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ON REPRESENTATIONS OF THE GROUP $SU(n, 1)$(1)

BY

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ABSTRACT. A natural bijection is established between the set of equivalence classes of irreducible unitary representations of the group $G = SU(n, 1)$, which are not induced from a proper parabolic subgroup, and the set of equivalence classes of irreducible representations of a maximal compact subgroup.

1. **Introduction.** In [4] all irreducible subquotients of all elementary (= nonunitary principal series) representations of the group $G = SU(n, 1)$ and of its universal covering group were determined. We have also found all infinitesimal equivalences among these irreducible representations and in this way, owing to the subquotient theorem [1], [6], [9], we have described the set $\widehat{G}$ of all infinitesimal equivalence classes of irreducible quasi-simple representations of G. Furthermore, we have found a necessary and sufficient condition for a class $\pi \in \widehat{G}$ to contain a unitary representation and so the set $\hat{G}$ of all equivalence classes of unitary irreducible representations of G was completely described.

The aim of this paper is to give another description of the results of [4]. This description will use only very general terms and will not depend on almost any particular property of the group $SU(n, 1)$. Furthermore, it will provide us with very natural parametrizations of the sets $\widehat{G}$ and $\hat{G}$.

We shall also describe all elementary representations in which a given unitary irreducible nonelementary representation occurs as a subquotient. Using Blattner's conjecture (which is proven to be true for linear groups acting on hermitian symmetric spaces [8]) it will be very easy to identify the discrete series representations. This will, especially, give us all possible imbeddings of a discrete series representation as a subquotient of an elementary representation.

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2. **Statements of the results.** In the following G will denote the group $SU(n, 1)$, $\mathfrak{g}$ its Lie algebra, $\mathfrak{g}_c$ $(= \mathfrak{sl}(n+1, \mathbf{C}))$ the complexification of $\mathfrak{g}$, $\mathfrak{G}$ the universal enveloping algebra of $\mathfrak{g}$, $\mathfrak{Z}$ the center of $\mathfrak{G}$. For any Cartan subalgebra

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$\mathfrak{h}$ of $\mathfrak{g}_c$, let $S(\mathfrak{h})^W$ denote the algebra of Weyl group invariants in the symmetric algebra $S(\mathfrak{h})$ over $\mathfrak{h}$, and let $\varphi_{\mathfrak{h}}$ be the canonical isomorphism of $\mathfrak{Z}$ onto $S(\mathfrak{h})^W$ [9]. $S(\mathfrak{h})$ being identified with the polynomials on the dual space $\mathfrak{h}^*$ of $\mathfrak{h}$, let, for $\lambda \in \mathfrak{h}^*$, χ_λ denote the element of Hom($\mathfrak{Z}$, **C**) obtained by composing the evaluation at λ with $\varphi_{\mathfrak{h}}$: $\chi_\lambda(z) = \varphi_{\mathfrak{h}}(z)(\lambda)$, $z \in \mathfrak{Z}$ (the notation here differs from that in [4, p. 26]). Then $\lambda \longmapsto \chi_\lambda$ defines a bijection from the set of Weyl group orbits in $\mathfrak{h}^*$ onto Hom($\mathfrak{Z}$, **C**). For any $\lambda \in \mathfrak{h}^*$ let $[\lambda]$ denote the Weyl group orbit containing λ. Furthermore, if π is a quasi-simple representation of G, denote by $[\pi]$ the Weyl group orbit in $\mathfrak{h}^*$ which corresponds to the infinitesimal character of π.

Let (K, A, N) be an Iwasawa decomposition of G and $(\mathfrak{k}, \mathfrak{a}, \mathfrak{n})$ the corresponding decomposition of $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{k}$. Denote by $\mathfrak{p}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form. Let R be the root system of $(\mathfrak{g}_c, \mathfrak{h}_c)$ and let W be its Weyl group. For $\alpha \in R$, $\mathfrak{g}_c^\alpha$ will denote the corresponding root subspace of $\mathfrak{g}_c$. Let R_K and R_P denote the sets of compact and noncompact roots, respectively; $R_K = \{\alpha \in R;\ \mathfrak{g}_c^\alpha \subset \mathfrak{k}_c\}$, $R_P = \{\alpha \in R;\ \mathfrak{g}_c^\alpha \subset \mathfrak{p}_c\}$. Then R_K is the root system of $(\mathfrak{k}_c, \mathfrak{h}_c)$. Fix an R_K-Weyl chamber C in $i\mathfrak{h}^*$ and denote by R_K^C the corresponding positive roots in R_K. Let $\mathcal{C}$ be the set of all R-Weyl chambers in $i\mathfrak{h}^*$ contained in C. For any $D \in \mathcal{C}$ denote by R^D the corresponding positive roots in R. Furthermore, put

$$R_P^D = R^D \backslash R_K^C = R^D \cap R_P,$$

$$\rho_K = \frac{1}{2} \sum_{\alpha \in R_K^C} \alpha, \qquad \rho_P^D = \frac{1}{2} \sum_{\alpha \in R_P^D} \alpha.$$

$\hat{K}$, the set of equivalence classes of finite-dimensional irreducible representations of K, will be regarded as a subset of the closure of C by identifying any $q \in \hat{K}$ with its maximal weight.

If π is an admissible quasi-simple representation of G and $q \in \hat{K}$, let $(\pi : q)$ denote the multiplicity of q in $\pi|K$; furthermore, put, as in [4], $\Gamma(\pi) = \{q \in \hat{K};\ (\pi : q) > 0\}$.

DEFINITION. Let π be an admissible quasi-simple representation of G, $q \in \hat{K}$, $D \in \mathcal{C}$.

(i) q is called a D-fundamental weight of π if $q \in \Gamma(\pi)$ and $[\pi] = [q + \rho_K - \rho_P^D]$.

(ii) q is called a D-corner of π if $q \in \Gamma(\pi)$ and $q - \alpha \notin \Gamma(\pi)$ $\forall \alpha \in R_P^D$.

(iii) q is called a D-fundamental corner of π if it is a D-fundamental weight and a D-corner of π.

(iv) q is called a fundamental corner of π if it is a D-fundamental corner of π for some $D \in \mathcal{C}$.

REMARK. This definition is, of course, inspired by Blattner's conjecture for the K-multiplicities of a discrete series representation (see [7]).

Let M be the centralizer of A in K. For $p \in \hat{M}$ and $\lambda \in \mathfrak{a}_c^*$ let $\pi^{p,\lambda}$ denote the corresponding elementary representation of G ($\pi^{p,\lambda}$ is induced by the representation $man \longmapsto \exp((\lambda + \rho)(\log a))p(m)$ of MAN, $\rho(H) = \frac{1}{2}\operatorname{tr}(\operatorname{ad} H|\mathfrak{n})$, $H \in \mathfrak{a}$).

Let $\widehat{G}^e$ be the set of all elements in $\widehat{G}$ containing some elementary representation. Put

$$\widehat{G}^0 = \widehat{G}\backslash\widehat{G}^e, \quad \hat{G}^e = \widehat{G}^e \cap \hat{G}, \quad \hat{G}^0 = \hat{G} \cap \widehat{G}^0.$$

The following theorems give the description and parametrizations of these sets.

THEOREM 1. *$\pi^{p,\lambda}$ is reducible if and only if there exist $q \in \hat{K}$ and $D \in \mathcal{C}$ such that q is a D-fundamental weight of $\pi^{p,\lambda}$ but is not a D-corner of $\pi^{p,\lambda}$.*

This theorem, together with the fact that two irreducible elementary representations $\pi^{p,\lambda}$ and $\pi^{p',\lambda'}$ are infinitesimally equivalent if and only if (p, λ) and (p', λ') are conjugated by the action of the Weyl group of $(\mathfrak{g}, \mathfrak{a})$, completely describes $\widehat{G}^e$.

For any $\pi \in \widehat{G}^0$ let $F(\pi)$ be the set of all fundamental corners of π.

