcal{C}|$, and the full subcategory determined by an equivalence class is called a *component* of $\mathcal{C}$. $\mathcal{C}$ is *connected* if it has only one component. $Z\mathcal{C}$ denotes the *additivization* of $\mathcal{C}$, or in other words, the ringoid whose objects are those of $\mathcal{C}$, and whose hom group $Z\mathcal{C}(A, B)$ is the free abelian group on $\mathcal{C}(A, B)$. The inclusion $\mathcal{C} \longrightarrow Z\mathcal{C}$ induces an isomorphism of categories mod $Z\mathcal{C} \simeq \mathrm{Ab}^{\mathcal{C}}$. The *affinization* of $\mathcal{C}$ (aff $\mathcal{C}$) is the (nonadditive) subcategory of $Z\mathcal{C}$ consisting of those morphisms whose integer coefficients sum to 1. We have $\mathcal{C} \subset \mathrm{aff}\, \mathcal{C}$, with equality if and only if $\mathcal{C}$ is a preordered set.

A $\mathcal{C}$*-set* is a covariant functor $F$: $\mathcal{C} \longrightarrow$ Sets. As with modules, if $x \in FA$ and $\alpha \in \mathcal{C}(A, B)$, then $\alpha x$ denotes $F(\alpha)(x)$. Composing $F$ with the free abelian group functor Sets $\longrightarrow$ Ab, we obtain a $Z\mathcal{C}$-module $ZF$. If we take only those elements whose integer coefficients sum to 1, we obtain an aff $\mathcal{C}$-set denoted by aff $F$.

A $\mathcal{C}$-set is *indecomposable* if it is not empty (that is, at least one of its values is not empty), and it cannot be written as a disjoint union (coproduct) of two nonempty $\mathcal{C}$-sets. If $F$ is any $\mathcal{C}$-set and if $x$ and $x'$ are elements of (values of) $F$, write $x \sim x'$ if there is a system of equations

$$
\begin{aligned}
\alpha_1 x &= \alpha_2 x_1,\\
\alpha_3 x_1 &= \alpha_4 x_2,\\
&\ \ \vdots\\
\alpha_{2n-1} x_{n-1} &= \alpha_{2n} x'.
\end{aligned}
\tag{1}
$$

Then $\sim$ is an equivalence relation on the elements of $F$, and each equivalence class determines an indecomposable sub $\mathcal{C}$-set of $F$. Therefore every $\mathcal{C}$-set is the disjoint union of its indecomposable sub $\mathcal{C}$-sets. The colimit of $F$ has one element for each indecomposable sub $\mathcal{C}$-set. Thus $F$ is indecomposable if and only if $\operatorname{colim} F = 1$. In particular, all representables $\mathcal{C}(A,\ )$ are indecomposable. A colimit of a connected diagram of indecomposables (with connected index category $\mathcal{D}$) is again indecomposable, as seen from

$$\operatorname{colim}_{\mathcal{C}} \operatorname{colim}_{\mathcal{D}} F = \operatorname{colim}_{\mathcal{D}} \operatorname{colim}_{\mathcal{C}} F = \operatorname{colim}_{\mathcal{D}} 1 = 1.$$

If $\{x_i\}$ is a family of elements of $FC$, then a morphism $\beta$ *filters* the family if $\beta x_i$ is independent of $i$. We shall consider the following conditions on a $\mathcal{C}$-set $F$.

I. For every $x \in FC$, $x' \in FC'$, there are morphisms $\beta$, $\beta'$ in $\mathcal{C}$ such that $\beta x = \beta' x'$.

II. Every pair in $FC$ is filtered in $\mathcal{C}$.

III. Every finite family in $FC$ is filtered in aff $\mathcal{C}$.

IV. Every pair in $FC$ is filtered in aff $\mathcal{C}$.

V. For every pair $x_1, x_2$ in $FC$, there is a sequence of equations

$$\begin{aligned}\beta_1 x_{i_1} &= \beta_2 x_{j_2},\\ \beta_2 x_{i_2} &= \beta_3 x_{j_3},\\ &\vdots\\ \beta_n x_{i_n} &= \beta_1 x_{j_1}\end{aligned}$$

where $\beta_i \in \mathcal{C}$, $i_k$ and $j_k$ are 1 or 2, and $\Sigma_{k=1}^{n}(j_k - i_k) = 1$.

