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DYADIC METHODS IN THE MEASURE THEORY OF NUMBERS

BY

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ABSTRACT. Some new theorems in metric diophantine approximation are obtained by dyadic methods. We show for example that if $m_1, m_2, \ldots,$ are distinct integers with $m_n = O(n^p)$ then $\Sigma_{n<N} e(m_n x) = O(N^{1-q})$ except for a set of x of Hausdorff dimension at most $(p + 4q - 1)/(p + 2q)$; and that for any sequence of intervals $I_1, I_2, \ldots$ in $[0, 1)$ the number of solutions of $\{x^n\} \in I_n$ $(n < N)$ is a.e. asymptotic to $\Sigma_{n<N} |I_n|$ $(x > 1)$.

1. **Introduction.** Let $g_1(x), g_2(x), \ldots$ be a sequence of differentiable functions on the finite interval $[\alpha, \beta]$. Throughout the paper we assume that $g_1'(x)$ and $g_k'(x) - g_j'(x)$ are positive and monotonic increasing in $[\alpha, \beta]$ whenever $k > j \geqslant 1$. We also assume that there are numbers $C > 0$, $c > 0$ and a, $0 \leqslant a < 1$, such that

$$g_k'(x) - g_j'(x) \geqslant c \tag{1}$$

whenever $j \geqslant 1$ and $k \geqslant j + Cj^a$.

Let f be an integrable function on $[0, 1)$. Extend f to the real line by periodicity. We write

$$S(f, m, n, x) = \frac{1}{n} \sum_{k=m+1}^{m+n} f(g_k(x)) - \int_0^1 f(y)\,dy$$

for $m \geqslant 0$, $n \geqslant 1$. Let F be the family of indicator functions of intervals in $[0, 1)$. We write

$$D(m, n, x) = \sup_{f \in \mathrm{F}} |S(f, m, n, x)|. \tag{2}$$

$D(m, n, x)$ is the discrepancy of $g_{m+1}(x), \ldots, g_{m+n}(x)$ (modulo 1).

THEOREM 1. *Suppose that for some* $p \geqslant 1 - a$,

$$g_k'(\beta) << k^p. \tag{3}$$

Then if $0 < q < \frac{1}{2}(1 - a)$, *we have*

$$D(0, n, x) = O(n^{-q}) \tag{4}$$

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except for a set of x in $[\alpha, \beta]$ *of Hausdorff dimension at most*

$$(p + 4q - \tfrac{1}{2}(1-a))/(p + 2q + \tfrac{1}{2}(1-a)). \tag{5}$$

Constants implied by $\ll$ will be independent of all variables $j, k, x, \ldots$ appearing in the inequality. In [1] I obtained instead of (5) the bound

$$(p + 5q - (1-a))/(p + 2q) \qquad (0 < q < (1-a)/3) \tag{6}$$

which is sharper for small q. J. W. S. Cassels [4] and P. Erdös and J. F. Koksma [7] showed that (4) holds almost everywhere in $[\alpha, \beta]$ (in the Lebesgue sense) for $0 < q < \tfrac{1}{2}(1-a)$.

Throughout the paper $\rho(n), \eta_1(n), \eta_2(n), \ldots$ are positive functions of $n = 1, 2, \ldots$ such that

$$\sum \eta_i(n)/n < \infty, \qquad \sum \rho(n)/n < \infty,$$

$n\rho(n)$ is nondecreasing for large n, and $\eta_i(n)$ is nonincreasing for $n = 1, 2, \ldots$.

THEOREM 2. *Let f be a square-integrable function on* $[0, 1)$ *such that*

$$f(x) \sim \sum_{-\infty}^{\infty} c_k e^{2\pi i k x}, \qquad r_m = \sum_{k=m+1}^{\infty} |c_k|^2 \ll \eta_1(m) \tag{7}$$

for $m = 1, 2, \ldots$. *Then*

$$S(f, 0, n, x) = o(1) \tag{8}$$

for almost all x in $[\alpha, \beta]$.

The case $a = 0$ of Theorem 2 is due to Koksma [10]. Koksma's theorem does not apply to $g_j(x) = xj^{1-a}$ $(0 < a < 1)$, for example.

We write $|J|$ for the Lebesgue measure of a real set J.

THEOREM 3. *Suppose that for* $n = 1, 2, \ldots$

$$\sum_{k=1}^{n} g_k'(x) \ll g_n'(x) \tag{9}$$

for $x \in [\alpha, \beta]$. *Let* J_k *be a set in* $[0, 1)$, *the union of* t_k *disjoint intervals, for* $k = 1, 2, \ldots$. *Let* $\Psi(n) = \Sigma_{k=1}^{n} |J_k|$. *Let* $N(n, x)$ *denote the number of integers* k, $1 \leqslant k \leqslant n$, *for which the fractional part* $\{g_k(x)\}$ *falls in* J_k.