THEOREM 2. (i) *For every $\pi \in \widehat{G}^0$, $F(\pi)$ has either one or two elements.*

(ii) *Let Ω be the set of all nonordered pairs (q, q'), $q, q' \in \hat{K}$. $\pi \longmapsto F(\pi)$ defines an injection of $\widehat{G}^0$ into Ω (the image of $\pi \in \widehat{G}^0$ with only one fundamental corner q being (q, q)).*

Let Ω_0 be the image of this injection.

(iii) *Ω_0 is the set of all pairs $(q, q') \in \Omega$ with the property that there exist $D, D' \in \mathcal{C}$ such that $[q + \rho_K - \rho_P^D] = [q' + \rho_K - \rho_P^{D'}]$, $q + \rho_K - \rho_P^D \in \bar{D}'$ and $q + \rho_K - \rho_P^{D'} \in \bar{\bar{D}}$. Here $\bar{D}$ denotes the closure of D in C and $\bar{\bar{D}}$ is the closure of D in $i\mathfrak{h}^*$. Furthermore, for any $(q, q') \in \Omega_0$, D and D' are well determined.*

Hence, if we denote by $\pi(q, q')$ the unique element of $\widehat{G}^0$ with the property $F(\pi(q, q')) = \{q, q'\}$ ($(q, q') \in \Omega_0$) then we get

$$\widehat{G}^0 = \{\pi(q, q'); (q, q') \in \Omega_0\}.$$

We give also another description of $\widehat{G}^0$ which is in a sense more convenient because of the nontransparent characterization of the set Ω_0.

THEOREM 3. *For $D \in \mathcal{C}$ let $\hat{K}^D$ denote the set of all $q \in \hat{K}$ such that there is $\pi \in \widehat{G}^0$ which has q as a D-fundamental corner. Put $\Omega_1 = \{(q, D); q \in \hat{K}^D, D \in \mathcal{C}\}$.*

(i) *For any $(q, D) \in \Omega_1$ there is a unique element $\pi(q, D)$ in $\widehat{G}^0$ which has q as a D-fundamental corner.*

(ii) $\widehat{G}^0 = \{\pi(q, D); (q, D) \in \Omega_1\}$.

(iii) *If* $(q, D) \in \Omega_1$, *then there exists at most one* $(q', D') \in \Omega_1$ *different from* (q, D) *such that* $\pi(q, D) = \pi(q', D')$. *Then* $F(\pi(q, D)) = \{q, q'\}$.

(iv) *If D has a wall in common with C then* $\hat{K}^D = \hat{K}$. *If D has not a wall in common with C, then there are two roots* α_D, β_D *in* R_P^D *perpendicular to the walls of D intersecting C and* $\hat{K}^D$ *is the set of all* $q \in \hat{K}$ *such that either* $(\alpha_D + \beta_D|q) \neq 0$ *or* $(\alpha|q + \rho_K - \rho_P^D) \neq 0$ $\forall \alpha \in R_P$.

By Proposition 11.4 in [4] we have the following description of $\hat{G}^e$:

THEOREM 4. *Identify* $\mathfrak{a}_c^*$ *with* **C** *so that* $\rho = n$. *For any* $p \in \hat{M}$ *put*

$$\lambda_p = \min\{\lambda \geq 0; \pi^{p,\lambda} \text{ reducible}\}.$$

Then

$$\hat{G}^e = \{\pi^{p,\lambda}; p \in \hat{M}, \lambda \in i(0, +\infty) \cup [0, \lambda_p)\}.$$

We shall give also the explicit value of λ_p for any $p \in \hat{M}$ (it is determined in [3] for "generic" p's).

THEOREM 5. (i) $\pi \in \widehat{G}^0$ *is unitary if and only if it has only one fundamental corner, i.e.,* Card $F(\pi) = 1$.

(ii) *For every* $q \in \hat{K}$ *there is a unique* $\pi_q \in \hat{G}^0$ *with* $F(\pi_q) = \{q\}$. $q \mapsto \pi_q$ *is a bijection of* $\hat{K}$ *onto* $\hat{G}^0$.

Furthermore, we shall prove

THEOREM 6. (i) π_q *is a discrete series representation if and only if* $q + \rho_K - \rho_P^D \in D$ *for some* $D \in \mathcal{C}$; *this D is then uniquely determined.*

(ii) *Every* $\pi \in \hat{G}^0$ *which is not in the discrete series occurs as a subquotient of some* π^{p,λ_p}, $p \in \hat{M}$.

REMARK. Statement (ii) can also be deduced from the results of [3] and [5].

3. **Irreducibility of elementary representations.** G is the group $SU(n, 1)$ of all complex square matrices g of order $n + 1$ with $\det g = 1$ and $g\Gamma g^* = \Gamma = \operatorname{diag}(1, \ldots, 1, -1)$. We choose K to be the subgroup of all unitary matrices in G. $\mathfrak{g}_c$ is naturally identified with $\mathfrak{sl}(n + 1, \mathbf{C})$. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$ (and $\mathfrak{k}$) of all diagonal matrices in $\mathfrak{g}$. Let e_{ij} be the matrix with 1 on the place (i, j) and 0 elsewhere $(1 \leq i, j \leq n + 1)$. Put $h_i = e_{ii} - e_{i+1,i+1}$ $(1 \leq i \leq n)$. Then $h_1, \ldots, h_n$ is a basis of $\mathfrak{h}_c$. We identify $\mathfrak{h}_c^*$ with the set

$$\mathbf{C}_0^{n+1} = \left\{s \in \mathbf{C}^{n+1}; \sum_{j=1}^{n+1} s_j = 0\right\}$$

in such a way that $s(h_j) = s_j - s_{j+1}$, $1 \leqslant j \leqslant n$. Then $i\mathfrak{h}^* = \mathbf{C}_0^{n+1} \cap \mathbf{R}^{n+1} = \mathbf{R}_0^{n+1}$.

The root system of $(\mathfrak{g}_c, \mathfrak{h}_c)$ is

$$R = \{\alpha_{jk};\ 1 \leqslant j,\ k \leqslant n + 1, j \neq k\}$$

where $\alpha_{jk}(\mathrm{diag}(t_1, \ldots, t_{n+1})) = t_j - t_k$. Furthermore,

$$R_K = \{\alpha_{jk};\ 1 \leqslant j,\ k \leqslant n, j \neq k\}, \quad R_P = \{\alpha_{j,n+1}, \alpha_{n+1,j};\ 1 \leqslant j \leqslant n\}.$$

The Weyl group W of R is the group $\mathfrak{S}_{n+1}$ of permutations of coordinates.

We choose the R_K-Weyl chamber in $\mathbf{R}_0^{n+1}$ to be

$$C = \{s \in \mathbf{R}_0^{n+1};\ s_j > s_{j+1},\ 1 \leqslant j \leqslant n - 1\}.$$

Then

$$R_K^C = \{\alpha_{jk};\ 1 \leqslant j < k \leqslant n\} \quad \text{and} \quad \mathcal{C} = \{D_0, D_1, \ldots, D_n\},$$

where

$$\begin{aligned} D_0 &= \{s \in C;\ s_{n+1} > s_1\}, \\ D_j &= \{s \in C;\ s_j > s_{n+1} > s_{j+1}\}, \qquad 1 \leqslant j \leqslant n - 1, \\ D_n &= \{s \in C;\ s_n > s_{n+1}\}. \end{aligned}$$

Put R_P^j and ρ_P^j instead of $R_P^{D_j}$ and $\rho_P^{D_j}$, respectively. Then we have

$$R_P^j = \{\alpha_{k,n+1};\ 1 \leqslant k \leqslant j\} \cup \{\alpha_{n+1,k};\ j + 1 \leqslant k \leqslant n\}$$

and

$$(1) \qquad (\rho_K - \rho_P^j)_i = \begin{cases} n/2 - i, & 1 \leqslant i \leqslant j, \\ n/2 - i + 1, & j + 1 \leqslant i \leqslant n, \\ j - n/2, & i = n + 1. \end{cases}$$

$\hat{K}$ was in [4] identified with $((n + 1)^{-1}\mathbf{Z})^n_> = \{q \in ((n + 1)^{-1}\mathbf{Z})^n;\ q_j - q_{n+1} \in \mathbf{Z}_+,\ 1 \leqslant j \leqslant n - 1\}$. In the new parametrization of $\mathfrak{h}_c^*$ and identifying $\hat{K}$ with a subset of the closure of C we have $q = (q_1, \ldots, q_n, -\Sigma_{j=1}^n q_j)$.