As in ring theory, where a notion is defined first for modules and then for rings by considering a ring as a module over itself, we shall say that $\mathcal{C}$ itself satisfies any of the above conditions if all of its representables $\mathcal{C}(A, \ )$ satisfy that condition.

Clearly II ⇒ III ⇒ IV. Also V ⇒ IV, since

$$\beta = \sum_{k=1}^{n} (j_k - i_k)\beta_k$$

is easily seen to filter $x_1$ and $x_2$.

Note that there exists $\beta$ in aff $\mathcal{C}$ with $\beta x_1 = \beta x_2$ if and only if there is an equation

$$\sum_{i=1}^{n+1} \beta_i x_1 + \sum_{i=1}^{n} \beta_i' x_2 = \sum_{i=1}^{n+1} \beta_i x_2 + \sum_{i=1}^{n} \beta_i' x_1 \tag{2}$$

with $\beta_i, \beta_i'$ in $\mathcal{C}$. This would not be true, of course, if we were using coefficients other than integers.

LEMMA 2.1. *If $\mathcal{C}$ satisfies* I, *then $F$ satisfies* IV *if and only if it satisfies* V.

PROOF. The terms on either side of (2) must pair off. This gives a family of systems of the form

$$\begin{aligned}\gamma_1 x_{i_1} &= \gamma_2 x_{i_2},\\ &\vdots\\ \gamma_t x_{i_t} &= \gamma_1 x_{j_1}\end{aligned}$$

where the $t$'s sum to $2n+1$ and where $\gamma_k$ is a $\beta_i'$ if $i_k = 1$ and is a $\beta_i$ if $i_k = 2$. Using I for $\mathcal{C}$ we can combine two such families into one by multiplying each by an appropriate morphism and writing one under the other. Thus we can piece all the families together to obtain a single such family, at which point we will have $t = 2n + 1$. Now the $x_{i_k}$'s being the terms on the left of (2), we must have $\Sigma i_k = (n + 1) + 2n$. Likewise $\Sigma j_k = 2(n + 1) + n$. Hence $\Sigma(j_k - i_k) = 1$.

LEMMA 2.2. *The following are equivalent.*

(a) $\mathcal{C}$ *satisfies* I.

(b) *All indecomposable* $\mathcal{C}$*-sets satisfy* I.

(c) colim: $\text{Sets}^{\mathcal{C}} \longrightarrow \text{Sets}$ *preserves monomorphisms.*

(d) *Every nonempty sub* $\mathcal{C}$*-set of an indecomposable* $\mathcal{C}$*-set is again indecomposable.*

PROOF. (a) ⇒ (b). If $x$ and $x'$ are any elements of an indecomposable, then there is a system (1). Using condition I on $\mathcal{C}$, we can successively reduce the number of equations until there is just one, which is what is required.

(b) ⇒ (c). Let $F' \longrightarrow F$ be an inclusion. If $x$ and $x'$ are elements of $F'$ which are in the same indecomposable component of $F$, then we can write $\beta x = \beta' x'$ as elements of $F$, hence as elements of $F'$.

(c) ⇒ (d). If $F' \longrightarrow F$ is an inclusion with $F'$ nonempty and $\operatorname{colim} F = 1$, then $\operatorname{colim} F' = 1$ by (c).

(d) ⇒ (a). If $\mathcal{C}$ does not satisfy I, then for some object $A$ and some pair $\alpha, \alpha' \in \mathcal{C}(A,\ )$, the sub $\mathcal{C}$-set generated by $\alpha$ will not intersect that generated by $\alpha'$.

The reader may also verify the following proposition.

