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HOLOMORPHIC CONVEXITY OF COMPACT SETS IN ANALYTIC SPACES AND THE STRUCTURE OF ALGEBRAS OF HOLOMORPHIC GERMS

BY

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ABSTRACT. Let $(X, \mathcal{O}_X)$ be a reduced analytic space and let K be a compact, holomorphically convex subset of X. It is shown that analogs of Cartan's Theorems A and B are valid for coherent analytic sheaves on K. This result is applied to the study of the algebra of germs on K of functions holomorphic near K. In particular, characterizations of finitely generated ideals, prime ideals and homomorphisms are obtained.

1. **Introduction.** This paper deals with what might be called the "semi-local" theory of holomorphic functions: algebras of germs of holomorphic functions on a compact subset of an analytic space. More precisely, let $(X, \mathcal{O}_X)$ be a reduced analytic space for which the global holomorphic functions separate points, and let K be a compact subset of X. The space $\Gamma(K, \mathcal{O}_X)$ of sections is an algebra over C, with a natural projective limit topology. We may also view elements of $\Gamma(K, \mathcal{O}_X)$ as functions holomorphic near K, and we let $H_X(K)$ denote the closure of $\Gamma(K, \mathcal{O}_X)$ in $C(K)$; $H_X(K)$ is a Banach algebra, and the relationship between $\Gamma(K, \mathcal{O}_X)$ and $H_K(K)$, which we discuss in §2, allows the application of Banach algebra techniques, which prove useful in a number of places.

We say that K is holomorphically convex (in X) if every nonzero, continuous, complex-valued homomorphism of $\Gamma(K, \mathcal{O}_X)$ arises from evaluation at a point of K. Compact holomorphically convex sets share the most important properties of Stein spaces, in that analogs of Cartan's Theorems A and B are valid for them (Theorem 3.1). We also derive (Theorem 3.2) a strong converse to this result. These two theorems generalize to analytic spaces results obtained by F. R. Harvey and R. O. Wells, Jr. [15] for Stein manifolds. The techniques involved in the present paper are quite different from those employed by Harvey

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and Wells since, in this more general context, envelopes of holomorphy are unavailable. One effect of these theorems is that the analytic space X may be assumed, without loss of generality, to be a closed subvariety of some open set in $\mathbf{C}^n$.

In §4, we discuss the ideal theory of the ring $\Gamma(K, \mathcal{O}_X)$. Unlike the local rings ${}_x\mathcal{O}_X$, the ring $\Gamma(K, \mathcal{O}_X)$ need not be Noetherian, and one thrust of our investigation is the determination of those ideals which are finitely generated (Theorems 4.1–4.5). In particular, we obtain a form of the Nullstellensatz for finitely generated ideals (Theorem 4.2). As in any ring, the prime ideals in $\Gamma(K, \mathcal{O}_X)$ play a central role, and we derive a structure theorem for the prime ideals (Theorem 4.7). Among the applications of this result is an alternate proof (Theorem 4.10) of a result due to Y.-T. Siu [18] which characterizes those compact sets K for which the ring $\Gamma(K, \mathcal{O}_X)$ is Noetherian. We also give a number of examples which illustrate the possible pathologies in the non-Noetherian case.

In §5, we take up the study of algebra homomorphisms $T: \Gamma(K, \mathcal{O}_X) \longrightarrow \Gamma(L, \mathcal{O}_Y)$ where $(Y, \mathcal{O}_Y)$ is an analytic space and L is a compact subset of Y. In Theorem 5.1 and Corollaries 5.2 and 5.3, we show that all such homomorphisms arise from holomorphic maps; this generalizes some of the results obtained by the author in [22] and [23]. In case L is holomorphically convex and T is a surjection, we are able to obtain much more definitive results (Theorem 5.6).

Some of this work was announced in [24] and [25].

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2. **Preliminaries.** In this section we collect some definitions, notations and results which will be used later.

Let $(X, \mathcal{O}_X)$ be an analytic space (all analytic spaces are assumed to be reduced) and let K be a compact subset of X. The elements of $\Gamma(K, \mathcal{O}_X)$ are sections of the sheaf $\mathcal{O}_X$ over K, and we may identify them with germs on K of functions near K. We follow the common practice of not distinguishing between a function holomorphic near K and its germ on K. To avoid confusion in the only place it seems likely to occur, we will write ${}_xf$ for the germ of the function f at the point x in X, and $f(x)$ for its value at x. If U and V are open subsets of X with $K \subset U \subset V$, there are natural maps $\rho_{VU}: \Gamma(V, \mathcal{O}_X) \longrightarrow \Gamma(U, \mathcal{O}_X)$ and $\rho_U: \Gamma(U, \mathcal{O}_X) \longrightarrow \Gamma(K, \mathcal{O}_X)$; the family $\{\Gamma(U, \mathcal{O}_X); \rho_{VU}\}$, where U, V range over all open sets containing X, is a projective system and the projective limit is $\Gamma(K, \mathcal{O}_X)$. We equip $\Gamma(K, \mathcal{O}_X)$ with the finest locally convex topology which renders all the maps $\rho_U: \Gamma(U, \mathcal{O}_X) \longrightarrow \Gamma(K, \mathcal{O}_X)$ continuous.

By a result of Allan, Dales and McClure [26], an arbitrary subset E of

$\Gamma(K, \mathcal{O}_X)$ (convex or not) is closed in this topology if and only if $\rho_U^{-1}(E)$ is closed in $\Gamma(U, \mathcal{O}_X)$ for each open set U containing K. With this topology, $\Gamma(K, \mathcal{O}_X)$ is an algebra over $\mathbf{C}$ and a generalized LF space; we refer to [9] for further information about this topology.

We use $H_X(K)$ to denote the uniform closure in $C(K)$ of $\Gamma(K, \mathcal{O}_X)$. Then $H_X(K)$ is a commutative Banach algebra with unit. By the spectrum of $H_X(K)$ (respectively $\Gamma(K, \mathcal{O}_X)$) we mean the space Sp $H_X(K)$ (respectively Sp $\Gamma(K, \mathcal{O}_X)$) of continuous, nonzero homomorphisms of $H_X(K)$ into $\mathbf{C}$ (respectively, $\Gamma(K, \mathcal{O}_C)$ into $\mathbf{C}$), equipped with the Gelfand topology. We denote the Gelfand transform of an element f by $\hat{f}$. The following result summarizes some material from [15], [22] and [23].

Proposition 2.1. *Every homomorphism of* $\Gamma(K, \mathcal{O}_X)$ *into* $\mathbf{C}$ *is continuous. The natural map* Sp $H_X(K) \longrightarrow$ Sp $\Gamma(K, \mathcal{O}_X)$ *is a homeomorphism onto. In particular,* Sp $\Gamma(K, \mathcal{O}_X)$ *is a compact Hausdorff space.*

The chief significance of Proposition 2.1 is that it gives us access to the machinery of uniform algebras. For general information about uniform algebras we refer to the books by Gamelin [11] and Stout [19]. The book by Gunning and Rossi [14] and the lecture notes by Narasimhan [17] are good references for general information about holomorphic functions and analytic spaces. For the little bit of ring theory that is needed here, we refer to Nagata [16].

We say that the analytic space $(X, \mathcal{O}_X)$ is *separated* if for each x, y in X with $x \neq y$ we can find a function f in $\Gamma(X, \mathcal{O}_X)$ such that $f(x) \neq f(y)$. If $(X, \mathcal{O}_X)$ is separated then the evaluation map $K \longrightarrow$ Sp $\Gamma(K, \mathcal{O}_X)$ is a homeomorphism (it is always continuous); in this case we will suppress the mapping and simply view K as a subset of Sp $\Gamma(K, \mathcal{O}_X)$. The set K is said to be *holomorphically convex in* X if $K =$ Sp $\Gamma(K, \mathcal{O}_X)$. If K has a neighborhood basis in X consisting of open Stein subspaces, then it is easily seen to be holomorphically convex, but an example due to Bjork [5] shows that the converse need not hold. Some further information about holomorphically convex sets in Stein manifolds can be found in [15].

The following result gives a useful criterion for holomorphic convexity and illustrates the use of Banach algebra techniques.