(a) *If* $t_k = 1$ *for* $k = 1, 2, \ldots$, *then for every* $\epsilon > 0$,

$$N(n, x) = \Psi(n) + O(\Psi(n)^{1/2} \log^{(3+\epsilon)/2} \Psi(n)) \tag{10}$$

for almost all x in $[\alpha, \beta]$.

(b) *If* $t_k \ll \Psi(k)\rho([\Psi(k)])$, *then*

$$N(n, x) = \Psi(n) + o(\Psi(n)) \tag{11}$$

for almost all x *in* $[\alpha, \beta]$.

(c) *The conclusions* (a) *and* (b) *are valid if instead of* (9) *we suppose that* $g_k(x) = \lambda_k x$ *where* $0 < \lambda_1 < \lambda_2 < \cdots$ *are integers whose greatest common divisors satisfy*

$$\sum_{j=1}^{n} (\lambda_j, \lambda_n) << \lambda_n. \tag{12}$$

It is not hard to show that some restriction on t_k is necessary. Let $\epsilon > 0$; then

$$t_k << \exp(\Psi(k)^{1+\epsilon})$$

does not imply (11) almost everywhere when $g_k(x) = 2^k x$; see §5.

Many results are known for the case $t_k = 1$ $(k = 1, 2, \ldots)$. Cassels [3] showed that in case (9) holds, $N(n, x)$ is bounded or unbounded with $\Psi(n)$ for almost all x. W. Philipp [13] proved (10) in the particular case $g_k(x) = \lambda_k x$, where $0 < \lambda_1 < \lambda_2 < \cdots$ are real numbers satisfying $\inf_k \lambda_{k+1}/\lambda_k > 1$. W. J. LeVeque [11] proved (11) in case (c) when $|J_1| \geqslant |J_2| \geqslant \cdots$. W. M. Schmidt [14] and V. Ennola [6] proved related but more difficult theorems.

2. Proof of Theorem 1.

LEMMA 1. *Let* F *be a positive piecewise differentiable function on* $[\alpha, \beta]$. *Suppose* $F'(x) \leqslant A$ $(\alpha \leqslant x \leqslant \beta)$ *and* $\int_\alpha^\beta F(x)\,dx \leqslant B$. *Let*

$$E = \{x \in [\alpha, \beta] : F(x) \geqslant d > 0\}.$$

There is a covering of E *with intervals* $I_1, \ldots, I_q$ *such that for* $0 < \gamma \leqslant 1$,

$$\sum_{j=1}^{q} |I_j|^\gamma << (1 + ABd^{-2})^{1-\gamma} B^\gamma d^{-\gamma}.$$

The implied constant is absolute.

PROOF. Cover E with disjoint intervals $I_1, \ldots, I_q$, $|I_j| = d/2A$ for $j < q$, $|I_q| \leqslant d/2A$, each I_j meeting E. Clearly $F(x) \geqslant d/2$ on each I_j, so

$$\frac{d}{2}(q-1)\frac{d}{2A} \leqslant \frac{d}{2}\sum_{j=1}^{q} |I_j| \leqslant B,$$

and so by Hölder's inequality

$$\sum_{j=1}^{q} |I_j|^\gamma \leqslant q^{1-\gamma}\left(\sum_{j=1}^{q} |I_j|\right)^\gamma << (1 + ABd^{-2})^{1-\gamma}(Bd^{-1})^\gamma.$$

THEOREM 4. *Let* $D(m, n, x)$ *be defined as in* (2) *where* $\mathfrak{F}$ *denotes some family of functions on* $[0, 1)$. *Assume* $D(m, n, x)$ *is measurable for* $m \geqslant 0$, $n \geqslant 1$ *and*

(13) $$nD(m, n, x) \leqslant h(m, n, x) \quad \text{for } m \geqslant 0, n \geqslant 1, \alpha \leqslant x \leqslant \beta,$$

where $h(m, n, x)$ *is positive and piecewise differentiable on* $[\alpha, \beta]$. *Assume that for some* $\rho \geqslant 0, \sigma \geqslant 0, \nu \geqslant \rho + 1, \mu \geqslant \sigma + 1$,

(14) $$\int_\alpha^\beta h^2(m, n, x)\,dx \ll k^\rho n^{\nu-\rho}$$

and

(15) $$h(m, n, x)h'(m, n, x) \ll k^\sigma n^{\mu-\sigma} \qquad (\alpha \leqslant x \leqslant \beta)$$

for all $k \geqslant 1, 0 \leqslant m \leqslant k, 1 \leqslant n \leqslant k$. *Then if* $0 < \lambda < \min(\frac{1}{2}\mu, \frac{1}{4}(\nu + \mu))$, *we have*