As in [4] we choose $\mathfrak{a}$ to be spanned over $\mathbf{R}$ by $e_{n,n+1} + e_{n+1,n}$. $\hat{M}$ was identified with $((n + 1)^{-1}\mathbf{Z})^{n-1}_>$ and $\mathfrak{a}_c^* = \mathbf{C}$ with $\rho = n$. Let $\nu: \hat{M} \times \mathbf{C} \longrightarrow \mathbf{C}_0^{n+1}$ be defined by

$$\nu(p, \lambda)_i = \begin{cases} \tfrac{1}{2}(\lambda - \Sigma_{k=1}^{n-1} p_k), & i = 1, \\ p_{i-1} + n/2 - i + 1, & 2 \leqslant i \leqslant n, \\ -\tfrac{1}{2}(\lambda + \Sigma_{k=1}^{n-1} p_k), & i = n + 1. \end{cases}$$

Then ν is an injection and its image is

$$E = \left\{ s \in \mathbf{C}_0^{n+1};\, s_j \in \frac{1}{n+1}\mathbf{Z},\, s_k - s_{k+1} \in \mathbf{N},\, 2 \leqslant j \leqslant n,\, 2 \leqslant k \leqslant n-1 \right\}.$$

Denote the elementary representation $\pi^{p,\lambda}$ by π^s, $s = \nu(p, \lambda)$. Then $[\pi^s] = [s]$, i.e., the infinitesimal character of π^s is χ_s (see [4, p. 45]).

The result about reducibility of elementary representations from [4] (Theorems 7.5 and 8.7) now gets the form:

PROPOSITION 1. *π^s is reducible if and only if either $s_1 - s_j \in \mathbf{Z}\backslash\{0\}$ $\forall j \in \{2, \ldots, n\}$ or $s_{n+1} - s_j \in \mathbf{Z}\backslash\{0\}$ $\forall j \in \{2, \ldots, n\}$.*

PROOF OF THEOREM 1. Suppose that π^s is reducible. Interchanging s_1 and s_{n+1} if necessary (this corresponds to the action of the nontrivial element of the little Weyl group), we can suppose that $s_1 - s_j \in \mathbf{Z}\backslash\{0\}$ $\forall j \in \{2, \ldots, n\}$. We have $\Gamma(\pi^s) = \{q \in \hat{K}; p < q\}$ $(\nu(p, \lambda) = s)$, where $p < q$ means $q_i - p_i \in \mathbf{Z}_+$ and $p_i - q_{i+1} \in \mathbf{Z}_+$, $1 \leqslant i \leqslant n-1$. We have to show that there are $q \in \hat{K}$ and $j \in \{0, \ldots, n\}$ such that $p < q$, $[q + \rho_K - \rho_P^j] = [s]$ and $p < q - \alpha$ for some $\alpha \in R_P^j$. Let j be the smallest element of $\{1, \ldots, n-1\}$ such that $s_1 > s_{j+1}$ and put $j = n$ if $s_n > s_1$. Let $q \in \hat{K}$ be defined by

$$q_i = \begin{cases} p_i, & 1 \leqslant i \leqslant j-1, \\ s_1 + j - n/2, & i = j, \\ p_i - 1, & j+1 \leqslant i \leqslant n. \end{cases}$$

Then it is easily seen that $q \in \hat{K}$, $p < q$ and $\alpha_{j,n+1} \in R_P^j$, $q - \alpha_{j,n+1} \in \hat{K}$, $p < q - \alpha_{j,n+1}$. Furthermore, by (1)

$$q + \rho_K - \rho_P^j = (s_2, \ldots, s_j, s_1, s_{j+1}, \ldots, s_{n+1});$$

hence $[q + \rho_K - \rho_P^j] = [s]$.

Suppose now that there are $j \in \{0, \ldots, n\}$, $q \in \hat{K}$ and $\alpha \in R_P^j$ such that $q - \alpha \in \hat{K}$, $p < q$, $p < q - \alpha$ and $[q + \rho_K - \rho_P^j] = [s]$. Put $t = q + \rho_K - \rho_P^j$ and let $w \in W$ be such that $t = ws$. We have $s_2 > \cdots > s_n$ and $t_1 > \cdots t_j \geqslant t_{j+1} > \cdots > t_n$. Now, $p < q$ and the formulas relating p and q with s and t give

$$t_1 \geqslant s_2 > t_2 \geqslant s_3 > \cdots \geqslant s_j > t_j \geqslant s_{j+1} \geqslant t_{j+1} > s_{j+2} \geqslant \cdots > s_n \geqslant t_n.$$

These inequalities, together with the fact that w is a permutation, imply $\{s_1, s_{n+1}\} \subset \{t_j, t_{j+1}, t_{n+1}\}$. Interchanging s_1 and s_{n+1} if necessary, we have to consider the following possibilities:

(a) $s_1 = t_{n+1}$, $s_{n+1} = t_j$. Then $s_i = t_{i-1}$, $2 \leqslant i \leqslant j$, and $s_i = t_i$,

$j + 1 \leqslant i \leqslant n$. Therefore $q_i = p_i$, $1 \leqslant i \leqslant j - 1$, $q_i = p_{i-1}$, $j + 1 \leqslant i \leqslant n$, $q_j = s_{n+1} - n/2 + j$. Hence $p \not< q - \alpha$ for all $\alpha \in R_P^j$ except possibly $\alpha = \alpha_{j,n+1}$. So α in the assumption must be $\alpha_{j,n+1}$. This means $q_j - p_j \in \mathbf{N}$, or $s_{n+1} - s_{j+1} \in \mathbf{N}$. Furthermore, $p < q$ implies $p_{j-1} - q_j \in \mathbf{Z}_+$, or $s_j - s_{n+1} \in \mathbf{N}$. Hence, $s_{n+1} - s_k \in \mathbf{Z}\backslash\{0\}$ for all $k \in \{2, \ldots, n\}$, and π^s is reducible by Proposition 1.

(b) $s_1 = t_{n+1}$, $s_{n+1} = t_{j+1}$. Similarly as above we get $q_i = p_i$, $1 \leqslant i \leqslant j$, $q_i = p_{i-1}$, $j + 2 \leqslant i \leqslant n$, $q_{j+1} = s_{n+1} - n/2 + j$, and $\alpha = \alpha_{n+1,j+1}$. Furthermore, $p < q$ implies $s_{n+1} - s_{j+1} \in \mathbf{N}$, and $p < q - \alpha_{n+1,j+1}$ implies $s_{j+1} - s_{n+1} \in \mathbf{N}$. Hence, π^s is again reducible.

(c) $s_1 = t_j$, $s_{n+1} = t_{j+1}$. Then $q_i = p_i$, $1 \leqslant i \leqslant j - 1$, $q_i = p_{i-1}$, $j + 2 \leqslant i \leqslant n$, $q_j = s_1 - n/2 + j$, $q_{j+1} = s_{n+1} - n/2 + j$, $-\Sigma_{k=1}^n q_k = p_j + n - 2j$. By $p < q$ we have $s_j - s_1 \in \mathbf{N}$ and $s_{n+1} - s_{j+2} \in \mathbf{N}$. So we have to show that either $s_1 \neq s_{j+1}$ or $s_{n+1} \neq s_{j+1}$. Suppose that $s_1 = s_{n+1} = s_{j+1}$. Then $q = (p_1, \ldots, p_j, p_j, \ldots, p_{n-1})$. But then $p \not< q - \alpha$ $\forall \alpha \in R_P^j$ contradicting the assumptions. Q.E.D.