Proposition 2.2. *Let* X *be a closed subvariety of the open set* Ω *in* $\mathbf{C}^n$, *let* K *be a compact subset of* X *and let* ζ: Sp $\Gamma(K, \mathcal{O}_X) \longrightarrow \mathbf{C}^n$ *be given by* $\zeta(\varphi) = (\varphi(z_1), \ldots, \varphi(z_n))$. *If the range of* ζ *is a subset of* K, *then* K *is holomorphically convex.*

Proof. Observe first that $\zeta(x) = x$ for each x in K. Thus it suffices to show that ζ is one-to-one. To this end, let φ be in Sp $\Gamma(K, \mathcal{O}_X)$ and let f be in

$\Gamma(K, \mathcal{O}_X)$. Since Sp $\Gamma(K, \mathcal{O}_X) = $ Sp $H_X(K)$, ζ is actually a joint spectrum map and Allan's generalization of the usual functional calculus [1], [2] shows that there is an element g in $H_X(K)$ for which $\hat{g} = f \circ \zeta$. On the other hand, it is clear that $\hat{f}|K = f = \hat{g}\,|\,K$, and that K contains the Silov boundary of $H_X(K)$, so that in fact $\hat{f} = \hat{g}$ on Sp $\Gamma(K, \mathcal{O}_X)$. Thus, $\hat{f}(\varphi) = \varphi(f) = f \circ \zeta(\varphi) = \hat{f}(\zeta(\varphi))$. Since the Gelfand transforms of functions in $\Gamma(K, \mathcal{O}_X)$ separate the points of Sp $\Gamma(K, \mathcal{O}_X)$, we conclude that $\varphi = \zeta(\varphi)$ is a point of K, which completes the proof.

Note that the above argument remains valid if we replace $\Gamma(K, \mathcal{O}_X)$ by any subalgebra which contains the coordinate functions. Thus, Proposition 2.2 is a generalization of Theorem 3.7 of [21].

3. **Compact holomorphically convex sets.** In this section we establish the properties of compact holomorphically convex sets which are vital in succeeding sections. Our basic result is the following theorem, which shows that compact holomorphically convex sets enjoy a fundamental property of Stein spaces.

THEOREM 3.1. *Let $(X, \mathcal{O}_X)$ be a separated analytic space and let K be a compact, holomorphically convex subset of X. If $\mathcal{F}$ is a coherent analytic sheaf on K then*

(A) ${}_x\mathcal{F}$ *is generated by* $\Gamma(K, \mathcal{F})$ *as an* ${}_x\mathcal{O}_X$*-module for each x in K;*
(B) $H^q(K, \mathcal{F}) = 0$ *for all* $q \geqslant 1$.

PROOF. We begin by developing some machinery which will allow us to transfer both assertions into $\mathbf{C}^n$. Since K is compact, only finitely many of the irreducible branches of X meet K. As the assertions depend only on behavior near K, there is no loss in assuming that X is finite dimensional. By [14, Corollary C8, p. 223], there are functions $f_1, \ldots, f_N$ in $\Gamma(X, \mathcal{O}_X)$ such that the map $F = (f_1, \ldots, f_N): X \longrightarrow \mathbf{C}^N$ is one-to-one.

Let $\mathcal{F}$ be a coherent analytic sheaf on K; by a theorem of Cartan [8], $\mathcal{F}$ has an extension to a coherent analytic sheaf (which we continue to denote by $\mathcal{F}$) on an open set U_0 containing K. Let U_1 be any open, relatively compact subset of U_0 which contains K (U_1 will be specified later); then $F\,|\,U_1$ is a homeomorphism so that $W_1 = F(U_1)$ is a locally compact subset of $\mathbf{C}^N$. Hence there is an open subset Ω of $\mathbf{C}^N$ such that W_1 is a closed subset of Ω. Thus, $F\,|\,U_1\colon U_1 \longrightarrow \Omega$ is a proper mapping and W_1 is a closed subvariety of Ω.

Since Sp $\Gamma(K, \mathcal{O}_X) = $ Sp $H_X(K) = K$, $F(K)$ is the joint spectrum of the elements $f_1, \ldots, f_N$ in $H_X(K)$. By a fundamental lemma of Arens and Calderón [3], there are functions $f_{N+1}, \ldots, f_M$ in $H_X(K)$ and a polynomial polyhedron $P \subset \mathbf{C}^M$ such that $G(K) \subset P$ and $\pi(P) \subset \Omega$, where $G = (f_1, \ldots, f_M)$ and

$\pi\colon \mathbf{C}^M \longrightarrow \mathbf{C}^N$ is the projection on the first N coordinates. Since $\Gamma(K, \mathcal{O}_X)$ is dense in $H_X(K)$, we can in fact choose the functions $f_{N+1}, \ldots, f_M$ to be holomorphic in some neighborhood U_2 of K, and we may assume that $U_2 \subset U_1$. Set $W_2 = F(U_2)$, $W_3 = G(U_2)$. As above, we see that W_2 and W_3 are closed subvarieties of open subsets of $\mathbf{C}^N$ and $\mathbf{C}^M$, respectively. Observe that π is a holomorphic homeomorphism of W_3 onto W_2, but need not be a biholomorphism.

Let $\mathcal{E} = F_*\mathcal{F}$, the direct image sheaf on W_1. By Grauert's Direct Image Theorem [12], $\mathcal{E}$ is a coherent analytic sheaf. Set $V = \pi^{-1}(W_1) \cap P$, so that V is a closed subvariety of P, and let $\pi^{-1}\mathcal{E}$ be the inverse image sheaf on V. Let $\mathcal{A}$ be the sheaf on V associated with the presheaf

$$\mathcal{A}_0(Q) = \pi^{-1}\mathcal{E}(Q) \otimes \mathcal{O}_V(Q)$$

where the tensor products are taken as $\pi^{-1}\mathcal{O}_{W_1}$-modules. The sheaf $\mathcal{A}$ is the so-called "analytic pull-back" and, by a theorem of Grauert and Remmert [13], is a coherent analytic sheaf.

Since the maps $F \mid U_2$, G and $\pi \mid W_3$ are all homeomorphisms, we see that $(\pi|W_3)^{-1}\mathcal{E} = G_*\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_{W_3}$-modules. We define a sheaf map $\psi\colon \mathcal{A} \mid W_3 \longrightarrow (\pi|W_3)^{-1}\mathcal{E}$ as follows. If y is in W_3 and α is in ${}_y(\mathcal{A}\,|W_3)$ then since ${}_y\mathcal{A} = {}_y(\pi^{-1}\mathcal{E}) \otimes {}_y\mathcal{O}_V$, α has a representation of the form $\alpha = \Sigma[(\beta_i \otimes \gamma_i)|W_3]$ where $\beta_i \in {}_y(\pi^{-1}\mathcal{E})$ and $\gamma_i \in {}_y\mathcal{O}_V$. Note that $\pi^{-1}\mathcal{E} \mid W_3 = (\pi|W_3)^{-1}\mathcal{E}$. Set $\psi(\alpha) = \Sigma\,(\gamma_i|W_3)(\beta_i|W_3)$. It is easy to see that ψ is a well-defined homomorphism of sheaves of $\mathcal{O}_V$-modules. If Q is an open subset of W_3 and $h \in (G_*\mathcal{F})(Q)$, set $\bar{h} = h \circ (\pi|W_3)^{-1} \circ \pi$ and note that $\bar{h}$ is a section of the sheaf $\pi^{-1}\mathcal{E}$ over the open set $\pi^{-1}(\pi(Q)) \cap P$ in V. A direct computation shows that $\psi(\bar{h} \otimes 1) = h$, so that ψ is a surjection.

We are now ready to prove (A). Let x be a point of K and set $y = G(x)$. Since $\mathcal{A}$ is a coherent analytic sheaf and V is a Stein space, Cartan's Theorem A implies that ${}_y\mathcal{A}$ is generated by global sections (as an $\mathcal{O}_V$-module). Hence ${}_y(G_*\mathcal{F})$ is generated, as an $\mathcal{O}_V$-module, by sections in $\Gamma(G(K), G_*\mathcal{F})$, and ${}_x\mathcal{F}$ is generated as a $G^{-1}\mathcal{O}_V$-module (and a fortiori as an $\mathcal{O}_X$-module) by sections in $\Gamma(K, \mathcal{F})$.