(16) $$D(0, n, x) = O(n^{\lambda-1})$$

except for a set of x of Hausdorff dimension at most

(17) $$(\mu + \nu - 4\lambda)/(\mu - 2\lambda).$$

PROOF. For $s \geqslant 1$ write

$$H(s, x) = \sum_{t=1}^{s} \sum_{u=0}^{2^{s-t}-1} h^2(u2^t, 2^{t-1}, x).$$

We have

$$\int_\alpha^\beta H(s, x)\,dx \ll \sum_{t=1}^{s} 2^{s-t}2^{s\rho}2^{t(\nu-\rho)} \ll s2^{\nu s}$$

while

$$H'(s, x) \ll \sum_{t=1}^{s} \sum_{u=1}^{2^{s-t}-1} |h(u2^t, 2^{t-1}, x)h'(u2^t, 2^{t-1}, x)|$$
$$\ll \sum_{t=1}^{s} 2^{s-t}2^{s\sigma}2^{t(\mu-\sigma)} \ll s2^{\mu s}.$$

We may assume $\lambda > \frac{1}{2}\nu$. Let γ be a real number, $1 - (2\lambda - \nu)/(\mu - 2\lambda) < \gamma < 1$. Let

$$E_s = \{x \in [\alpha, \beta] : H(s, x) \geqslant s^{-1}2^{2\lambda s}\}$$

for $s = 1, 2, \ldots$. By Lemma 1 there is a covering of E_s with intervals I_{sj}, $1 \leqslant j \leqslant q_s$, such that

$$\sum_j |I_{sj}|^\gamma \ll (s^4 2^{s(\nu+\mu-4\lambda)})^{1-\gamma} s^{2\gamma} 2^{s(\nu-2\lambda)\gamma}$$
$$\ll s^4 2^{s(\nu+\mu-4\lambda-\gamma(\mu-2\lambda))}.$$

Let

$$W = \bigcap_{m=1}^{\infty} \bigcup_{s=m}^{\infty} E_s,$$

then because the series $\Sigma_s \Sigma_j |I_{sj}|^\gamma$ converges, there is a covering of W by intervals $I_1, I_2, \ldots$ for which $\Sigma |I_j|^\gamma$ is arbitrarily small; so W has dimension $\leqslant \gamma$, and hence $\leqslant (\nu + \mu - 4\lambda)/(\mu - 2\lambda)$.

It remains to show that if $x \notin W$, (16) holds. For $s > s_0(x)$, $x \notin E_s$. Suppose $n > 2^{s_0}$, say $2^{s-1} \leqslant n < 2^s$, then $s > s_0$. Choose an f in $\mathcal{F}$. Using the dyadic representation of n, we can write

$$nS(f, 0, n, x) = \sum_t \sum_{u_t 2^t + 1}^{u_t 2^t + 2^{t-1}} \left(f(g_k(x) - \int_0^1 f(y)\,dy \right)$$

where each u_t is an integer, $0 \leqslant u_t < 2^{s-t}$, and t takes some of the values $1, \ldots, s$. By definition of $D(m, n, x)$ and (13),

$$nD(0, n, x) \leqslant \sum_t h(u_t 2^t, 2^{t-1}, x)$$

and by the Cauchy-Schwarz inequality,

$$n^2 D^2(0, n, x) \leqslant s \sum_t h^2(u_t 2^t, 2^{t-1}, x) \leqslant sH(s, x).$$

But $x \notin E_s$ and so

$$nD(0, n, x) < 2^{\lambda s} \leqslant 2^\lambda n^\lambda \qquad (n > 2^{s_0})$$

so that (16) holds, and Theorem 4 is proved.

To deduce Theorem 1 we require a smooth majorant for $nD(m, n, x)$. Write $e(x) = e^{2\pi i x}$.

LEMMA 2. *The discrepancy of* $g_{m+1}(x), \ldots, g_{m+n}(x)$ *satisfies*

$$nD(m, n, x) \ll \frac{n}{u+1} + \sum_{k=1}^{u} \frac{1}{k} \left| \sum_{j=m+1}^{m+n} e(kg_j(x)) \right|$$

for every integer $u \geqslant 1$. *The implied constant is absolute.*

PROOF. This is Theorem III of [8].