4. Irreducible nonelementary representations. $\widehat{G}^0$ was parametrized in [4, §10] as follows.

$$\widehat{G}^0 = \{\pi_{j,r}; r \in T_j, 0 \leqslant j \leqslant n\} \cup \{\pi_{j,k,r}; r \in T_{j,k}, 0 \leqslant j < k \leqslant n\}.$$

Here T_j is the set of all $r \in ((n + 1)^{-1}\mathbf{Z})^n_{\geqslant}$ such that $r_j > r_{j+1}$ ($r_0 = \infty$, $r_{n+1} = -\infty$) and one of the following conditions is satisfied:

(a) $r_k + \Sigma_{i=1}^n r_i = j + k - n$ for some $k \in \{1, \ldots, j - 1\}$;

(b) $r_{k+1} + \Sigma_{i=1}^n r_i = j + k - n$ for some $k \in \{j + 1, \ldots, n - 1\}$;

(c) $r_j \geqslant 2j - n - \Sigma_{i=1}^n r_i \geqslant r_{j+1}$.

Furthermore, T_{jk} is the set of all $r \in ((n + 1)^{-1}\mathbf{Z})^{n+1}_{\geqslant}$ such that $r_j > r_{j+1}$, $r_{k+1} > r_{k+2}$ and $\Sigma_{i=1}^{n+1} r_i = j + k - n$ ($r_0 = +\infty$, $r_{n+2} = -\infty$).

We simplify this in the following way. Let, for $0 \leqslant j \leqslant k \leqslant n$, S_{jk} be the set of all $r \in ((n + 1)^{-1}\mathbf{Z})^{n+1}_{\geqslant}$ such that $\Sigma_{i=1}^{n+1} r_i = j + k - n$ and either $r_j > r_{j+1}$ or $r_{k+1} > r_{k+2}$. Then $T_{jk} \subset S_{jk}$. Let r be in T_j such that $r_k + \Sigma_{i=1}^n r_i = j + k - n$ for some $k \in \{1, \ldots, j - 1\}$. Put

$$r' = (r_1, \ldots, r_k, r_k, \ldots, r_j, r_{j+1}, \ldots, r_n).$$

Then $r' \in S_{kj}$. Let $r \in T_j$ be such that $r_{k+1} + \Sigma_{i=1}^n r_i = j + k - n$ for some $k \in \{j + 1, \ldots, n - 1\}$. Now put

$$r' = (r_1, \ldots, r_j, r_{j+1}, \ldots, r_{k+1}, r_{k+1}, \ldots, r_n).$$

Then $r' \in S_{jk}$. Finally, if $r \in T_j$, $r_j \geqslant 2j - n - \Sigma_{i=1}^n r_i \geqslant r_{j+1}$, put

$$r' = \left(r_1, \ldots, r_j, 2j - n - \sum_{i=1}^{n} r_i, r_{j+1}, \ldots, r_n\right) \in S_{jj}.$$

With the obvious notation we get

$$\widehat{G}^0 = \{\pi_{j,k,r}; r \in S_{jk}, 0 \leqslant j \leqslant k \leqslant n\}.$$

Furthermore, with the notation from [4, §10]:

$$\begin{aligned} \Gamma(\pi_{j,k,r}) = \{q \in \hat{K}; (q_1, \ldots, q_j) &< (\infty, r_1, \ldots, r_j), \\ (q_{j+1}, \ldots, q_k) &< (r_{j+1}, \ldots, r_{k+1}), \\ (q_{k+1}, \ldots, q_n) &< (r_{k+2}, \ldots, r_{n+1}, -\infty)\}. \end{aligned} \tag{2}$$

Using the imbeddings of $\pi_{j,k,r}$'s as subquotients of elementary representations in the proof of Proposition 10.2 in [4] we find $[\pi_{j,k,r}] = [s]$ where

$$s_i = \begin{cases} r_i + n/2 - i, & 1 \leqslant i \leqslant j, \\ r_i + n/2 - i + 1, & j + 1 \leqslant i \leqslant k + 1, \\ r_i + n/2 - i + 2, & k + 2 \leqslant i \leqslant n + 1. \end{cases} \tag{3}$$

From (2) and (3) we get easily

PROPOSITION 2. *Let $0 \leqslant j \leqslant k \leqslant n$ and $r \in S_{jk}$.*

(i) $\lambda^j(r) = (r_1, \ldots, r_k, r_{k+2}, \ldots, r_{n+1})$ *is the unique j-fundamental corner of* $\pi_{j,k,r}$.

(ii) $\lambda^k(r) = (r_1, \ldots, r_j, r_{j+2}, \ldots, r_{n+1})$ *is the unique k-fundamental corner of* $\pi_{j,k,r}$.

(iii) *For* $i \in \{0, \ldots, n\}$, $i \neq j$, $i \neq k$, $\pi_{j,k,r}$ *has no i-fundamental corners.*

PROOF OF THEOREM 2. From Proposition 2, we see that $F(\pi_{j,k,r}) = \{\lambda^j(r), \lambda^k(r)\}$ and (i) is proven.

(ii) Suppose that $0 \leqslant j \leqslant k \leqslant n$, $0 \leqslant l \leqslant m \leqslant n$, $r \in S_{jk}$, $s \in S_{lm}$ and $\{\lambda^j(r), \lambda^k(r)\} = \{\lambda^l(s), \lambda^m(s)\}$. We can suppose $j \leqslant l$. Then we have the following possibilities:

(a) $j \leqslant k \leqslant l \leqslant m$. Then it follows $r_i = s_i$ for $1 \leqslant i \leqslant j$ and $m + 2 \leqslant i \leqslant n + 1$, $r_i = s_{i-1}$ for $k + 2 \leqslant i \leqslant l + 1$, and $r_{j+1} = \cdots = r_{k+1} = s_{j+1} = \cdots = s_k = a \geqslant r_{l+2} = \cdots = r_{m+1} = s_{l+1} = \cdots = s_{m+1} = b$. Hence, $m + l - j - k = \Sigma_{i=1}^{n+1} s_i - \Sigma_{i=1}^{n+1} r_i = b - a$. This yields $a = b$ and $j = k = l = m$. Hence, $\pi_{j,k,r} = \pi_{l,m,s}$.

(b) $j \leqslant l \leqslant k \leqslant m$. Then $r = s$; hence, $0 = \Sigma_{i=1}^{n+1} s_i - \Sigma_{i=1}^{n+1} r_i = (m - k) + (l - j)$, i.e., $l = j$, $m = k$. Therefore $\pi_{j,k,r} = \pi_{l,m,s}$.

(c) $j \leqslant l \leqslant m < k$. It follows $r = s$ and $s_{j+1} = \cdots = s_{l+1}$, $s_{m+1} = \cdots = s_{k+1}$. Now, $s_{m+1} = s_{m+2}$ and $s \in S_{l,m}$ implies $s_l > s_{l+1}$; hence $l = j$.

Furthermore, $0 = \Sigma_{i=1}^{n+1} s_i - \Sigma_{i=1}^{n+1} r_i = l + m - j - k = m - k$; therefore $m = k$ and again $\pi_{j,k,r} = \pi_{l,m,s}$.

Therefore $\pi \longmapsto F(\pi)$ defines an injection of $\widehat{G}^0$ into Ω and (ii) is proven.

(iii) Let Ω_0 be the image of this injection and let Ω_0' be the subset of Ω defined by the properties in (iii). We have to show $\Omega_0 = \Omega_0'$.

Let $\pi = \pi_{j,k,r} \in \widehat{G}^0$. Then $F(\pi) = (\lambda^j(r), \lambda^k(r))$. Put $t = \lambda^j(r) + \rho_K - \rho_P^j$, $s = \lambda^k(r) + \rho_K - \rho_P^k$. Obviously, $[t] = [s]$. We have either $r_j > r_{j+1}$ or $r_{k+1} > r_{k+2}$. It is easy to check that $t \in \bar{\bar{D}}_k$, $s \in \bar{\bar{D}}_j$. Furthermore, $r_j > r_{j+1}$ implies $t \in \bar{D}_k$ and $r_{k+1} > r_{k+2}$ implies $s \in \bar{D}_j$. Hence, $F(\pi) \in \Omega_0'$ and $\Omega_0 \subset \Omega_0'$.