PROPOSITION 2.3. *If $F$ is a* $\mathcal{C}$*-set which satisfies* I (*in particular, if $F$ is indecomposable and* $\mathcal{C}$ *satisfies* I), *then $F$ satisfies any of the conditions* II *to* V *that* $\mathcal{C}$ *does.*

3. **Characterization of exact colimits.** A category $\mathcal{C}$ is *filtered* if $\mathcal{C}$ satisfies II and every pair of objects map to a common object. The homological interest in filtered categories lies in the fact that they are precisely the categories relative to which the colimit functor $\text{Sets}^{\mathcal{C}} \longrightarrow \text{Sets}$ preserves finite limits. It follows that if $\mathcal{A}$ is any algebraic category [7], then the colimit functor $\mathcal{A}^{\mathcal{C}} \longrightarrow \mathcal{A}$ preserves finite limits whenever $\mathcal{C}$ is filtered. In particular this is true when $\mathcal{A} = \text{Ab}$, in which case, of course, finite limit preserving is equivalent to monomorphism preserving. However, in this case we see from the fact that any colimit is the coproduct of the colimits of restrictions to components, plus the fact that coproducts in Ab preserve monomorphisms, that colim: $\text{Ab}^{\mathcal{C}} \longrightarrow \text{Ab}$ preserves monomorphisms providing only that the components of $\mathcal{C}$ are filtered. The original aim of the present work was to find out if the converse was true, as Oberst conjectured [9]. We observe that the components of $\mathcal{C}$ are filtered if and only if $\mathcal{C}$ satisfies I and II.

If $M \in \mathrm{Ab}^{\mathcal{C}}$, then the colimit of $M$ is given by

$$\operatorname{colim}_{\mathcal{C}} M = \bigoplus_{A \in |\mathcal{C}|} MA/X \tag{1}$$

where $X$ is the subgroup of the coproduct generated by elements of the form $x - \alpha x$ with, say, $x \in MA$, $\alpha \in \mathcal{C}(A, B)$, and so $\alpha x \in MB$. Note that if $\Sigma r_i \alpha_i$ has integer coefficients $r_i$ summing to 1, then

$$x - \left(\sum r_i \alpha_i\right) x = \sum r_i (x - \alpha_i x).$$

It follows that if $M$ is considered as an object of $\mathrm{Ab}^{\mathrm{aff}\,\mathcal{C}}$, then

$$\operatorname{colim}_{\mathcal{C}} M = \operatorname{colim}_{\mathrm{aff}} M.$$

This yields the "if" part of the following theorem, which is close to being a restatement of [4, Theorem 1].

THEOREM 3.1. $\mathrm{Colim}_{\mathcal{C}}\colon \mathrm{Ab}^{\mathcal{C}} \longrightarrow \mathrm{Ab}$ *is exact if and only if the components of* aff $\mathcal{C}$ *are filtered.*

For the converse, we reexpress the colimit as

$$\operatorname{colim}_{\mathcal{C}} M = \Delta Z \otimes_{Z\mathcal{C}} M \tag{2}$$

where $\Delta Z$ is the constant functor at $Z$ over $\mathcal{C}^{\mathrm{op}}$. It is easy to see the natural isomorphism between the right sides of (1) and (2) using the definition of the tensor product. Thus we see that $\operatorname{colim}_{\mathcal{C}}$ is exact if and only if $\Delta Z$ is flat. But $\Delta Z$ is flat (if and) only if the monomorphism is pure in the exact sequence

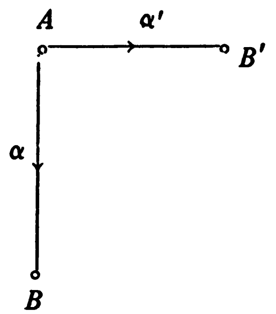
$$0 \longrightarrow K \longrightarrow \bigoplus_{C \in \mathcal{C}} Z\mathcal{C}(\ \ , C) \xrightarrow{\epsilon} \Delta Z \longrightarrow 0$$

where $\epsilon_B$ sums coefficients, and hence $KB$ is the group of all formal linear combinations $\Sigma r_i \beta_i$ with domain $\beta_i = B$ for all $i$ and $\Sigma r_i = 0$.