In order to establish (B), we work with Čech cohomology, following the usual notation and terminology, as set forth in [14], for example. Let $\mathcal{U}$ be an open cover of K (which we may assume to be finite) with nerve $\mathcal{N}(\mathcal{U})$, and let τ be an element of $C^q(\mathcal{N}(\mathcal{U}), \mathcal{F})$, the family of q-cochains of $\mathcal{F}$ relative to $\mathcal{N}(\mathcal{U})$. If τ is a cocycle, we wish to establish the existence of a refinement $\mathcal{V}$ of $\mathcal{U}$ and refining map $\sigma\colon \mathcal{V} \longrightarrow \mathcal{U}$ such that $\sigma^*\tau$, which is an element of $C^q(\mathcal{N}(\mathcal{V}), \mathcal{F})$, is actually a coboundary.

Observe first of all that there is no loss in assuming that $\mathcal{U} = \{Q_1, \ldots, Q_j\}$

is actually a family of relatively compact open subsets of U_0; set $U_1 = \bigcup Q_i$. Then $\mathcal{W} = \{F(Q_1), \ldots, F(Q_j)\}$ is an open cover of W_1 and $\mathcal{W}' = \{\pi^{-1}F(Q_1) \cap P, \ldots, \pi^{-1}F(Q_j) \cap P\}$ is an open cover of V. Since a cochain is just an indexed family of sections of a sheaf over certain intersections of elements of the cover, there are induced cochains $F_*\tau$ in $C^q(\mathcal{N}(\mathcal{W}), \mathcal{E})$, $\pi^{-1}F_*\tau$ in $C^q(\mathcal{N}(\mathcal{W}'), \pi^{-1}\mathcal{E})$ and $\pi^{-1}F_*\tau \otimes 1$ in $C^q(\mathcal{N}(\mathcal{W}'), \mathcal{A})$. Direct computations verify that these cochains are actually cocycles. Since $\mathcal{A}$ is a coherent analytic sheaf on V, Cartan's Theorem B implies the existence of a refinement $\mathcal{W}'' = \{H_1, H_2, \ldots, H_i\}$ of $\mathcal{W}''$, and a refining map σ_0: $\mathcal{W}'' \longrightarrow \mathcal{W}'$ such that $\alpha = \sigma_0^*[(\pi^{-1}F_*\tau) \otimes 1]$ is a coboundary. Let $\mathcal{V}' = \{H_1 \cap W_3, \ldots, H_i \cap W_3\}$, and observe that $(\alpha|W_3) \in C^q(\mathcal{N}(\mathcal{V}'), \mathcal{A}|W_3)$ and is a coboundary. The homomorphism ψ: $\mathcal{A}|W_3 \longrightarrow (\pi|W_3)^{-1}\mathcal{E}$ induces a homomorphism

$$\psi_*: C^q(\mathcal{N}(\mathcal{V}'), \mathcal{A}|W_3) \longrightarrow C^q(\mathcal{N}(\mathcal{V}'), (\pi|W_3)^{-1}\mathcal{E}),$$

which may be seen to preserve cocycles and coboundaries; hence $\psi_*(\alpha|W_3)$ is a coboundary.

Let $\mathcal{V} = \{G^{-1}(H_1 \cap W_3), \ldots, G^{-1}(H_i \cap W_3)\}$; evidently, $\mathcal{V}$ is an open cover of K which refines $\mathcal{U}$, and the refining map σ_0 induces a refining map σ: $\mathcal{V} \longrightarrow \mathcal{U}$. Since G is a homeomorphism and $G_*\mathcal{F} = (\pi|W_3)^{-1}\mathcal{E}$, we have the induced homomorphism

$$G_*^{-1}: C^q(\mathcal{N}(\mathcal{V}'), (\pi|W_3)^{-1}\mathcal{E}) \longrightarrow C^q(\mathcal{N}(\mathcal{V}), \mathcal{F})$$

which again preserves cocycles and coboundaries. A direct computation now shows that $G_*^{-1}[\psi_*(\alpha|W_3)] = \sigma^*\tau$, so that $\sigma^*\tau$ is a coboundary. This establishes (B).

The proof of Theorem 3.1 which we have given is completely different, and much more complicated, than that given by Harvey and Wells [15] for compact holomorphically convex subsets of a Stein manifold. This is because Harvey and Wells were able to exploit the fact that an open subset of a Stein manifold has an envelope of holomorphy which is again a Stein manifold. Since an open subset of an analytic space (even a Stein space) need not have a nice envelope of holomorphy, we, of necessity, were forced to use other techniques.

The following is a strong converse to Theorem 3.1. Note that we do not need to assume anything about the supply of functions in $\Gamma(X, \mathcal{O}_X)$.

THEOREM 3.2. *Let $(X, \mathcal{O}_X)$ be an analytic space and let K be a compact subset of X. Assume that $H^1(K, \mathcal{F}) = 0$ for all coherent analytic subsheaves $\mathcal{F}$ of $(\mathcal{O}_X|K)^M$, for each M. Then there is an open set U, containing K, which is biholomorphically equivalent with a closed subvariety of an open set in some $\mathbf{C}^N$. The set K is holomorphically convex in U.*

PROOF. As in the proof of Theorem 20 [14, p. 246], it follows that

(i) for each x, y in K with $x \neq y$, there is a function f in $\Gamma(K, \mathcal{O}_X)$ such that $f(x) \neq f(y)$ (i.e., $\Gamma(K, \mathcal{O}_X)$ separates the points of K);

(ii) for each x in K there are functions $f_1, \ldots, f_n$ in $\Gamma(K, \mathcal{O}_X)$ such that $(f_1, \ldots, f_n)$ maps a neighborhood of x biholomorphically onto a closed subvariety of some polydisk in $\mathbf{C}^n$ (i.e., $\Gamma(K, \mathcal{O}_X)$ provides coordinates on K).

By utilizing the compactness of K and $K \times K$, we can find a finite family $\{g_1, g_2, \ldots, g_N\}$ in $\Gamma(K, \mathcal{O}_X)$ which separates points and provides coordinates on K. Hence there is an open set U' containing K, on which the functions $g_1, \ldots, g_N$ are analytic, separate points and provide coordinates. Thus, if U is open, relatively compact in U' and contains K, $(g_1, \ldots, g_N)$ maps U biholomorphically onto a closed subvariety of some open set in $\mathbf{C}^N$.

In order to show that K is holomorphically convex in U, observe first that the map $K \longrightarrow \operatorname{Sp} \Gamma(K, \mathcal{O}_X)$ is injective. If φ is in $\operatorname{Sp} \Gamma(K, \mathcal{O}_X)$ and not in the range of this map, then for each x in K there is a function h in the kernel of φ such that $h(x) \neq 0$. A compactness argument enables us to find functions $h_1, \ldots, h_M$ in the kernel of φ with no common zero on K. Let μ: $(\mathcal{O}_X|K)^M \longrightarrow (\mathcal{O}_X|K)$ be the sheaf homomorphism defined by

$$\mu(\alpha_1, \ldots, \alpha_M) = \sum h_i \alpha_i.$$

Then μ is surjective; since $H^1(K, \ker \mu) = 0$ by assumption, it follows that μ_*: $\Gamma(K, (\mathcal{O}_X|K)^M) \longrightarrow \Gamma(K, (\mathcal{O}_X|K))$ is also surjective. Hence there are functions $h'_1, \ldots, h'_M$ in $\Gamma(K, \mathcal{O}_X)$ such that $\Sigma h_i h'_i = 1$. But this contradicts the fact that the kernel of φ is a proper ideal, and this contradiction completes the proof.

As a consequence of Theorems 3.1 and 3.2 we see that, in the study of compact holomorphically convex sets and the algebra $\Gamma(K, \mathcal{O}_X)$, there will be no loss of generality if we assume X to be a closed subvariety of an open subset of some $\mathbf{C}^N$. This assumption will occasionally be convenient. When we speak of a compact holomorphically convex set in an analytic space which is not assumed to be separated, we simply mean a compact set which satisfies the hypotheses of Theorem 3.2, and hence has a neighborhood which is separated, and in which it is holomorphically convex in the sense we have previously used.

We now proceed to draw several consequences of Theorems 3.1 and 3.2. The first one follows by the argument used in the last paragraph of the proof of Theorem 3.2.