Take $(q, q') \in \Omega_0'$ and let $j, k \in \{0, \ldots, n\}$ be such that $[q + \rho_K - \rho_P^j] = [q' + \rho_K - \rho_P^k]$, $q + \rho_K - \rho_P^j \in \bar{D}_k$, $q' + \rho_K - \rho_P^k \in \bar{\bar{D}}_j$. Put $t = q + \rho_K - \rho_P^j$, $s = q' + \rho_K - \rho_P^k$. Suppose first $j \leqslant k$. Then $t \in \bar{D}_k$, $s \in \bar{\bar{D}}_j$ and $[t] = [s]$ implies $t_i = s_i$ for $1 \leqslant i \leqslant j$ and $k + 1 \leqslant i \leqslant n$, $t_i = s_{i-1}$ for $j + 2 \leqslant i \leqslant k$, $t_{j+1} = s_{n+1}$ and $t_{n+1} = s_k$. By (1), this yields $q_i = q_i'$ for $1 \leqslant i \leqslant j$ and $k + 1 \leqslant i \leqslant n$, $q_i = q_{i-1}'$ for $j + 2 \leqslant i \leqslant k$, $q_{j+1} + \Sigma_{i=1}^n q_i' = j + k - n$ and $q_k' + \Sigma_{i=1}^n q_i = j + k - n$. Put $r = (q_1, \ldots, q_k, q_k', q_{k+1}, \ldots, q_n)$. Then $r \in ((n + 1)^{-1}\mathbf{Z})_>^{n+1}$, $\Sigma_{i=1}^{n+1} r_i = j + k - n$ and $t_j > t_{j+1}$ implies $r_j > r_{j+1}$. Hence, $r \in S_{jk}$. Obviously, $\lambda^j(r) = q$, $\lambda^k(r) = q'$; therefore, $F(\pi_{j,k,r}) = (q, q')$ and $(q, q') \in \Omega_0$.

Suppose now $k \leqslant j$. Then, similarly as above, we find that $r = (q_1, \ldots, q_k, q_{k+1}', q_{k+1}, \ldots, q_n) \in S_{kj}$, $\lambda^k(r) = q'$, $\lambda^j(r) = q$; hence, $F(\pi_{k,j,r}) = (q, q')$ and again $(q, q') \in \Omega_0$. Thus, $\Omega_0' = \Omega_0$. Finally, the injectivity in (ii) shows that j and k are well determined by (q, q') and Theorem 2 is proven.

Proof of Theorem 3. Let $\hat{K}^j$ be the set of all $q \in \hat{K}$ such that there is $\pi \in \widehat{G}^0$ which has q as a j-fundamental corner. We have $\Omega_1 = \{(q, j); q \in \hat{K}^j, 0 \leqslant j \leqslant n\}$. Statement (iv) of Theorem 3 is equivalent to

Lemma 1. *$\hat{K}^0 = \hat{K}^n = \hat{K}$. For $1 \leqslant j \leqslant n - 1$, $\hat{K}^j$ is the set of all $q \in \hat{K}$ such that either $q_j > q_{j+1}$ or $(\alpha | q + \rho_K - \rho_P^j) \neq 0$ $\forall \alpha \in R_P$.*

Proof. Obviously, $\hat{K}^j = \{\lambda^j(r); r \in S_{kj}, 0 \leqslant k < j\} \cup \{\lambda^j(r); r \in S_{jk}, j \leqslant k \leqslant n\}$.

(a) Suppose first $j = 0$. Then $\hat{K}^0 = \{\lambda^0(r); r \in S_{0k}, 0 \leqslant k \leqslant n\}$. Take any $q \in \hat{K}$. Then $q_i + n - i > q_{i+1} + n - (i + 1)$, $1 \leqslant i \leqslant n - 1$. Hence, there is unique $k \in \{0, \ldots, n\}$ such that $q_k = n - k \geqslant -\Sigma_{i=1}^n q_i > q_{k+1} + n - (k + 1)$ $(q_0 = \infty, q_{n+1} = -\infty)$. Put $r = (q_1, \ldots, q_k, -\Sigma_{i=1}^n q_i + k - n, q_{k+1}, \ldots, q_n)$. Then $r \in S_{0k}$ and $\lambda^0(r) = q$. Hence, $\hat{K}^0 = \hat{K}$.

(b) Suppose now $j = n$. Then for $q \in \hat{K}$ we see that there is unique $k \in \{0, \ldots, n\}$ such that $q_k - k \geqslant -\Sigma_{i=1}^n q_i > q_{k+1} - (k + 1)$. Now, $r = (q_1, \ldots, q_k, -\Sigma_{i=1}^n q_i + k, q_{k+1}, \ldots, q_n)$ is in S_{kn} and $\lambda^n(r) = q$. Hence, $\hat{K}^n = \hat{K}$.

(c) Finally, suppose $1 \leqslant j \leqslant n - 1$. Put

$$\hat{K}_j = \{q \in \hat{K}; \text{ either } q_j > q_{j+1} \text{ or } (\alpha | q + \rho_K - \rho_P^j) \neq 0 \ \forall \alpha \in R_P\}.$$

Let $0 \leqslant k < j$, $r \in S_{kj}$. Put $q = \lambda^j(r)$. We have either $r_k > r_{k+1}$ or $r_{j+1} > r_{j+2}$. If $r_{j+1} > r_{j+2}$, then $q_j > q_{j+1}$ and $q \in \hat{K}_j$. Suppose $r_{j+1} = r_{j+2}$, i.e., $q_j = q_{j+1}$. Then $r_k > r_{k+1}$. This means $q_k + \Sigma_{i=1}^n q_i - k - j + n > 0$. Furthermore, $q_{k+1} + \Sigma_{i=1}^n q_i - k - j + n = r_{k+2} - r_{k+1} \leqslant 0$. Now, for $1 \leqslant l \leqslant k$,

$$(\alpha_{l,n+1}|q + \rho_K - \rho_P^j) = q_l + \sum_{i=1}^n q_i - l - j + n$$

$$\geqslant q_k + \sum_{i=1}^n q_i - k - j + n > 0;$$

for $k + 1 \leqslant l \leqslant j$,

$$(\alpha_{l,n+1}|q + \rho_K - \rho_P^j) = q_l + \sum_{i=1}^n q_i - l - j + n$$

$$\leqslant q_{k+1} + \sum_{i=1}^n q_i - k - j + n - 1 < 0;$$

for $j + 1 \leqslant l \leqslant n$,

$$(\alpha_{l,n+1}|q + \rho_K - \rho_P^j) = q_l + \sum_{i=1}^n q_i - l - j + n + 1$$

$$< q_{k+1} + \sum_{i=1}^n q_i - k - j + n \leqslant 0.$$

Hence, $(\alpha|q + \rho_K - \rho_P^j) \neq 0 \ \forall \alpha \in R_P$, and $q \in K_j$.

Quite similarly, we find that for $j < k \leqslant n$, $r \in S_{jk}$, $q = \lambda^j(r)$ is in $\hat{K}_j$.

Finally, if $r \in S_{jj}$ and $q = \lambda^j(r)$, then either $r_j > r_{j+1}$ or $r_{j+1} > r_{j+2}$, so in any case $r_j > r_{j+2}$, i.e., $q_j > q_{j+1}$.

This proves that $\hat{K}^j \subset \hat{K}_j$, $1 \leqslant j \leqslant n - 1$.

Take now $q \in \hat{K}_j$. There is unique $k \in \{0, \ldots, n\}$ such that $q_k + n - j - k \geqslant -\Sigma_{i=1}^n q_i > q_{k+1} + n - j - k - 1$. Put $r = (q_1, \ldots, q_k, -\Sigma_{i=1}^n q_i + j + k - n, q_{k+1}, \ldots, q_n)$. Then $r \in ((n+1)^{-1}\mathbf{Z})_>^{n+1}$ and $\Sigma_{i=1}^{n+1} r_i = j + k - n$. If we can show that $r \in S_{jk}$ (or S_{kj} in case $k < j$) we shall have $q = \lambda^j(r)$; hence, $q \in \hat{K}^j$.

Suppose first $k < j$. If $q_j > q_{j+1}$, then $r_{j+1} > r_{j+2}$; hence, $r \in S_{kj}$. If $q_j = q_{j+1}$, then it must be $(\alpha|q + \rho_K - \rho_P^j) \neq 0 \, \forall \alpha \in R_P$. But $r_k - r_{k+1} = (\alpha_{k,n+1}|q + \rho_K - \rho_P^j)$. Hence $r_k > r_{k+1}$, and again $r \in S_{kj}$.