LEMMA 3. *Let* $p(x)$ *be a real periodic integrable function of* x *with period* 1 *and suppose*

$$\int_0^1 p(x)\,dx = 0, \qquad P = \max_t \left| \int_0^t p(x)\,dx \right|.$$

Let g *be a real function on* $[\alpha, \beta]$ *with monotonic derivative and* $\min_y |g'(y)| = G > 0$. *Then*

$$\left| \int_\alpha^\beta p(g(y)\,dy) \right| \leqslant \frac{2P}{G}.$$

PROOF. This is Lemma 1 of [4].

LEMMA 4.

$$\int_\alpha^\beta \left|\sum_{j=m+1}^{m+n} e(kg_j(x))\right|^2 dx \ll n(m+n)^a \left(1 + \frac{\log(n+1)}{k}\right)$$

$$(m \geq 0, n \geq 1, k \geq 1).$$

PROOF. This is a slight variant of Lemma 3.1 of [1] using Lemma 3 above instead of the particular case $g(x) = \lambda x$ of Lemma 3.

PROOF OF THEOREM 1. Let θ be a positive number specified below. We apply Theorem 4 with

$$h(m,n,x) = K\left(n^{1-\theta} + \sum_{k<n^\theta} \frac{1}{k}\left(\left|\sum_{j=m+1}^{m+n} \cos kg_j(x)\right| + \left|\sum_{j=m+1}^{m+n} \sin kg_j(x)\right|\right)\right)$$

which by Lemma 2 majorizes $nD(m,n,x)$ for a suitable $K > 0$. By Lemma 4 and Minkowski's inequality,

$$\left(\int_\alpha^\beta h^2(m,n,x)dx\right)^{1/2} \ll n^{1-\theta} + n^{1/2}(m+n)^{a/2} \sum_{k<n^\theta}\left(\frac{1}{k} + \frac{\log^{1/2}(n+1)}{k^{3/2}}\right)$$

$$\ll n^{1-\theta} + n^{1/2}(m+n)^{a/2}\log(n+1)$$

while by (3)

$$h(m,n,x)h'(m,n,x) \ll n\log(n+1) \sum_{k<n^\theta}\sum_{j=m+1}^{m+n} g_j'(\beta)$$

$$\ll n^{2+\theta}(m+n)^p \log(n+1).$$

We now assume that $0 < \theta \leq \frac{1}{2}(1-a)$. In the notation of Theorem 4, we may take $\rho = a, \nu - \rho = 2 + \epsilon - 2\theta - a \geq 1, \sigma = p$ and $\mu - \sigma = 2 + \theta + \epsilon \geq 1$. Here ϵ is an arbitrary positive number. The condition

$$\lambda < \min\left(\frac{\mu}{2}, \frac{\nu+\mu}{4}\right) = \min\left(\frac{2+\theta+p+\epsilon}{2}, \frac{4+p+2\epsilon-\theta}{4}\right)$$

is certainly satisfied if $\lambda \leq 1$ and $0 \leq \theta \leq (1-a)/2 \leq p$. Obviously the quantity

$$\frac{\mu+\nu-4\lambda}{\mu-2\lambda} = \frac{4+p+2\epsilon-\theta-4\lambda}{2+\theta+p+\epsilon-2\lambda}$$

is minimised by choosing the largest $\theta, \theta = \frac{1}{2}(1-a)$. Taking ϵ arbitrarily small, (16) holds outside a set of x of dimension at most

$$(4 + p - \tfrac{1}{2}(1 - a) - 4\lambda)/(2 + p + \tfrac{1}{2}(1 - a) - 2\lambda).$$

Substituting $\lambda = 1 - q$ we obtain Theorem 1.

Another application of Theorem 4 is following estimate of an exponential sum.

THEOREM 5. *Under the hypothesis of Theorem* 1,

$$\sum_{k=1}^{n} e(g_k(x)) = O(n^{1-q})$$

except for a set of x *in* $[\alpha, \beta]$ *of dimension at most*

$$(p + 4q - (1 - a))/(p + 2q). \tag{18}$$

PROOF. Apply Theorem 4 with F consisting of $e(x)$ and $h(m, n, x) = nD(m, n, x)$. By Lemma 4

$$\int_\alpha^\beta h^2(m, n, x)dx \ll k^a n \log(n + 1)$$

and

$$h(m, n, x)h'(m, n, x) \ll k^p n^2$$

for integers m, n, k, $0 \leqslant m \leqslant k$, $1 \leqslant n \leqslant k$. Take $\rho = a$, $\nu - \rho = 1 + \epsilon$, $\sigma = p$, $\mu - \sigma = 2$, and $\lambda = 1 - q$. The condition

$$\lambda < \min\left(\frac{\mu}{2}, \frac{\mu+\nu}{4}\right) = \min\left(\frac{2+p}{2}, \frac{2+p+1+a+\epsilon}{4}\right)$$

is satisfied because $p \geqslant 1 - a$, and we obtain the required result by substitution in (17).