COROLLARY 3.3. *If K is a compact holomorphically convex subset of X, then the maximal ideals of $\Gamma(K, \mathcal{O}_X)$ are precisely the sets $M_X = \{f$ in $\Gamma(K, \mathcal{O}_X)$: $f(x) = 0\}$ for some x in K.*

COROLLARY 3.4. *If K is a compact subset of X, then K is holomorphically convex if and only if each connected component of K is holomorphically convex.*

PROOF. Assume that K is holomorphically convex. If K_0 is a connected component of K and $\mathcal{F}$ is a coherent analytic sheaf on K_0, then the theorem of Cartan [8] cited previously allows us to extend $\mathcal{F}$ to a coherent analytic sheaf on some open set U containing K_0. Let U' be any open set in X with $K_0 \subset U' \subset U$, and such that the boundary of U' misses K. Let $\mathcal{F}'$ be the sheaf which is $\mathcal{F}$ on U' and 0 on $X - \bar{U}'$. Then $\mathcal{F}'$ is a coherent analytic sheaf on K so that $H^1(K, \mathcal{F}') = 0$. On the other hand $H^1(K, \mathcal{F}') = H^1(K \cap U', \mathcal{F})$, and U' may be chosen to be an arbitrarily small neighborhood of K_0. Since cohomology commutes with inverse limits [6], it follows that $H^1(K_0, \mathcal{F}) = 0$, as desired.

Conversely, assume that each connected component of K is holomorphically convex and let $\mathcal{F}$ be a coherent analytic sheaf on K. Let K^* be the quotient of K by the connectedness relation and $\varphi: K \longrightarrow K^*$ the natural map. Since $\varphi^{-1}(p)$ is a connected component of K for each p in K^*, $H^q(\varphi^{-1}(p), \mathcal{F}) = 0$ for all $q \geqslant 1$. Then by the Vietoris-Begle Theorem [6, II 11.1, p. 54] $H^q(K, \mathcal{F}) \cong H^q(K^*, \varphi_* \mathcal{F})$ for all $q \geqslant 1$. But K^* is totally disconnected and compact, so that every sheaf on K^* is soft. Hence $H^q(K, \mathcal{F}) \cong H^q(K^*, \varphi_* \mathcal{F}) = 0$ for all $q \geqslant 1$, which completes the proof.

THEOREM 3.5. *Let K be a compact holomorphically convex subset of X, let U be an open set containing K and let V be a closed subvariety of U which meets K. Then $K \cap V$ is a holomorphically convex subset of X and of V, and the natural map $\rho: \Gamma(K, \mathcal{O}_X) \longrightarrow \Gamma(K \cap V, \mathcal{O}_V)$ is onto.*

PROOF. It will be convenient to make use of the remark following Theorem 4.2 (which of course does not depend on the present theorem), and to assume that X is a closed subvariety of the open set $\Omega \subset \mathbf{C}^N$; as was previously noted, this involves no loss of generality. The natural map $\sigma: \Gamma(K, \mathcal{O}_X) \longrightarrow \Gamma(K \cap V, \mathcal{O}_X)$ has an adjoint $\sigma^*: \operatorname{Sp} \Gamma(K \cap V, \mathcal{O}_X) \longrightarrow \operatorname{Sp} \Gamma(K, \mathcal{O}_X)$. Since $\operatorname{Sp} \Gamma(K, \mathcal{O}_X) = K$, it is easy to see that $\sigma^*(\varphi) = (\varphi(z_1), \ldots, \varphi(z_n))$ for each φ in $\operatorname{Sp} \Gamma(K \cap V, \mathcal{O}_X)$. If $\sigma^*(\varphi) = x$ is in K but not in V, then the remark following Theorem 4.2 provides a function in f in $\Gamma(K, \mathcal{O}_X)$ for which $f|V \equiv 0$ but $f(x) \neq 0$. Arguing as in Proposition 2.2, we see that $\hat{f}(\varphi) = f \circ \sigma^*(\varphi) \neq 0$ but, that $f \equiv 0$ on $K \cap V$, which is absurd. Hence the range of σ^* is a subset of $K \cap V$, so that $K \cap V$ is indeed a holomorphically convex subset of X. That $K \cap V$ is holomorphically convex in V follows in precisely the same way.

If $\mathcal{I}(V)$ is the ideal sheaf of V, then the following sequence is exact:

$$0 \longrightarrow \mathcal{I}(V) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X/\mathcal{I}(V) \longrightarrow 0.$$

Since $\mathcal{I}(V)$ is coherent, $H^1(K, \mathcal{I}(V)) = 0$ so that the section map $\Gamma(K, \mathcal{O}_X) \longrightarrow \Gamma(K, \mathcal{O}_X/\mathcal{I}(V))$ is onto; but $\Gamma(K, \mathcal{O}_X/\mathcal{I}(V)) = \Gamma(K \cap V, \mathcal{O}_V)$, so that ρ is onto.

4. **Ideal theory.** As was mentioned in the introduction, the ring $\Gamma(K, \mathcal{O}_X)$ need not be Noetherian. The simplest example of this phenomenon arises upon taking $X = \mathbf{C}$, $K = \{0, 1, \frac{1}{2}, \ldots\}$ and letting I be the ideal of all functions in $\Gamma(K, \mathcal{O}_X)$ which vanish in some neighborhood of the origin; it is easy to see that I is not finitely generated. It is therefore of primary importance to identify the finitely generated ideals, and this we now proceed to do.

Throughout this section, we assume that $(X, \mathcal{O}_X)$ is a separated analytic space and that K is a compact holomorphically convex subset of X. If E is a nonempty subset of $\Gamma(K, \mathcal{O}_X)$ and $x \in K$, we let E_x be the ideal in ${}_x\mathcal{O}_X$ generated by $\{{}_xf: f \in E\}$, and set $\mathcal{S}(E) = \bigcup E_x$. It is easy to see that $\mathcal{S}(E)$ is an ideal subsheaf of $\mathcal{O}_X|K$. Our basic result is the following.

THEOREM 4.1. *If E is a nonempty subset of $\Gamma(K, \mathcal{O}_X)$ then $\Gamma(K, \mathcal{S}(E))$ is the ideal in $\Gamma(K, \mathcal{O}_X)$ which is generated by E. The restriction of the operator $\mathcal{S}$ to the family of finitely generated ideals in $\Gamma(K, \mathcal{O}_X)$ is a bijection onto the family of coherent ideal subsheaves of $\mathcal{O}_X|K$; its inverse is the map $\mathcal{J} \longrightarrow \Gamma(K, \mathcal{J})$.*

PROOF. If J is the ideal in $\Gamma(K, \mathcal{O}_X)$ generated by E, it is clear that $\mathcal{S}(J) = \mathcal{S}(E)$, so we need to show that $J = \Gamma(K, \mathcal{S}(J))$. Let g be in $\Gamma(K, \mathcal{S}(J))$. For each x in K, there are elements $f_1, f_2, \ldots, f_k$ in J and $\gamma_1, \gamma_2, \ldots, \gamma_k$ in ${}_x\mathcal{O}_X$ such that ${}_xg = \Sigma f_i\gamma_i$. Compactness of K then allows us to choose a finite subset $H = \{h_1, h_2, \ldots, h_n\}$ of J such that ${}_yg \in H_y$ for each y in K. Define a sheaf homomorphism μ: $(\mathcal{O}_X|K)^n \longrightarrow (\mathcal{O}_X|K)$ by setting $\mu(\alpha_1, \ldots, \alpha_n) = \Sigma h_i\alpha_i$; then we have the short exact sequence $0 \longrightarrow \ker \mu \longrightarrow (\mathcal{O}_X|K)^n \longrightarrow \mathcal{S}(H) \longrightarrow 0$. Since $H^1(K, \ker \mu) = 0$, the induced map μ^*: $\Gamma(K, (\mathcal{O}_X|K)^n) \longrightarrow \Gamma(K, \mathcal{S}(H))$ is surjective. Since $g \in \Gamma(K, \mathcal{S}(H))$, we can find functions $f_1, \ldots, f_n$ in $\Gamma(K, \mathcal{O}_X)$ such that $g = \Sigma f_ih_i$, so that $g \in J$. Thus $\Gamma(K, \mathcal{S}(J)) \subset J$, and the reverse containment is clear.