Similarly, in the case $k > j$, $q_j > q_{j+1}$ implies $r_j > r_{j+1}$, and $(\alpha|q + \rho_K - \rho_P^j) \neq 0 \ \forall \alpha \in R_P$ gives $r_{k+2} - r_{k+1} = (\alpha_{k+1,n+1}|q + \rho_K - \rho_P^j) \neq 0$.

Finally, suppose $k = j$. Then $q_j > q_{j+1}$ means $r_j > r_{j+2}$; hence either

$r_j > r_{j+1}$ or $r_{j+1} > r_{j+2}$, i.e., $r \in S_{jj}$. On the other hand, $r_j - r_{j+1} = (\alpha_{j,n+1} \mid q + \rho_K - \rho_P^j)$. Hence, in any case $q \in \hat{K}_j$ implies $r \in S_{jj}$.

Therefore, $\hat{K}_j \subset \hat{K}^j$ and Lemma 1 is proven.

Let us return now to the proof of Theorem 3. By the definition of $\hat{K}^j$, for any $(q, j) \in \Omega_1$ there is $\pi \in \widehat{G}^0$ such that q is j-fundamental corner of π. Then $\pi = \pi_{j,k,r}$ (or $\pi = \pi_{k,j,r}$) for some $k \in \{0, \ldots, n\}$, where $r \in S_{jk}$ (or $r \in S_{kj}$). From the proof of Lemma 1 we see that this k is well determined by q and j. But then r is also well determined. Hence, this π is unique. Therefore, (i) is proven and (ii) follows immediately. (iii) follows from Theorem 2. Q.E.D.

5. Nonelementary unitary representations.

Proof of Theorem 5. Let T_j $(0 \leq j \leq n)$ and T_{jk} $(0 \leq j < k \leq n)$ be defined as at the beginning of §4. Let $r \in T_{jk}$. From [4, Proposition 11.4(iii)] we know that $\pi_{j,k,r}$ is unitary if and only if $r_{j+1} = \cdots = r_{k+1}$. Let $r \in T_j$. By Proposition 11.4(ii) in [4], $\pi_{j,r}$ is unitary if and only if

$$(4) \qquad r_j - l_1(r) \geq -\sum_{i=1}^{n} r_i - n + j \geq r_{j+1} - l_2(r) + 1,$$

where

$$l_1(r) = \max\{l \in \{1, \ldots, j\}; r_{l-1} > r_l\} \qquad (r_0 = \infty),$$

$$l_2(r) = \min\{l \in \{j+1, \ldots, n\}; r_l > r_{l+1}\} \qquad (r_{n+1} = -\infty).$$

Suppose first that $r_k + \Sigma_{i=1}^n r_i = j + k - n$ for some $k \in \{1, \ldots, j-1\}$. Then $\pi_{j,r} = \pi_{k,j,r'}$, for $r' = (r_1, \ldots, r_k, r_k, r_{k+1}, \ldots, r_n)$. Now $0 > r_{j+1} - r_k + k + 1 - l_2(r) = (r_{j+1} - l_2(r) + 1) - (-\Sigma_{i=1}^n r_i - n + j)$; hence the second inequality in (4) is automatically satisfied. Suppose that the first is also satisfied. Then $r_j - l_1(r) \geq r_k - k$, i.e., $0 \geq r_j - r_k \geq l_1(r) - k$. Hence, $k \geq l_1(r)$, or $r_k = r_{k+1} = \cdots = r_j$. Conversely, if $r_k = \cdots = r_j$, then $k \geq l_1(r)$; hence, $r_j - l_1(r) \geq r_j - k = r_k - k = -\Sigma_{i=1}^n r_i - n + j$.

Therefore, $\pi_{j,r}$ is unitary if and only if $r_k = \cdots = r_j$, or equivalently $r'_{k+1} = \cdots = r'_{j+1}$.

Suppose now that $r_{k+1} + \Sigma_{i=1}^n r_i = j + k - n$ for some $k \in \{j+1, \ldots, n-1\}$. Then $\pi_{j,r} = \pi_{j,k,r'}$ with $r' = (r_1, \ldots, r_{k+1}, r_{k+1}, r_{k+2}, \ldots, r_n)$. Similarly as above we find that the unitarity of $\pi_{j,r}$ is equivalent to $r'_{j+1} = \cdots = r'_{k+1}$.

Finally, if $r_j \geq 2j - n - \Sigma_{i=1}^n r_i \geq r_{j+1}$, then (4) is satisfied and $\pi_{j,r}$ is unitary.

The conclusion is that for $0 \leq j \leq k \leq n$ and $r \in S_{jk}$, $\pi_{j,k,r}$ is unitary if and only if $r_{j+1} = \cdots = r_{k+1}$. But this is precisely equivalent to $\lambda^j(r) = \lambda^k(r)$, that is to the fact that $\pi_{j,k,r}$ has only one fundamental corner. This proves statement (i) of Theorem 5.

Put

$$R_{jk} = \{r \in S_{jk}; r_{j+1} = \cdots = r_{k+1}\}, \quad \hat{G}^0_{jk} = \{\pi_{j,k,r}; r \in R_{jk}\}.$$

Then $\hat{G}^0$ is the disjoint union of $\hat{G}^0_{jk}$, $0 \leqslant j \leqslant k \leqslant n$. For $0 \leqslant j \leqslant k \leqslant n$, let $\hat{K}_{jk}$ denote the set of all $q \in \hat{K}$ with the property: there is $\pi \in \hat{G}^0$ such that q is j-fundamental corner and k-fundamental corner of π and q is not i-fundamental corner of π if $i \neq j$ and $i \neq k$. Obviously, $\hat{K}_{jk} = \{\lambda^j(r); r \in R_{jk}\}$. Furthermore, if $r, s \in R_{jk}$ and $\lambda^j(r) = \lambda^j(s)$, then $r_i = s_i$ for $1 \leqslant i \leqslant n+1$, $i \neq k+1$, and $r_{k+1} = -\Sigma_{i \neq k+1} r_i + j + k - n = -\Sigma_{i \neq k+1} s_i + j + k - n = s_{k+1}$. Hence, $r = s$. Therefore, $r \mapsto \lambda^j(r)$ is a bijection of R_{jk} onto $\hat{K}_{jk}$. So, to prove statement (ii) of Theorem 5 we have to show that $\hat{K}$ is the disjoint union of $\hat{K}_{jk}$, $0 \leqslant j \leqslant k \leqslant n$.

First of all, we easily see that

$$\hat{K}_{jk} = \left\{q \in \hat{K}; q_{j+1} = \cdots = q_k = -\sum_{i=1}^{n} q_i + k + j - n, \ q_j > q_{k+1}\right\},$$

$$(5) \qquad\qquad 0 \leqslant j < k \leqslant n,$$

$$\hat{K}_{jj} = \left\{q \in \hat{K}; q_j \geqslant -\sum_{i=1}^{n} q_i + 2j - n \geqslant q_{j+1}, \ q_j > q_{j+1}\right\}, \quad 0 \leqslant j \leqslant n.$$

Suppose $q \in \hat{K}_{jk} \cap \hat{K}_{lm}$, $0 \leqslant j < k \leqslant n$, $0 \leqslant l < m \leqslant n$. By (5) we have either $j \notin \{l+1, \ldots, m-1\}$ or $k \notin \{l+1, \ldots, m+1\}$ and either $l \notin \{j+1, \ldots, k-1\}$ or $m \notin \{j+1, \ldots, k-1\}$. We can suppose $j \leqslant l$. Then we have the following possibilities.

(a) $k \leqslant m$. Then $\Sigma_{i=1}^n q_i = -q_k + k + j - n = -q_m + l + m - n$; hence, $0 \leqslant q_k - q_m = k + j - l - m \leqslant 0$. Therefore $q_k = q_m$, $k = m$ and $j = l$.