One can obtain $(p + 4q - (1 - a))/(p + q)$ instead of (18) by the method of [1], but this is obviously less sharp.

3. **Proof of Theorem** 2. We follow [10]. It is enough to prove the theorem for real functions f with mean value 0. All functions f, f_n in this section are real square integrable, with period one and mean value 0.

LEMMA 5. *If*

$$\int_\alpha^\beta n^2 S^2(f, m, n, x)dx \ll (m + n)^a n^{2-a}\eta_2(n) \tag{19}$$

for $m \geqslant 0$, $n \geqslant 1$, *then*

$$S(f, 0, n, x) = o(1)$$

for almost all x *in* $[\alpha, \beta]$.

PROOF. This is a special case of Theorem 7 of [9], which like Theorem 4 is proved by a "dyadic method".

LEMMA 6. *Suppose that*

$$\int_0^1 (f(y) - f_n(y))^2 dy << \eta_3(n) \tag{20}$$

and

$$\int_\alpha^\beta n^2 S^2(f_n, m, n, x)\, dx << (m+n)^a n^{2-a} \eta_4(n). \tag{21}$$

Then $S(f, 0, n, x) = o(1)$ *for almost all* x *in* $[\alpha, \beta]$.

PROOF

$$\int_\alpha^\beta n^2 S^2(f, m, n, x)\, dx \leq 2 \int_\alpha^\beta n^2 S^2(f_n, m, n, x)\, dx$$

$$+ 2 \int_\alpha^\beta n^2 S^2(f - f_n, m, n, x)\, dx$$

$$<< (m+n)^a n^{2-a} \eta_4(n) + n \sum_{j=m+1}^{m+n} \int_\alpha^\beta (f - f_n)^2 (g_j(x))\, dx,$$

applying (20) and the Cauchy-Schwarz inequality. Now apply Lemma 3 with $p(x) = (f - f_n)^2(x) - \int_0^1 (f - f_n)^2 dx = (f - f_n)^2 - M_n$, say. Then $P \leq M_n$, and

$$\int_\alpha^\beta (f - f_n)^2 (g_j(x)\, dx \leq \left(\frac{2}{g_j'(\alpha)} + \beta - \alpha\right) M_n << \eta_3(n),$$

$$\int_\alpha^\beta n^2 S^2(f, m, n, x) dx << (m+n)^a n^{2-a} (\eta_3(n) + \eta_4(n))$$

and the lemma follows from Lemma 5.

LEMMA 7. *Let* $n \geq 1$, *and let* $t = t(n)$ *and* $v = v(n)$ *be positive integers and* $\gamma = \gamma(n) > 0$. *Then given* f *there is an* f_n *with the following properties.*

$$\int_0^1 (f(y) - f_n(y))^2 dy << r_v + v^4 \gamma^4, \tag{22}$$

$$\int_\alpha^\beta n^2 S^2(m, n, x)\, dx \tag{23}$$

$$<< n(m+n)^a (\log(n+1) \log(t+1) + t) + n^2 \gamma^{-2} r_t t^{-1}.$$

PROOF. Let $f_n(x) = (1/2\gamma) \int_{-\gamma}^{\gamma} f(x+y)\, dy$, then f_n has period 1 and mean value 0. Integrating term by term,

$$f_n(x) = \sum_{k=-\infty}^{\infty} C_k e^{2\pi i k x}, \quad C_k = \frac{\sin 2\pi k \gamma}{2\pi k \gamma} c_k \qquad (k = 1, 2, \ldots)$$

and $C_0 = 0$, $C_{-k} = \bar{C}_k$. By the Cauchy-Schwarz inequality,

$$\sum_{k=t+1}^{\infty} |C_k| \leq \frac{1}{2\pi\gamma} \left(\sum_{t+1}^{\infty} |c_k|^2 \sum_{t+1}^{\infty} \frac{1}{k^2} \right)^{1/2} << \gamma^{-1} r_t^{1/2} t^{-1/2}. \tag{24}$$