LEMMA 3.2. *If* $\operatorname{colim}_{\mathcal{C}}$ *is exact, then* $\mathcal{C}$ *satisfies* I.

PROOF. We consider the diagram in $\mathcal{C}$

(3)



and we regard $\alpha - \alpha'$ as an element of $KA$. Writing $\alpha - \alpha' = 1_B\alpha - 1_{B'}\alpha'$, we then see by Cohn's criterion that we can write

$$\alpha - \alpha' = \sum_{i=1}^{n} r_i\beta_i\alpha - \sum_{j=1}^{m} r'_j\beta'_j\alpha'$$

where $\Sigma r_i = 0 = \Sigma r'_j$. Taking terms involving $\alpha$ to one side and those involving $\alpha'$ to the other, we have that neither side is zero since the coefficients on either side sum to 1. Hence there must be a term on one side equal to a term on the other, which gives the result.

It suffices now to prove II for aff $C$, since given a diagram (3) in aff $C$, we can complete it to a not necessarily commutative square in aff $C$ by the lemma, and then make the square commutative using II for aff $C$. Thus consider $\alpha_1, \alpha_2 \in$ aff $C(A, B)$. Then $\alpha_1 - \alpha_2 \in KA$, and so since $\alpha_1 - \alpha_2 = 1_B(\alpha_1 - \alpha_2)$, we see again by Cohn's criterion that we can write

$$\alpha_1 - \alpha_2 = \sum_{i=1}^{n} r_i\beta_i(\alpha_1 - \alpha_2)$$

where $\Sigma r_i = 0$. This can be rewritten

$$\sum_{i=0}^{n} s_i\beta_i(\alpha_1 - \alpha_2) = 0 \tag{4}$$

where $s_i = -r_i$ for $i > 0$, $s_0 = 1$, and $\beta_0 = 1_B$. Now although $\Sigma s_i = 1$, $\Sigma s_i\beta_i$ is not in general a morphism of aff $C$ since the codomains of the $\beta_i$ are not necessarily the same. However using Lemma 3.2, we can find for each object which appears as the codomain of a $\beta_i$ a morphism $\gamma$ to a common object, and composing each $\beta_i$ in (4) with the appropriate $\gamma$ does not change the validity of (4). Thus we may assume that $\Sigma s_i\beta_i$ is a morphism of aff $C$, and so the theorem is proved.

In practice one uses the following proposition to show that the components of aff $C$ are filtered.

PROPOSITION 3.3. *The components of* aff $C$ *are filtered if and only if* $C$ *satisfies* I *and* III.

PROOF. Suppose $C$ satisfies I and III. Then aff $C$ will satisfy I providing we can show it satisfies II. Let $\alpha_1, \alpha_2 \in$ aff $C(A, B)$. Then the sum of coefficients in $\alpha_1 - \alpha_2$ is zero, and so if $\beta$ is any morphism of aff $C$ such that $\beta\gamma$ is the same for all $\gamma$ in $C$ appearing in $\alpha_1 - \alpha_2$, then $\beta(\alpha_1 - \alpha_2) = 0$, as required.

Now suppose that the components of aff $C$ are filtered. Then $C$ clearly satisfies III. To see that it satisfies I, consider the diagram (3) in $C$, and complete it to a commutative square in aff $C$. We obtain an equation with coefficients summing to 1 on both sides. The result then follows just as in Lemma 3.2.