(b) $k > m$. Then $m \in \{j+1, \ldots, k-1\}$; hence, $l \notin \{j+1, \ldots, k-1\}$, i.e., $l = j$. This implies $-q_{j+1} + k + j - n = \Sigma_{i=1}^n q_i = -q_{j+1} + j + m - n$, i.e., $m = k$.

Hence, in any case $j = l$ and $k = m$.

Suppose now $q \in \hat{K}_{jj} \cap \hat{K}_{lm}$, $0 \leqslant j \leqslant n$, $0 \leqslant l < m \leqslant n$. By (5) we have $j \notin \{l+1, \ldots, m-1\}$, i.e., the following possibilities.

(a) $j \leqslant l$. Then $-\Sigma_{i=1}^n q_i + 2j - n \geqslant q_{j+1} \geqslant q_{l+1} = -\Sigma_{i=1}^n q_i + l + m - n$; hence, $l \geqslant j \geqslant \frac{1}{2}(l+m)$, i.e., $l \geqslant m$, a contradiction.

(b) $j \geqslant m$. Then $-\Sigma_{i=1}^n q_i + 2j - n \leqslant q_j \leqslant q_m = -\Sigma_{i=1}^n q_i + l + m - n$ which gives again $l \geqslant m$.

Suppose finally $q \in \hat{K}_{jj} \cap \hat{K}_{kk}$, $0 \leqslant j \leqslant k \leqslant n$. Suppose $j < k$. Then by (5), $0 \geqslant q_k - q_{j+1} \geqslant 2(k-j) > 0$, which is impossible. Hence, $j = k$.

Therefore, $(j, k) \neq (l, m)$ implies $\hat{K}_{jk} \cap \hat{K}_{lm} = \varnothing$.

Now, we shall show $\hat{K} = \bigcup_{0<j<k<n} \hat{K}_{jk}$. Take $q \in \hat{K}$ and suppose $q \notin \hat{K}_{jj}$ for any $j \in \{0, \ldots, n\}$. $q \notin \hat{K}_{00}$ implies $q_1 > -\Sigma_{i=1}^n q_i - n$; $q \notin \hat{K}_{nn}$ implies $-\Sigma_{i=1}^n q_i - n > q_n - 2n$; $q \notin \hat{K}_{jj}$ for any $j \in \{1, \ldots, n-1\}$ implies that if $q_j - 2j \geqslant -\Sigma_{i=1}^n q_i - n \geqslant q_{j+1} - 2j$, $j \in \{1, \ldots, n-1\}$, then $q_j = q_{j+1}$. Hence, there are $0 \leqslant j < k \leqslant n$ such that $q_j > q_{j+1} = \cdots = q_k > q_{k+1}$ ($q_0 = \infty$, $q_{n+1} = -\infty$) and $q_k - 2j > -\Sigma_{i=1}^n q_i - n > q_k - 2k$.

Suppose that $q_k - 2j > -\Sigma_{i=1}^n q_i - n \geqslant q_k - k - j$. Then there is $s \in \{j+1, \ldots, k\}$ such that $-\Sigma_{i=1}^n q_i - n = q_k - j - s$. Then $q_{j+1} = \cdots = q_s = -\Sigma_{i=1}^n q_i + s + j - n$, $q_j > q_s \geqslant q_{s+1}$; hence $q \in \hat{K}_{js}$.

Suppose now that $q_k - k - j > -\Sigma_{i=1}^n q_i - n > q_k - 2k$. Then there is $s \in \{j+1, \ldots, k-1\}$ such that $-\Sigma_{i=1}^n q_i - n = q_k - k - s$. Then $q_{s+1} = \cdots = q_k = -\Sigma_{i=1}^n q_i + k + s - n$; hence $q \in \hat{K}_{sk}$.

This proves assertion (ii) of Theorem 5.

6. **Discrete series representations.** As $G = SU(n, 1)$ is a real form of a simply connected complex semisimple Lie group, the set of discrete series representations is parametrized by $\bigcup_{j=0}^n (\hat{K} \cap D_j)$ [2]. Let θ_λ denote the discrete series representation corresponding to $\lambda \in \hat{K} \cap D_j$, $j \in \{0, \ldots, n\}$. By Blattner's conjecture (which is proven for the hermitian symmetric case in [8]), if $\lambda \in \hat{K} \cap D_j$, then θ_λ has $\lambda - \rho_K + \rho_P^j$ as a j-fundamental corner. Hence $\theta_\lambda = \pi_q$ for $q = \lambda - \rho_K + \rho_P^j$. Then $q + \rho_K - \rho_P^j = \lambda \in D_j$. Conversely, if $q \in \hat{K}$ and $j \in \{0, \ldots, n\}$ are such that $\lambda = q + \rho_K - \rho_P^j \in D_j$, then $\theta_\lambda = \pi_q$. This proves assertion (i) in Theorem 6.

7. **Subquotients of reducible elementary representations.** In the following we write $\pi_{j,k}(r)$ instead of $\pi_{j,k,r}$. Using Theorem 7.5 in [4] and the notation preceding this theorem, we easily find

PROPOSITION 3. (i) *Let $p \in \hat{M}$ be such that $\pi^{p,0}$ is reducible (i.e., $0 \in K(p)$). Put $j = j(p, 0)$. Then $\pi^{p,0}$ is direct sum of the following two irreducible representations*:

$$\pi_{j,j}(p_1, \ldots, p_{j-1}, j - \tfrac{1}{2}s(p), j - \tfrac{1}{2}s(p), p_j, \ldots, p_{n-1}),$$

$$\pi_{j-1,j-1}(p_1, \ldots, p_{j-1}, j - \tfrac{1}{2}s(p) - 1, j - \tfrac{1}{2}s(p) - 1, p_j, \ldots, p_{n-1}).$$

(ii) *Let $p \in \hat{M}$ and $\lambda > 0$ be such that $\lambda \in K(p)$, $-\lambda \in S(p)$, $j(p, \lambda) = j \leqslant k \leqslant n-1$, $-\lambda = s_k(p)$. Then $\pi^{p,\lambda}$ has the following two irreducible subquotients*:

$$\pi_{j,k}(p_1, \ldots, p_{j-1}, j + k - s(p) - p_k, p_j, \ldots, p_k, p_k, \ldots, p_{n-1}),$$

$$\pi_{j-1,k}(p_1, \ldots, p_{j-1}, j + k - s(p) - p_k - 1, p_j, \ldots, p_k, p_k, \ldots, p_{n-1}).$$

(iii) *Let $p \in \hat{M}$ and $\lambda > 0$ be such that $-\lambda \in S(p)$, $\lambda \in K(p)$, $j(p, -\lambda) = j$,*

$\lambda = s_k(p)$, $1 \leqslant k \leqslant j-1 \leqslant n-1$. *Then* $\pi^{p,\lambda}$ *has the following two irreducible subquotients*:

$$\pi_{k,j}(p_1, \ldots, p_k, p_k, \ldots, p_{j-1}, j+k-s(p)-p_k, p_j, \ldots, p_{n-1}),$$

$$\pi_{k,j-1}(p_1, \ldots, p_k, p_k, \ldots, p_{j-1}, j+k-s(p)-p_k-1, p_j, \ldots, p_{n-1}).$$

(iv) *Let* $p \in \hat{M}$ *and* $\lambda > 0$ *be such that* $\lambda \in K(p)$, $-\lambda \in K(p)$. *Put* $j(p, \lambda) = j$, $j(p, -\lambda) = k$ $(1 \leqslant j \leqslant k \leqslant n)$. *Then* $\pi^{p,\lambda}$ *has the following irreducible subquotients*:

$$\pi_{j,k}(p_1, \ldots, p_{j-1}, j - \tfrac{1}{2}s(p) + \tfrac{1}{2}\lambda, p_j, \ldots, p_{k-1}, k - \tfrac{1}{2}s(p) - \tfrac{1}{2}\lambda, p_k, \ldots, p_{n-1}),$$

$$\pi_{j,k-1}(p_1, \ldots, p_{j-1}, j - \tfrac{1}{2}s(p) + \tfrac{1}{2}\lambda, p_j, \ldots, p_{k-1}, k - \tfrac{1}{2}s(p) - \tfrac{1}{2}\lambda - 1, p_k, \ldots, p_{n-1}),$$

$$\pi_{j-1,k}(p_1, \ldots, p_{j-1}, j - \tfrac{1}{2}s(p) + \tfrac{1}{2}\lambda - 1, p_j, \ldots, p_{k-1}, k - \tfrac{1}{2}s(p) - \tfrac{1}{2}\lambda, p_k, \ldots, p_{n-1}),$$

$$\pi_{j-1,k-1}(p_1, \ldots, p_{j-1}, j - \tfrac{1}{2}s(p) + \tfrac{1}{2}\lambda - 1, p_j, \ldots, p_{k-1}, k - \tfrac{1}{2}s(p) - \tfrac{1}{2}\lambda - 1, p_k, \ldots, p_{n-1}).$$

($\pi_{j,k-1}(\cdots)$ *does not appear if* $k = j$.)