Note also that $\Sigma_{k=1}^{\infty} |C_k|^2 \leq r_0$. Now

$$\int_0^1 (f(y) - f_n(y))^2 dy = 2 \sum_{k=1}^{\infty} |c_k|^2 \left(1 - \frac{\sin 2\pi k\gamma}{2\pi k\gamma}\right)^2$$

$$\ll r_v + \sum_{k=1}^{v} |c_k|^2 k^4 \gamma^4 \ll r_v + v^4\gamma^4,$$

and

$$nS(f_n, m, n, x) = \sum_{k=-\infty}^{\infty} C_k \sum_{j=m+1}^{m+n} e(kg_j(x)),$$

$$|nS(f_n, m, n, x)| \leq 2 \sum_{k=1}^{t} |C_k| \left| \sum_{j=m+1}^{m+n} e(kg_j(x)) \right| + 2n \sum_{k=t+1}^{\infty} |C_k|$$

and by the Cauchy-Schwarz inequality and (24),

$$n^2 S^2(f_n, m, n, x) \ll \sum_{k=1}^{t} |C_k|^2 \sum_{k=1}^{t} \left| \sum_{j=m+1}^{m+n} e(kg_j(x)) \right|^2 + n^2\gamma^{-2} r_t t^{-1}.$$

Integrating and using Lemma 4, we obtain the following estimate, which implies the lemma.

$$\int_\alpha^\beta n^2 S^2(f_n, m, n, x) dx \ll \sum_{k=1}^{t} n(m+n)^a \left(1 + \frac{\log(n+1)}{k}\right) + n^2\gamma^{-2} r_t t^{-1}.$$

PROOF OF THEOREM 2. By Lemmas 6 and 7, it suffices to choose for $n = 1, 2, \ldots$ the numbers $v = v(n)$, $t = t(n)$ and $\gamma = \gamma(n)$ so that

$$r_v + v^4\gamma^4 \ll \eta_3(n), \tag{25}$$

(26) $n(m+n)^a(\log(n+1)\log(t+1) + t) + n^2\gamma^{-2} r_t t^{-1} \ll n^{2-a}(m+n)^a \eta_4(n)$.

Choose $t = [n^\lambda]$ where $0 < \lambda < 1 - a$; $\gamma = n^{-\mu}$ where $0 < \mu < \lambda/2$, and $v = [n^\theta]$, where $0 < \theta < \mu$. Then since $\Sigma \eta_1([n^\theta])/n$ converges by the integral test, (25) follows; (26) is true also. This completes the proof of Theorem 2.

4. Proof of Theorem 3. This depends on a general lemma whose proof is another variant of the dyadic method. The underlying idea is due to Schmidt [14].

LEMMA 8. *Let $E_1, E_2, \ldots$ be measurable sets in $[\alpha, \beta]$ with indicator functions $X_1, X_2, \ldots$; let $\psi(1), \psi(2)$, be numbers in $[0, 1]$ such that $\Psi(n) = \Sigma_{j=1}^{n} \psi(j) \to \infty$. Define $h_j(x) = X_j(x) - \psi(j)$. If*

$$\sum_{j=1}^{k} \left| \int_\alpha^\beta h_j h_k dx \right| \ll \psi(k) \tag{27}$$

then for every $\epsilon > 0$

$$\sum_{j=1}^{n} h_j(x) = O(\Psi^{1/2}(n) \log^{(3+\epsilon)/2} \Psi(n))$$

for almost all x in $[\alpha, \beta]$.

PROOF. This is a repetition almost verbatim of the proof of Theorem 3 of [12], starting at equation (3).

LEMMA 9. *Let* $J_1, J_2, \ldots$ *be intervals in* $[0, 1)$ *and let*

$$E_j = \{x \in [\alpha, \beta] : \{g_j(x)\} \in J_j\}.$$

Define h_j *as in Lemma* 8 *with* $\psi(j) = |J_j|$. *Then for* $j \geqslant 1, k \geqslant 1$,

$$\left|\int_\alpha^\beta h_j(x) h_k(x)\, dx\right| \leqslant |J_k| \left(\frac{6}{g_k'(\alpha)} + 4\int_\alpha^\beta \frac{g_j'(x)}{g_k'(x)}\, dx\right).$$

PROOF. This is a slight variant of Lemma 2 of [3].

PROOF OF THEOREM 3 (a). Let $h_j(x)$ be as in Lemma 9.

It suffices to show that (27) holds with $\psi(k) = |J_k|$.

But by Lemma 9 and (9),

$$(28)\quad \sum_{j=1}^{k} \left|\int_\alpha^\beta h_j(x) h_k(x)\, dx\right| \leqslant |J_k| \left(\frac{6k}{g_k'(\alpha)} + 4\int_\alpha^\beta \sum_{j=1}^{n} \frac{g_j'(x)}{g_k'(x)}\, dx\right) \ll |J_k|.$$

(It is obvious that $g_k'(\alpha) \gg k$.)

LEMMA 10. *Let* $h_j(x)$ *be defined as in Lemma* 8 *for* $j = 1, 2, \ldots$. *Assume instead of* (27) *that*

$$\int_\alpha^\beta \left(\sum_{j=1}^{n} h_j(x)\right)^2 dx \ll \Psi^2(n)\rho([\Psi(n)]),$$

then

$$(29)\qquad \sum_{j=1}^{n} h_j(x) = o(\Psi(n))$$

for almost all x in $[\alpha, \beta]$.