REMARK. If $R$ is an arbitrary nonzero ring, then one can form $R\mathcal{C}$ in the obvious way, and again define $\mathrm{aff}_R\ \mathcal{C}$ to be the subcategory of morphisms with coefficients in $R$ summing to 1. For commutative $R$, Theorem 3.1 and its proof are still valid, $\mathrm{colim}_\mathcal{C}$ being understood to go from $(\mathrm{mod}\,R)^\mathcal{C}$ to $\mathrm{mod}\,R$. Proposition 3.3 holds also. On the other hand, it is not clear whether Theorem 4.1 below ever holds unless $R$ contains $\mathbf{Z}$ as an abelian group retract.

4. **Fixed points.** If $F$ is a $\mathcal{C}$-set and $\tau: F \longrightarrow F$, then an element $x \in F\mathcal{C}$ is a *fixed point* for $\tau$ if $\tau x = x$. The proof of the following theorem is based on the proof of [4, Theorem 2].

THEOREM 4.1. *If $F$ satisfies* IV *and $\tau$ is any endomorphism of $F$, then $\tau$ has a fixed point. In fact, if $x$ is any element of $F$, then $\gamma x$ is a fixed point for some $\gamma \in \mathcal{C}$.*

PROOF. In equation (2) of §2, set $x_1 = x$ and $x_2 = \tau x$. We may assume there is no cancellation of the form $\beta_i x = \beta_j' x$, for then $\beta_i \tau x = \beta_j' \tau x$ and we can simply delete the terms involving $\beta_i$ and $\beta_j'$. It follows that

$$\sum_{i=1}^{n+1} \beta_i x = \sum_{i=1}^{n+1} \tau\beta_i x$$

and

$$\sum_{i=1}^{n} \beta_i' x = \sum_{i=1}^{n} \tau\beta_i' x.$$

The first equation gives rise to a family of equations of the form $\gamma x = \tau^r \gamma x$ where the $r$'s sum to $n+1$ and where $\gamma$ is one of the $\beta_i$. Likewise the second equation gives rise to a family of the form $\gamma' = \tau^s \gamma' x$ where the $s$'s sum to $n$. Hence for any integer $m > 1$, there is an equation of the form $\gamma x = \tau^l \gamma x$ where $m \nmid l$. We remark that such an equation implies $\gamma x = \tau^{dl} \gamma x$ for any positive integer $d$.

Now select an equation $\gamma x = \tau^m \gamma x$ (not necessarily one of the above) with $m$ positive and minimal. We shall show $m = 1$. Supposing otherwise, we can replace $x$ by $\gamma x$ in the above, and we find an equation of the form $\delta\gamma x = \tau^l \delta\gamma x$ where $m \nmid l$. Let $c$ and $d$ be any positive integers. If $cm > dl$, then

$$\delta\gamma x = \tau^{cm}\delta\gamma x = \tau^{cm-dl}\tau^{dl}\delta\gamma x = \tau^{cm-dl}\delta\gamma x.$$

Similarly if $cm < dl$, we have $\delta\gamma x = \tau^{dl-cm}\delta\gamma x$. It follows that if $t = (l, m)$, then $\delta\gamma x = \tau^t \delta\gamma x$ where $t < m$ since $m \nmid l$. This contradicts minimality of $m$, and proves the theorem.

By induction we obtain:

COROLLARY 4.2. *If F satisfies* IV *and* $\tau_1, \ldots, \tau_n$ *are any endomorphisms of F, then for every x in F there is a* $\gamma$ *in* $\mathcal{C}$ *such that* $\tau_i\gamma x = \gamma x$ *for all i.*

COROLLARY 4.3. *If* $\mathcal{C}$ *satisfies* IV, *then any finite family of endomorphisms of the same object can be filtered in* $\mathcal{C}$. *Hence if* $\mathcal{C}(B, A)$ *is nonempty, then any finite family in* $\mathcal{C}(A, B)$ *can be filtered in* $\mathcal{C}$.

A *weak terminal object* in a category is an object to which all other objects map. Any connected category which satisfies I and which has only a finite number of objects has a weak terminal object.