From this we can easily identify all elementary representations in which a given $\pi \in \widehat{G}^0$ appears as subquotient. Especially, for unitary representations we get the following statement (notation is from §5):

THEOREM 7. (i) *If* $q \in \hat{K}_{jk}$, $0 \leqslant j < k \leqslant n$, *then* π_q *appears as a subquotient in at most four different elementary representations* $\pi^{p,\lambda}$, $p \in \hat{M}$, $\lambda \geqslant 0$. *These are* $\pi^{p,\lambda}$ *for the following values of* p *and* λ:

(1) $p = (q_1, \ldots, q_{j-1}, q_{j+1}, \ldots, q_n)$, $\lambda = q_j - q_k + k - j$ (*only if* $j \geqslant 1$).

(2) $p = (q_1, \ldots, q_{j-1}, q_{j+1}, \ldots, q_k, q_k, q_{k+2}, \ldots, q_n)$, $\lambda = q_j - q_{k+1} + k - j$ (*only if* $1 \leqslant j \leqslant k \leqslant n-1$).

(3) $p = (q_1, \ldots, q_j, q_{j+2}, \ldots, q_n)$, $\lambda = k - j$.

(4) $p = (q_1, \ldots, q_j, q_{j+1}, \ldots, q_k, q_{k+2}, \ldots, q_n)$, $\lambda = q_{j+1} - q_{k+1} + k - j$ (*only if* $k \leqslant n-1$).

(*Notice, that in cases* $q_j = q_{j+1}$ *or* $q_k = q_{k+1}$ *some of these elementary representations coincide.*)

(ii) *If* $q \in \hat{K}_{jj}$, $0 \leqslant j \leqslant n$, *then* π_q *appears as a subquotient in at most three different elementary representations* $\pi^{p,\lambda}$, $p \in \hat{M}$, $\lambda > 0$. *These are* $\pi^{p,\lambda}$ *for the following values of* p *and* λ:

(1) $p = (q_1, \ldots, q_{j-1}, 2j - n - \Sigma_{i=1}^n q_i, q_{j+2}, \ldots, q_n)$, $\lambda = q_j - q_{j+1}$ (*only if* $1 \leqslant j \leqslant n-1$).

(2) $p = (q_1, \ldots, q_{j-1}, q_{j+1}, \ldots, q_n)$, $\lambda = q_j - 2j + n + \Sigma_{i=1}^n q_i$ (*only if* $j \geqslant 1$).

(3) $p = (q_1, \ldots, q_j, q_{j+2}, \ldots, q_n)$, $\lambda = 2j - n - \Sigma_{i=1}^n q_i - q_{j+1}$ (*only if* $j \leqslant n-1$).

(*Notice that in cases* $q_j = 2j - n - \Sigma_{i=1}^n q_i$ *or* $q_{j+1} = 2j - n - \Sigma_{i=1}^n q_i$ *some of these elementary representations coincide.*)

We can prove now statement (ii) of Theorem 6. If $q \in \hat{K}$ is such that π_q is not a discrete series representation, then we have the following possibilities:

(a) $q \in \hat{K}_{jj}$, $q_j = 2j - n - \Sigma_{i=1}^n q_i$, $j \geqslant 1$. By Theorem 7(ii) π_q appears as a subrepresentation of $\pi^{p,0}$, $p = (q_1, \ldots, q_{j-1}, q_{j+1}, \ldots, q_n)$; obviously $\lambda_p = 0$.

(b) $q \in \hat{K}_{jj}$, $q_{j+1} = 2j - n - \Sigma_{i=1}^n q_i$, $j \leqslant n-1$. By Theorem 7(ii) π_q is a subrepresentation of $\pi^{p,0}$, $p = (q_1, \ldots, q_j, q_{j+2}, \ldots, q_n)$ and then $\lambda_p = 0$.

(c) $q \in \hat{K}_{jk}$, $0 \leqslant j < k \leqslant n$. Then, by Theorem 7(i), π_q appears as a subquotient of $\pi^{p,k-j}$, $p = (q_1, \ldots, q_j, q_{j+2}, \ldots, q_n)$. We have to prove that $\lambda_p = k - j$. To do so, we use the notation from [4, §7]. We have $q_{j+1} = \cdots = q_k = -\Sigma_{i=1}^n q_i + j + k - n$. Hence, $s(p) = j + k - 2q_k$ and $s_i(p) = j + k - 2i$ for $j + 1 \leqslant i + k - 1$. Therefore, $K(p)$ is contained in $k - j + 2\mathbf{Z}$ and $\{k-j-2, k-j-4, \ldots, j-k+2\} \cap K(p) = \varnothing$. Hence

$$\lambda_p = \min\{\lambda \geqslant 0; \lambda \in K(p) \cup (-K(p))\} \geqslant k - j.$$

On the other hand, $\pi^{p,k-j}$ is reducible; hence, $\lambda_p = k - j$. This proves Theorem 6.

Finally, we state a theorem giving the exact value of λ_p for any $p \in \hat{M}$. This can be easily deduced from Theorems 7.5 and 8.5 in [4]. We use the notation from [4, §7], and for $p \in \hat{M} = ((n+1)^{-1}\mathbf{Z})_{>}^{n-1}$ we put $p_j = \infty$ for $j \in \mathbf{Z}$, $j \leqslant 0$, and $p_j = -\infty$ for $j \in \mathbf{Z}$, $j \geqslant n$.

THEOREM 8. (i) *Let* $p \in \hat{M}$ *be such that* $S(p) = \{s_1(p), \ldots, s_{n-1}(p)\} \subset 2\mathbf{Z}$. *If* $s_j(p) \neq 0 \forall j \in \{1, \ldots, n-1\}$, *then* $\lambda_p = 0$. *If* $s_j(p) = 0$ *for some* $j \in \{1, \ldots, n-1\}$, *then*

$$\lambda_p = \max\{2k + 2; k \in \mathbf{Z}_+, p_{j-k} = p_{j+k}\}.$$

(ii) *Let* $p \in \hat{M}$ *be such that* $S(p) = \{s_1(p), \ldots, s_{n-1}(p)\} \subset 2\mathbf{Z} + 1$. *If* $\{1, -1\} \not\subset S(p)$, *then* $\lambda_p = 1$. *If* $s_j(p) = 1$, $s_{j+1}(p) = -1$ *for some* $j \in \{1, \ldots, n-2\}$, *then* $\lambda_p = \max\{2k + 3; k \in \mathbf{Z}_+, p_{j-k} = p_{j+k+1}\}$.

REMARK. When this paper was finished, I heard about the preprint *On an infinitesimal characterization of the discrete series* by T. J. Enright and V. S.

Varadarajan. In this paper, they have constructed for any $q \in \hat{K}$ and $D \in \mathcal{C}$ an irreducible admissible representation $\pi(q, D)$ of G with the property that q is a D-fundamental corner of $\pi(q, D)$. The only assumption about the group is rank G = rank K. Our results show that these representations exhaust all of $\widehat{G}^0$ in the case $G = SU(n, 1)$. Furthermore, it is not difficult to check that if $q \in \hat{K}\backslash\hat{K}^j$ then $\pi(q, D^j)$ is equivalent to some irreducible elementary representation having a fundamental corner and all such elementary representations are also exhausted.

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