PROOF. There is a sequence of integers $1 \leqslant m_1 < m_2 < \cdots$ such that $m_{r+1}/m_r \to 1$ as $r \to \infty$ and

$$\sum_{r=1}^{\infty} \rho(m_r) < \infty.$$

For a neat proof of this see [5, p. 312]. Let $n_1 < n_2 < \cdots$ be integers such that $[\Psi(n_r)] = m_r$, then for every $\epsilon > 0$,

$$\left|\left\{x \in [\alpha, \beta] : \left|\sum_{j=1}^{n_r} h_j(x)\right| > \epsilon\Psi(n_r)\right\}\right| \ll \epsilon^{-2}\rho(m_r).$$

From this inequality we readily deduce that for almost all x in $[\alpha, \beta]$,

$$(30)\qquad \sum_{j=1}^{n_r} h_j(x) = o(\Psi(n_r)) \quad \text{as } r \to \infty.$$

In view of $\Psi(n_{r+1}) \sim \Psi(n_r)$, (30) implies (29) and Lemma 10 is proved.

PROOF OF THEOREM 3(b). Write $J_k = J_k^{(1)} \cup \cdots \cup J_k^{(t_k)}$ where the $J_k^{(r)}$ are disjoint intervals in [0, 1). Define E_k and h_k as in Lemma 9, then

$$h_k(x) = \sum_{r=1}^{t_k} h_k^{(r)}(x)$$

where $h_k^{(r)}$ is defined like h_k with $J_k^{(r)}$ replacing J_k. Thus

$$\int_\alpha^\beta h_j(x)h_k(x)\,dx = \sum_{r=1}^{t_j}\sum_{s=1}^{t_k}\int_\alpha^\beta h_j^{(r)}h_k^{(s)}\,dx$$
$$\leqslant t_j \sum_{s=1}^{t_k} |J_k^{(s)}| \left(\frac{6}{g_k'(\alpha)} + 4\int_\alpha^\beta \frac{g_j'(x)\,dx}{g_k'(x)}\right)$$

and

$$\sum_{k=1}^{n}\sum_{j=1}^{n}\left|\int_\alpha^\beta h_j(x)h_k(x)\,dx\right| << \Psi(n)\rho([\Psi(n)])\sum_{k=1}^{n}|J_k|$$
$$<< \Psi^2(n)\rho([\Psi(n)])$$

by the hypotheses on t_j and $\rho(n)$. Theorem 3(b) now follows on application of Lemma 10.

LEMMA 11. *Let* $0 < \lambda_1 < \lambda_2 < \cdots$ *be a sequence of integers. Let* J_1, $J_2, \ldots$ *be intervals in* [0, 1). *Define* h_j *as in Lemma* 9 *with* $g_j(x) = \lambda_j x$. *Then for* $j \geqslant 1, k \geqslant 1$,

$$\left|\int_0^1 h_j(x)h_k(x)\,dx\right| \leqslant \frac{2(\lambda_j, \lambda_k)}{\lambda_k}|J_k|. \tag{31}$$

PROOF. This is essentially the result of section 1 of [11]. For a different proof see Lemma 10 of [14] where Schmidt's argument easily adapts to prove our lemma.

Theorem 3(c) can now be proved in the same way as parts (a) and (b) by using (31) instead of (27). Lemma 8 can also be used to prove

THEOREM 6. *Let* $0 \leqslant \psi_r(j) \leqslant 1$ $(r = 1, \ldots, h, j = 1, 2, \ldots)$ *where* h *is a positive integer. Suppose* $N(n, x_1, \ldots, x_h, \theta_1, \ldots, \theta_h)$ *denotes the number of solutions of* $\{jx_r - \theta_r\} < \psi_r(j)$ *for* $r = 1, 2, \ldots, h$ *with* $1 \leqslant j \leqslant n$. *Let* $\epsilon > 0$, *then for almost all* $(x_1, \ldots, x_h, \theta_1, \ldots, \theta_h)$ *in the sense of Lebesgue measure in* R^{2h},

$$N(n, x_1, \ldots, x_h, \theta_1, \ldots, \theta_h) = \Psi(n) + O(\Psi^{1/2}(n)\log^{3/2+\epsilon}\Psi(n)).$$

Here $\Psi(n) = \Sigma_{j=1}^n \psi_1(j) \cdots \psi_h(j)$.

This is a quantitative version of a theorem of Cassels [2] that $N(n, x_1, \ldots, x_h, \theta_1, \ldots, \theta_h)$ is bounded or unbounded with $\Psi(n)$ almost everywhere. The proof is obtained by adjoining Lemma 8 to the calculations in [2], and may be left to the reader.