If J is finitely generated, with generators $h_1, \ldots, h_n$, then $\mathcal{S}(J) = \mathcal{S}(\{h_1, \ldots, h_n\})$ is the image of a sheaf homomorphism μ: $(\mathcal{O}_X|K)^n \longrightarrow (\mathcal{O}_X|K)$ as above, so that $\mathcal{S}(J)$ is coherent. Conversely, if $\mathcal{J}$ is a coherent ideal subsheaf of $(\mathcal{O}_X|K)$, set $J = \Gamma(K, \mathcal{J})$. If x is in K, then Theorem 3.1(A), combined with the definition of coherence, shows that there are elements $f_1, \ldots, f_k$ in J such that ${}_y\mathcal{J}$ is generated by ${}_yf_1, \ldots, {}_yf_k$ for all y in K near x. As before, compactness of K enables us to find functions $h_1, \ldots, h_n$ in J such that ${}_y\mathcal{J}$ is generated by ${}_yh_1, \ldots, {}_yh_n$ for each y in K. Since $\mathcal{S}(J)$ is certainly a subsheaf of $\mathcal{J}$, it follows that $S(\mathcal{J}) = \mathcal{J}$, and an argument identical with that of the first paragraph show that $h_1, \ldots, h_n$ generate J, which completes the proof.

If R is a commutative ring and I is an ideal of R, recall that the *radical of* I is the ideal

$$\text{rad}(I) = \{r\colon r^n \in I, \text{ some } n = 1, 2, \ldots\}.$$

The ideal I is *radical* if $\text{rad}(I) = I$. By virtue of the Nullstellensatz, the radical ideals in ${}_x\mathcal{O}_X$ correspond to the germs of varieties through X. In the ring $\Gamma(K, \mathcal{O}_X)$, we shall show that it is the finitely generated radical ideals which correspond to varieties.

If U is an open subset of X which contains K and V is a closed subvariety of U, then by the *ideal of* V we mean

$$\text{id}(V) = \{f \text{ in } \Gamma(K, \mathcal{O}_X)\colon f|V \equiv 0 \text{ near } K\}.$$

If $\mathcal{I}(V)$ is the ideal sheaf of V, observe that $\text{id}(V) = \Gamma(K, \mathcal{I}(V))$. We frequently suppress mention of U and simply refer to V as a variety near K.

The following result is a Nullstellensatz for finitely generated ideals.

THEOREM 4.2. *The ideal* $J \subset \Gamma(K, \mathcal{O}_X)$ *is the ideal of a variety near* K *if and only if* J *is finitely generated and radical.*

PROOF. If V is a variety near K then $\text{id}(V) = \Gamma(K, \mathcal{I}(V))$ is certainly radical, and is finitely generated by Theorem 4.1. Conversely, if J is radical and generated by functions $f_1, \ldots, f_m$ which are holomorphic on the open set U containing K, set $V = \{u \text{ in } U\colon f_i(u) = 0, i = 1, 2, \ldots, m\}$. If $g \in \text{id}(V)$ and $x \in K$, the usual Nullstellensatz implies that there is an integer n such that ${}_xg^n \in J_x$. Compactness of K allows us to conclude that there is an integer N such that ${}_yg^N \in J_y$ for each y in K. In view of Theorem 4.1, $g^N \in J$; since J is radical, it follows that $g \in J$ so that $\text{id}(V) \subset J$, and the reverse containment is obvious.

We remark that if $g_1, \ldots, g_n$ are generators for $\text{id}(V)$, then the set of common zeros of $g_1, \ldots, g_n$ coincides with V in a neighborhood of K (since ${}_xg_1, \ldots, {}_xg_n$ generates ${}_x\mathcal{I}(V)$ for each x in K). As a special case of Theorem 4.2, note that every maximal ideal is finitely generated.

The proof of the following corollary involves only a slight modification of the proof of Theorem 4.2, and is omitted.

COROLLARY 4.3. *The radical of a finitely generated ideal in* $\Gamma(K, \mathcal{O}_X)$ *is again finitely generated.*

If the converse of Corollary 4.3 were true, it would simplify considerably the search for finitely generated ideals. To see that it is not true, set $X = \mathbf{C}^2$,

$K = \{(0, 0)\} \cup \{(1/n, 0): n = 1, 2, \ldots\}$ and let J be the ideal of all functions f in $\Gamma(K, \mathcal{O}_X)$ such that f/w is holomorphic near K and f/w^2 is holomorphic near $(0, 0)$. It is easy to see that J is not finitely generated while rad(J) is the principal ideal generated by w. The converse of Corollary 4.3 does hold, however, for an important class of ideals.

COROLLARY 4.4. *Let J be an ideal in $\Gamma(K, \mathcal{O}_X)$ which is contained in a unique maximal ideal. If* rad(J) *is finitely generated, so is J.*

PROOF. Assume that J is contained in the maximal ideal $\{f: f(x) = 0\}$ and that rad(J) = id(V); clearly $V \cap K = \{x\}$. If we choose $g_1, \ldots, g_n$ in J which generate J_x then these functions vanish at x; since rad(J_x) is the ideal in ${}_x\mathcal{O}_X$ which corresponds to V, the Nullstellensatz shows that $g_1, \ldots, g_n$ have no other common zero in K close to x. Hence for y in K near x, J_y is generated by ${}_yg_1, \ldots, {}_yg_n$, and we see that $\mathcal{S}(J)$ is coherent; an application of Theorem 4.1 completes the proof.

PROPOSITION 4.5. *If U is an open subset of X which contains K and A is a nonempty subset of $\Gamma(U, \mathcal{O}_X)$ then the ideal in $\Gamma(K, \mathcal{O}_X)$ generated by A is finitely generated.*

PROOF. As we have observed, there is no loss in assuming that X is a closed subvariety of the open set $\Omega \subset \mathbf{C}^n$. For each x in K choose a polydisk P_x centered at x and contained in Ω for which $P_x \cap X \subset U$. Every function in A then has an extension to P_x; let $\widetilde{A}$ be the totality of all such extensions. Let M be the ideal in ${}_x\mathcal{O}_{\mathbf{C}^n}$ generated by $\widetilde{A}$; M is then generated by a finite number of functions $\widetilde{a}_1, \ldots, \widetilde{a}_k$ in A. An application of Theorem D2 in [14, p. 82] gives a closed polydisk D_x centered about x and contained in P_x, such that each $\widetilde{a}$ in $\widetilde{A}$ has a representation $\widetilde{a} = \Sigma \beta_i \widetilde{a}_i$ valid near D_x. If we extract a finite subcover of the cover $\{D_x\}$ of K, we may see that there are functions $a_1, \ldots, a_m$ in A such that ${}_ya_1, \ldots, {}_ya_m$ generate A_y for each $y \in K$. It follows from Theorem 4.1 that $a_1, \ldots, a_m$ generate the same ideal in $\Gamma(K, \mathcal{O}_X)$ as A.

A similar idea yields the following topological result.

THEOREM 4.6. *Every ideal in $\Gamma(K, \mathcal{O}_X)$ is closed.*

PROOF. As in Proposition 4.5, we may assume that X is a closed subvariety of the open set $\Omega \subset \mathbf{C}^n$. Let J be an ideal in $\Gamma(K, \mathcal{O}_X)$, U an open set containing K and $\rho_U: \Gamma(U, \mathcal{O}_X) \to \Gamma(K, \mathcal{O}_X)$ the natural map; by the result of Allan, Dales and McClure [26] mentioned earlier, we only need show that $\rho_U^{-1}(J)$ is closed. If it were not, there would be functions $f, f_1, \ldots$ in $\Gamma(U, \mathcal{O}_X)$ with f_i in $\rho_U^{-1}(J)$ for each i, f not in $\rho_U^{-1}(J)$ and f_i converging to f, uniformly on compact subsets of U. Thus there is a point x in K such that ${}_xf \notin J_x$. Let P

be an open polydisk centered at x and contained in Ω, such that $P \cap X \subset U$. The natural map $\Gamma(P, \mathcal{O}_{\mathbf{C}^n}) \to \Gamma(P \cap X, \mathcal{O}_X)$ is onto (since P is Stein) and open (by the Open Mapping Theorem for Fréchet spaces) so we can find functions $g, g_1, \ldots$ in $\Gamma(P, \mathcal{O}_{\mathbf{C}^n})$ such that ${}_x(g|P \cap X) = {}_xf, {}_x(g_i|P \cap X) = {}_xf_i$ and g_i converges to g, uniformly on compact subsets of P. If $\mathcal{M}$ is the ideal in ${}_x\mathcal{O}_{\mathbf{C}^n}$ generated by ${}_xg_1, {}_xg_2, \ldots$, then ${}_xg \notin \mathcal{M}$, which contradicts the usual Closure of Modules Theorem (see [14, p. 85]).