COROLLARY 4.4. *If* $\mathcal{C}$ *satisfies* I *and* IV *and is connected with a weak terminal object, then* $\mathcal{C}$ *is filtered.*

PROOF. To filter $\alpha_1, \alpha_2 \in \mathcal{C}(A, B)$, since $B$ maps to the weak terminal object $T$, it suffices to assume $B = T$. By I we can write $\beta\alpha = \beta'\alpha'$ in $\mathcal{C}$. By Corollary 4.3, $\beta$ and $\beta'$ can be filtered in $\mathcal{C}$. Hence $\alpha$ and $\alpha'$ can be filtered in $\mathcal{C}$.

From Proposition 2.3 and Theorem 4.1, it follows that if $\mathcal{C}$ satisfies I and IV, then $\mathcal{C}$ has the *fixed point property*—that is, any endomorphism of an indecomposable $\mathcal{C}$-set has a fixed point. (One cannot ask a decomposable to always have fixed points, for if $F_1$ maps to $F_2$ and $F_2$ maps to $F_1$, then $F_1 \oplus F_2$ cannot have a fixed point.) Adámek and Reiterman [1] prove by another method that if $\mathcal{C}$ satisfies I and V, then $\mathcal{C}$ has the fixed point property. Using an argument which we shall not attempt to improve on, they also establish the converse. Thus I and IV, I and V, and the fixed point property are all equivalent.

PROPOSITION 4.5. *Suppose that* $\mathcal{C}$ *satisfies* I *and* V, *and let F be an indecomposable* $\mathcal{C}$-*set with a finite bound on the cardinalities of the values FC. Then F satisfies* II.

PROOF. If $x_1, x_2 \in FC$, write $x_1 \sim x_2$ if $\beta x_1 = \beta x_2$ for some $\beta$ in $\mathcal{C}$. Using I for $\mathcal{C}$, it is easily seen that this is a congruence relation. The quotient $\mathcal{C}$-set $G$ has the property that if $\beta x_1 = \beta x_2$ with $\beta \in \mathcal{C}$, then $x_1 = x_2$. Let $|GC|$ be maximal. We wish to show it is 1. Note that if $\alpha \in \mathcal{C}(C, D)$, then $\alpha$ acts as a bijection on $G$. Let $x_1, x_2 \in GC$. As $G$ is indecomposable and $\mathcal{C}$ satisfies I, $G$ satisfies I, and so we can write $\alpha_1 x_1 = \alpha_2 x_2$. Then by V for $\mathcal{C}$, we can write

$$\begin{aligned} \beta_1\alpha_{i_1} &= \beta_2\alpha_{j_2}, \\ &\vdots \\ \beta_n\alpha_{i_n} &= \beta_1 x_{j_1}, \end{aligned} \qquad \sum(j_k - i_k) = 1.$$

Relative to $G$, let $\theta = \alpha_1^{-1}\alpha_2$. Then the above system becomes

$$\beta_1 = \beta_2\theta^{j_2 - i_1},$$
$$\vdots$$
$$\beta_n = \beta_1\theta^{j_1 - i_n}$$

which gives $\beta_1 = \beta_1\theta$. As $\beta_1$ is an injection on $G$, we find $\theta =$ identity, and so $x_1 = x_2$.

COROLLARY 4.6. *Suppose that* C *satisfies* I *and* V *and that for each object* $A$ *there is a finite upper bound on the cardinalities of the sets* $C(A, X)$. *Then* C *satisfies* II.

The corollary was proved by Adámek and Reiterman and in fact the proof of Proposition 4.5 is their proof. Corollaries 4.4 and 4.6 give one a good idea what a category must look like if it is to have the fixed point property but is not filtered. As stated at the beginning, the category of order preserving injections of finite ordinals, which is certainly not filtered (two morphisms can be filtered only if they are equal), apparently does even better by having a filtered affinization. What is needed now is an example of a category with the fixed point property whose affinization is not filtered, or in other words, one satisfying I and IV, but not III.

ADDED IN PROOF. The paper [1] contains a claim (before Theorem 2.2) that I and V imply III "easily by Lemma 1" of [6]. We do not believe it.

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