5. **Two examples.** We first give a new example of a sequence of integers $0 < \lambda_1 < \lambda_2 < \cdots$ satisfying (12). Several examples were given in [11]; in particular (12) holds if λ_j has $O(1)$ divisors.

Let $p_1, \ldots, p_t$ be a given finite set of primes and let (λ_j) be the sequence of integers $p_1^{x_1} \cdots p_t^{x_t}$ ($x_1 \geq 0, \ldots, x_t \geq 0$ integers) enumerated in increasing order. We fix $\lambda_k = p_1^{a_1} \cdots p_t^{a_t}$, and let $\lambda_j = p_1^{x_t} \cdots p_t^{x_t} \leq \lambda_k$.

For reasons of symmetry we need only sum $(\lambda_j, \lambda_k)\lambda_k^{-1}$ over those λ_j with

$$x_1 > a_1, \ldots, x_q > a_q, x_{q+1} \leq a_{q+1}, \ldots, x_t \leq a_t \tag{32}$$

where $0 \leq q < t$. Let $M = M(x_{q+1}, \ldots, x_t)$ be the number of lattice points $(x_1, \ldots, x_q)$ which satisfy (32) and

$$(x_1 - a_1)\log p_1 + \cdots + (x_q - a_q)\log p_q$$
$$\leq (a_{q+1} - x_{q+1})\log p_{q+1} + \cdots + (a_t - x_t)\log p_t.$$

We have

$$M \ll ((a_{q+1} - x_{q+1}) + \cdots + (a_t - x_t))^q$$
$$\ll (a_{q+1} - x_{q+1} + 1)^q \cdots (a_t - x_t + 1)^q$$

where the implied constant depends only on $p_1, \ldots, p_t$. But clearly, where λ_j takes all values permitted by (32),

$$\sum_j (\lambda_j, \lambda_k)\lambda_k^{-1} = \sum_{0 \leq x_r \leq a_r (r=q+1,\ldots,t)} M(x_{q+1}, \ldots, x_t) p_{q+1}^{x_{q+1}-a_{q+1}} \cdots p_t^{x_t - a_t}$$
$$\ll \prod_{r=q+1}^{t} \sum_{y=0}^{a_r} (y+1)^q p_r^{-y} \ll 1,$$

which establishes (12).

Our other example is a complement to Theorem 3.

Theorem 7. *If $0 < \lambda_1 < \lambda_2 < \cdots$ is a sequence of integers such that $\lambda_j | \lambda_{j+1}$ ($j \geq 1$) there is a finite union J_j of disjoint intervals in* $[0, 1)$ *for $j = 1, 2, \ldots$ such that $\Psi(n) = \sum_{j=1}^n |J_j| \longrightarrow \infty$, but the inequalities*

$$\{\lambda_j x\} \in J_j \tag{33}$$

have at most finitely many solutions j for almost all real x.

Proof. Write $n^{-1}A = \{x \in [0, 1): \{nx\} \in A\}$ if A is a measurable set in $[0, 1)$ and n is a positive integer. Notice that $|n^{-1}A| = |A|$.

Let $1 < a < 2$. Define $I_k = [0, 1/2(k+1)^a]$ $(k = 0, 1, \ldots)$. If $k^2 < j \leqslant (k+1)^2$ for some $k \geqslant 0$ define

$$J_j = (\lambda_{(k+1)^2}/\lambda_j)^{-1} I_k.$$

Then J_j is a finite union of disjoint intervals in $[0, 1)$ and

$$\sum_j |J_j| = \sum_k \frac{(k+1)^2 - k^2}{2(k+1)^a} = \infty.$$

On the other hand (33) has infinitely many solutions precisely when

$$\{\lambda_{(k+1)^2} x\} \in I_k$$

has infinitely many solutions k, and Theorem 7 now follows from the Borel-Cantelli lemma.

Suppose in Theorem 7 that $\lambda_j = 2^j$. Then if $k^2 < j \leqslant (k+1)^2$, J_j consists of $2^{(k+1)^2 - j} < 2^{2j^{1/2}}$ intervals, while

$$\Psi(j) >> \sum_{r \leqslant k} r^{1-a} >> k^{2-a} >> j^{(2-a)/2}.$$

Thus in the notation of Theorem 3,

$$t_j << e^{\Psi(j)^{1+\delta}} \tag{34}$$

where $\delta(a) \longrightarrow 0$ as $a \longrightarrow 1$. It would be interesting to close the gap between (34) and $t_j << \Psi(j)\rho([\Psi(j)])$.

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