Theorems 4.1, 4.2 and 4.6 reveal that the ideal structures of the algebras $\Gamma(K, \mathcal{O}_X)$ and $\Gamma(X, \mathcal{O}_X)$ are quite different. For example, the latter algebra always contains nonclosed ideals, and the ideal of a variety (even of an irreducible subvariety of $\mathbf{C}^3$) need not be finitely generated (see the discussion and examples in [4]).

In the remainder of this section, we characterize the prime ideals in $\Gamma(K, \mathcal{O}_X)$ and give a number of applications, including a proof of Siu's theorem.

If U is a neighborhood of K, V is a closed subvariety of U and $p \in V \cap K$, we say that V is *essentially irreducible at* p if for each open set U' with $K \subset U' \subset U$, only one irreducible branch of $V \cap U'$ contains p. (Note that if W is an arbitrary subvariety of U and $x \in K \cap W$ then it is always possible to choose a smaller neighborhood U_1 of K such that each irreducible branch of $W \cap U_1$ which contains x is in fact essentially irreducible at x. For, if this were not so, the germ of W at x would have infinitely many irreducible branches.) If V is essentially irreducible at p, it is clear that $\{f: (f|V) \equiv 0 \text{ near } p\}$ is a prime ideal; we show below that all prime ideals are of this form.

If J is an ideal in $\Gamma(K, \mathcal{O}_X)$, we set

$$Z(J) = \{x \text{ in } K: f(x) = 0, \text{ all } f \text{ in } J\}.$$

Since every ideal is contained in a maximal ideal, $Z(J) \neq \emptyset$ unless $J = \Gamma(K, \mathcal{O}_X)$. We say that x in $Z(J)$ is a *generic point of* J if every function f in $\Gamma(K, \mathcal{O}_X)$ for which ${}_xf \in J_x$ actually belongs to J.

THEOREM 4.7. *If J is a prime ideal, then*:

(a) *each point of $Z(J)$ is a generic point of J*;

(b) *if $p \in Z(J)$ then there is a variety W near K, which is essentially irreducible at p, and such that $J = \{f: (f|W) \equiv 0$ near $p\}$*;

(c) *$Z(J)$ is connected.*

PROOF. Let J be a prime ideal and p a point of $Z(J)$. Using Corollary 4.3, we can find elements $f_1, \ldots, f_k$ of J (holomorphic in the open set U, say) which generate a radical ideal and such that ${}_pf_1, \ldots, {}_pf_k$ generate J_p. Set $W = \{u \text{ in } U: f_i(u) = 0, \text{ each } i\}$; then $f_1, \ldots, f_k$ generate id(W).

Let Q be any open set with $K \subset Q \subset U$ and let $W \cap Q = \bigcup W_j$ be the decomposition of $W \cap Q$ into irreducible branches (since K is compact only a finite number of these branches meet K). Suppose that p belonged to two of these branches, W_1 and W_2 say. Set $W' = \bigcup_{j \neq 1} W_j$. By the remark following Theorem 4.2, there is a neighborhood Q' of K and functions $a_1, \ldots, a_s, b_1, \ldots, b_t$ in $\Gamma(Q', \mathcal{O}_X)$ such that $W_1 \cap Q'$ is the set of common zeros of $a_1, \ldots, a_s$ and $W' \cap Q'$ is the set of common zeros of $b_1, \ldots, b_t$. Then we can find an i and j such that neither $(a_i|W')$ nor $(b_j|W_1)$ vanish identically near p. Since ${}_pf_1, \ldots, {}_pf_n$ generate J_p, neither a_i nor b_j belongs to J; on the other hand, $a_ib_j \in \mathrm{id}(W)$ which is a subset of J. Since J is prime, this contradiction shows that W is essentially irreducible at p.

It is clear that $(f|W) \equiv 0$ near p for each f in J; we need to show the converse. So let f be a holomorphic function on a neighborhood Q of K such that $(f|W) \equiv 0$ near p, and let W_1 be the irreducible branch of $W \cap Q$ which contains p; evidently $(f|W_1) \equiv 0$. If $W_1 = W$, then $f \in J$. Otherwise, let W' be the union of all the other irreducible branches of $W \cap Q$ which meet K. Arguing as before, we can find a function g in $\Gamma(K, \mathcal{O}_X)$ such that $(g|W') \equiv 0$ near K but $(g|W_1)$ is not identically zero near p. But then $fg \in J$ and $g \notin J$ so $f \in J$. This evidently proves (a) and (b).

To establish (c), let $H_1, H_2, \ldots$ be a decreasing fundamental sequence of neighborhoods of K, each of which is a subset of U. Let V_j be the irreducible branch of $W \cap H_j$ which contains p and let C_j be the connected component of $V_j \cap K$ which contains p. Straightforward arguments show that each C_j is compact, $C_j \supset C_{j+1}$ and $\bigcap C_j = Z(J)$, which yields (c).

Slight modifications of the above arguments lead to the following result.

COROLLARY 4.8. *Let x be a point of K and let I be an ideal in $\Gamma(K, \mathcal{O}_X)$ for which x is a generic point. Then*:

(a) *x is a generic point of* $\mathrm{rad}(I)$;

(b) *there is a variety V near K such that* $\mathrm{rad}(I) = \{f\colon (f|V) \equiv 0$ *near* $x\}$;

(c) $\mathrm{rad}(I)$ *is the finite intersection of prime ideals*;

(d) $Z(I) = Z(\mathrm{rad}\,(I))$ *is connected.*

Recall that the *Krull dimension* of a commutative ring is the maximal length of a chain of prime ideals.

COROLLARY 4.9. *The Krull dimension of* $\Gamma(K, \mathcal{O}_X)$ *is the maximum of* $\{\dim_p X\colon p \in K\}$.

PROOF. Consider a proper chain $I_0 \subset I_1 \subset \cdots \subset I_s$ of prime ideals, and let p be in $Z(I_s)$. Then p is a generic point of each I_j. If $W_0, W_1, \ldots, W_s$

are the varieties associated with $I_0, \ldots, I_s$ and p by Theorem 4.7 (all in the open set U containing K), then the irreducible branch of W_{j+1} which contains p is a proper subvariety of the irreducible branch of W_j which contains p. Hence s, which is the length of the sequence of prime ideals, does not exceed $\dim_p X$, so that the Krull dimension of $\Gamma(K, \mathcal{O}_X)$ does not exceed the maximum of $\{\dim_p X\colon p \in K\}$. The reverse inequality is straightforward.

It is appropriate at this point to describe an example which illustrates the possible pathology of prime ideals. Consider the mapping $\varphi\colon \mathbf{C}^2 \longrightarrow \mathbf{C}^3$ defined by

$$\varphi(z, w) = (z^2, z^3 - z, w).$$

Then φ is a proper map and $X = \varphi(\mathbf{C}^2)$ is a closed subvariety of $\mathbf{C}^3$. A simple computation shows that $\varphi(1, w) = \varphi(-1, w)$ for all w, but that φ separates every other pair of points in $\mathbf{C}^2$. Let $L \subset \mathbf{C}^2$ be the union of the line segments from $(-1, 0)$ to $(+1, 0)$, from $(0, 0)$ to $(0, 1)$ and from $(-1, 1/n)$ to $(0, 1/n)$ for each $n = 1, 2, \ldots$; set $K = \varphi(L)$. It may be seen that K is holomorphically convex in X (in fact it is rationally convex in $\mathbf{C}^3$). Set $p = (1, 0, 0) = \varphi(-1, 0) = \varphi(+1, 0)$ and $I = \{f \text{ in } \Gamma(K, \mathcal{O}_X)\colon {}_pf = 0\}$. Since X is essentially irreducible at p, I is a prime ideal; evidently, $Z(I) = K$. Note, however, that for each n, there is a function f_n in I such that the germ of f_n at $\varphi(-1, 1/n)$ is not zero. Hence the variety which Theorem 4.7 associates with I and the point p is X itself, while the variety associated to I and the point $\varphi(-1, 1/n)$ is not (in fact it is an irreducible branch of an appropriate neighborhood of K). Thus the dependence of the variety upon the particular choice of generic point is in fact necessary.

Using the characterization of prime ideals given by Theorem 4.7, we give a proof of Siu's Theorem [18], which is more elementary in the sense that we avoid any use of gap-sheaves.

THEOREM 4.10 (SIU [18]). *The ring $\Gamma(K, \mathcal{O}_X)$ is Noetherian if and only if $V \cap K$ has only a finite number of connected components for each variety V near K.*

PROOF. If $V \cap K$ has infinitely many connected components, at least one of them, call it L, is not open in $V \cap K$. It is clear that $I = \{f \text{ in } \Gamma(K \cap V, \mathcal{O}_V)\colon f \equiv 0 \text{ near } L\}$ is not finitely generated; since the map $\rho\colon \Gamma(K, \mathcal{O}_X) \longrightarrow \Gamma(K \cap V, \mathcal{O}_V)$ is onto, it follows that the ideal $\rho^{-1}(I) = \{f \text{ in } \Gamma(K, \mathcal{O}_X)\colon (f|V) \equiv 0 \text{ near } L\}$ is not finitely generated, either.

In order to establish the converse, note first that by Cohen's Theorem [16, p. 8] it suffices to show that each prime ideal is finitely generated. So let W be a variety which is essentially irreducible at a point p in K, and set

$J = \{f \text{ in } \Gamma(K, \mathcal{O}_X): (f|W) \equiv 0 \text{ near } p\}$. Since $\rho: \Gamma(K, \mathcal{O}_X) \longrightarrow \Gamma(K \cap W, \mathcal{O}_W)$ is onto and the kernel of ρ is id(W), which is finitely generated, it suffices to show that $I = \{g \text{ in } \Gamma(K \cap W, \mathcal{O}_W): {}_pg = 0\}$ is a finitely generated ideal in $\Gamma(K \cap W, \mathcal{O}_W)$.

Now let $\nu: N \longrightarrow W$ be the normalization of W and let $L = \nu^{-1}(W \cap K)$. By Lemma 4 of [18], L has only a finite number of connected components. We can choose a neighborhood Q of $K \cap W$ in W such that no two of these components lie in the same connected component of $\nu^{-1}(Q)$. Let H be the union of those components of $\nu^{-1}(Q)$ which contain a point of $\nu^{-1}(p)$; then $\nu|H$ is a proper mapping of H into Q so $\nu(H) = V$ is a closed subvariety of Q. On the other hand, if $g \in I$ then $g \circ \nu$ vanishes near $\nu^{-1}(p)$ and hence in a neighborhood of $H \cap L$. But then g vanishes near $(K \cap W) \cap V$, so that I is the ideal of a variety in $\Gamma(K \cap W, \mathcal{O}_W)$ and hence is finitely generated, which completes the proof.

We remark in passing, that if $V \cap K$ has infinitely many connected components then id(V) is not the intersection of a finite number of prime ideals.

In a sense, the rings $\Gamma(K, \mathcal{O}_X)$ are "bad" in that they may not be Noetherian. We conclude this section by showing that this defect disappears if we localize.

PROPOSITION 4.11. *The localization of* $\Gamma(K, \mathcal{O}_X)$ *at any maximal ideal is a Noetherian ring.*

PROOF. Let x be in K and consider the maximal ideal $M_x = \{f: f(x) = 0\}$. Let $\gamma: \Gamma(K, \mathcal{O}_X) \longrightarrow {}_x\mathcal{O}_X$ be the natural map, and R the subring of ${}_x\mathcal{O}_X$ consisting of those elements of the form $\gamma(f)\gamma(g)^{-1}$ where $g(x) \neq 0$. Straightforward arguments show that $\gamma: \Gamma(K, \mathcal{O}_X) \longrightarrow R$ is the localization of $\Gamma(K, \mathcal{O}_X)$ at M_x. To show that R is Noetherian, it suffices to show (by Cohen's Theorem [16, p. 8]) that each prime ideal is finitely generated. So let $P \subset R$ be a prime ideal. Evidently $\gamma^{-1}(P)$ is a prime ideal in $\Gamma(K, \mathcal{O}_X)$ and x belongs to $Z(\gamma^{-1}(P))$. Hence there is a variety V near K such that $\gamma^{-1}(P) = \{f: (f|V) \equiv 0 \text{ near } x\}$. Let $f_1, f_2, \ldots, f_s$ generate id(V). Let g belong to $\gamma^{-1}(P)$; we may suppose that g is holomorphic on a neighborhood U of K such that $V \cap U$ is a closed subvariety of U and $f_1, f_2, \ldots, f_s$ are holomorphic on U. Write $V \cap U = V_1 \cup V_2$ where V_1 is the irreducible branch that contains x and V_2 is the union of all the other irreducible branches. Clearly $(g|V_1) \equiv 0$; choose a function h in $\Gamma(K, \mathcal{O}_X)$ such that $h(x) \neq 0$ but $(h|V_2) \equiv 0$ near K. Then $gh \in \text{id}(V)$ so that $gh = \Sigma\alpha_i f_i$ and $\gamma(g)\gamma(h) = \Sigma\gamma(\alpha_i)\gamma(f_1)$. On the other hand, $\gamma(h)$ is a unit in R, so that $\gamma(g)$ belongs to the ideal in R generated by $\gamma(f_1), \ldots, \gamma(f_s)$. Since general properties of the localization insure that $\gamma(\gamma^{-1}(P))$ generates P, it follows that $\gamma(f_1), \ldots, \gamma(f_s)$ generate P, which completes the proof.

5. **Homomorphisms.** Throughout this section $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are analytic spaces, K is a compact holomorphically convex subset of X and L is a compact subset of Y. We wish to study homomorphisms $T: \Gamma(K, \mathcal{O}_X) \longrightarrow \Gamma(L, \mathcal{O}_Y)$. A holomorphic map φ of a neighborhood of L into X, for which $\varphi(L) \subset K$, defines by composition a unital homomorphism $T_\varphi: \Gamma(K, \mathcal{O}_X) \longrightarrow \Gamma(L, \mathcal{O}_X)$, which we call the *homomorphism induced by* φ. (Note that T_φ is determined by the germ of φ on L.) We begin by showing that, in fact, every unital homomorphism arises in this way. This result generalizes some of the work in [22] and [23]. (*Note.* In [22] and [23], there was an inadvertent omission of the comment that the homomorphisms under consideration were assumed to be unital.)

THEOREM 5.1. *Every unital homomorphism* $T: \Gamma(K, \mathcal{O}_X) \longrightarrow \Gamma(L, \mathcal{O}_Y)$ *is induced.*

PROOF. There is no loss in assuming that X is a closed subvariety of the open set $\Omega \subset \mathbf{C}^n$. Let $z_1, \ldots, z_n$ denote the coordinate functions, set $\varphi_j = T(z_j)$ for $j = 1, 2, \ldots, n$, and let $\varphi = (\varphi_1, \ldots, \varphi_n)$. If $y \in L$ then the map $f \longrightarrow T(f)(y)$ is a unital homomorphism of $\Gamma(K, \mathcal{O}_X)$ into $\mathbf{C}$ and so is just evaluation at some point x in K. Evidently the image of z_i under this map is just $\varphi_i(y)$, whence it follows that $x = \varphi(y)$ and that $\varphi(L) \subset K$. We are going to show that φ maps a neighborhood of L into X and induces T.

Let U be a neighborhood of K in X. As in the proof of Theorem 3.1, we can find an open set U' in X with $K \subset U' \subset U$, functions $f_{n+1}, \ldots, f_m$ in $\Gamma(U', \mathcal{O}_X)$ and a polynomial polyhedron P in $\mathbf{C}^m$ such that $\pi(P) \subset \Omega$, $\pi(P) \cap X \subset U$ and $G(K) \subset P$, where $\pi: \mathbf{C}^m \longrightarrow \mathbf{C}^n$ is the projection on the first n coordinates and $G = (z_1, \ldots, z_n, f_{n+1}, \ldots, f_m)$. Observe that G is a biholomorphism of U' with a closed subvariety of some open set in $\mathbf{C}^m$ and that its inverse is $\pi | G(U')$.

Let A be an arbitrary compact, polynomially convex subset of P which contains $G(K)$ and let $T_G: \Gamma(A, \mathcal{O}_{\mathbf{C}^m}) \longrightarrow \Gamma(K, \mathcal{O}_X)$ be the homomorphism induced by G. Set $S = T \circ T_G: \Gamma(A, \mathcal{O}_{\mathbf{C}^m}) \longrightarrow \Gamma(L, \mathcal{O}_Y)$. By results of [23] S is induced by the holomorphic map

$$\psi = (S(z_1), \ldots, S(z_m)) = (T(z_1), \ldots, T(z_n), T(f_{n+1}), \ldots, T(f_m)).$$

If $h \in \Gamma(P, \mathcal{O}_{\mathbf{C}^m})$ and $h|(\pi^{-1}(X) \cap P) \equiv 0$ then $T_G(h) = 0$ so $S(h) - h \circ \varphi = 0$ near L. Since P is a Stein manifold, we can find an open set P' with $A \subset P' \subset P$ and functions $h_1, h_2, \ldots, h_s$ in $\Gamma(P, \mathcal{O}_{\mathbf{C}^m})$ such that $\pi^{-1}(X) \cap P = \{p$ in $P': h_i(p) = 0$ for $i = 1, 2, \ldots, s\}$. Thus, ψ maps a neighborhood W of L into $\pi^{-1}(X) \cap P$. Since $\pi \circ \psi = \varphi$, it follows that φ maps W into X, and induces a homomorphism $T_\varphi: \Gamma(K, \mathcal{O}_X) \longrightarrow \Gamma(L, \mathcal{O}_Y)$.

Finally, if $g \in \Gamma(U, \mathcal{O}_X)$ then $g \circ \pi$ is a holomorphic function on $\pi^{-1}(X) \cap P$ and hence has an extension $\tilde{g}$ to P. Evidently, $T_G(\tilde{g}) = g$, so that

$$T(g) = T \circ T_G(\tilde{g}) = \tilde{g} \circ \psi = \tilde{g} \circ (T(z_1), \ldots, T(z_n), T(f_{n+1}), \ldots, T(f_m))$$
$$= g \circ \pi \circ (T(z_1), \ldots, T(z_n), T(f_{n+1}), \ldots, T(f_m)) = g \circ \varphi = T_\varphi(g).$$

Since U was arbitrary, this completes the proof.

It might be pointed out that the added complications in the above proof (beyond those in [22] and [23]) are due precisely to the fact that X may not be the set of common zeros of functions in $\Gamma(\Omega, \mathcal{O}_{\mathbf{C}^n})$.

COROLLARY 5.2. *If L is holomorphically convex then the induced homomorphism T_φ: $\Gamma(K, \mathcal{O}_X) \longrightarrow \Gamma(L, \mathcal{O}_Y)$ is an isomorphism if and only if φ is a biholomorphism of a neighborhood of L onto a neighborhood of K.*

If we note that an idempotent in $\Gamma(L, \mathcal{O}_Y)$ is just a function which is 1 near an open and closed subset and 0 near its complement, we arrive at the following description of all nonzero homomorphisms.

COROLLARY 5.3. *If T: $\Gamma(K, \mathcal{O}_X) \longrightarrow \Gamma(L, \mathcal{O}_Y)$ is a nonzero homomorphism, then there are an open and closed subset L' of L and a holomorphic mapping φ of a neighborhood of L' into X with $\varphi(L') \subset K$ such that $T(f) = f \circ \varphi$ near L' and $T(f) = 0$ near $(L - L')$ for each f in $\Gamma(K, \mathcal{O}_X)$.*

In case L is holomorphically convex and the homomorphism T: $\Gamma(K, \mathcal{O}_X) \longrightarrow \Gamma(L, \mathcal{O}_Y)$ is a surjection, we can obtain a great deal more information. We begin by distinguishing three types of surjections. If Y is a variety near K, $L = K \cap Y$ and T is the natural restriction, we say that T is of *type* 1 (see Theorem 3.5). If Y is an open subset of X, L is a compact holomorphically convex subset of K and T is the restriction and is a surjection, we say that T is of *type* 2. Finally, T is said to be of *type* 3 if it is an isomorphism.

Surjections of type 2 clearly arise if L is a connected component of K; that they may arise even if K is connected may be seen from the following example. In $\mathbf{C}^3$, let $X = \{(z, w, t): zw = 0\}$. If n is even, let C_n be a semicircle in $\{(z, w, t): z = 0\}$ joining $(0, 0, 1/n)$ and $(0, 0, 1/(n + 1))$; if n is odd let C_n be a semicircle in $\{(z, w, t): w = 0\}$ joining $(0, 0, 1/n)$ and $(0, 0, 1/(n + 1))$. Let K be the closure of the union of the C_n and $L = \{(0, 0, 0)\}$. It is easy to see that K is holomorphically convex (in fact it is polynomially convex in $\mathbf{C}^3$); that the restriction $\Gamma(K, \mathcal{O}_X) \longrightarrow \Gamma(L, \mathcal{O}_X)$ is onto may be seen from the following simple lemma, whose proof we omit.

LEMMA 5.4. *If L is a holomorphically convex subset of K then the following statements are equivalent*:

(i) $\Gamma(K, \mathcal{O}_X) \longrightarrow \Gamma(L, \mathcal{O}_X)$ *is onto*;

(ii) *if* $x \in (K - L)$ *then there is a neighborhood* U *of* K, *no irreducible branch of which contains* x *and meets* L;

(iii) *if* $x \in (K - L)$ *then there is a function* f *in* $\Gamma(K, \mathcal{O}_X)$ *such that* ${}_y f = 0$ *for all* y *in* L *but* $f(x) \neq 0$.

Now we can show that every surjection is in fact the composition of surjections of the type we have distinguished (and thus is a restriction followed by an isomorphism). We begin by noting, without proof, a simple lemma.

LEMMA 5.5. *If* T_φ: $\Gamma(K, \mathcal{O}_X) \longrightarrow \Gamma(L, \mathcal{O}_Y)$ *is a surjection and* L *is holomorphically convex, then* φ *is one-to-one in some neighborhood of* L.

THEOREM 5.6. *If* L *is holomorphically convex and* T: $\Gamma(K, \mathcal{O}_X) \longrightarrow \Gamma(L, \mathcal{O}_Y)$ *is a surjection, then* T *has a decomposition* $T = T_3 \circ T_2 \circ T_1$, *where each* T_i *is a surjection of type i.*

PROOF. Note first that T is unital and thus induced by a holomorphic map φ defined near L. In view of Lemma 5.5, there is no loss in assuming that φ is a one-to-one map of Y into X. We are going to find a variety Z near K which contains $\varphi(L)$ and such that φ induces an isomorphism $\Gamma(\varphi(L), \mathcal{O}_Z) \cong \Gamma(L, \mathcal{O}_Y)$; the rest will then be easy.

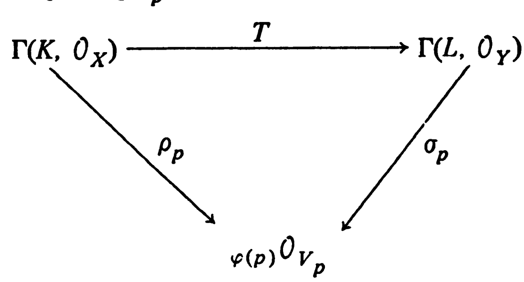
Let p be a point of L and γ_p: $\Gamma(L, \mathcal{O}_Y) \longrightarrow {}_p\mathcal{O}_Y$ the natural map. It is easy to see that $\ker(\gamma_p \circ T)$ is a radical ideal which has $\varphi(p)$ as generic point. By Corollary 4.8 there is a variety p near K such that $\varphi(p) \in K \cap V_p$ and

$$\ker(\gamma_p \circ T) = \{f\colon f|V_p \equiv 0 \text{ near } \varphi(p)\}.$$

If ρ_p: $\Gamma(K, \mathcal{O}_X) \longrightarrow {}_{\varphi(p)}\mathcal{O}_{V_p}$ is the natural map then

$$\ker(\rho_p) = \ker(\gamma_p \circ T) \supset \ker(T).$$

Hence there is a unique map σ_p which makes the following diagram commutative:



Since L is holomorphically convex, σ_p is induced by a map ψ: $U_p \longrightarrow Y$ with $\psi(\varphi(p)) = p$. Since T is induced by φ, it follows that $\rho_p = \sigma_p \circ T$ is induced by $\varphi \circ \psi$. On the other hand, ρ_p is induced by the inclusion. Hence there is a neighborhood U_p' of p in V_p on which $\varphi \circ \psi$ is the inclusion of U_p' in X.

Thus, by choosing a sufficiently small neighborhood Q_p of p in Y, we may conclude that $\varphi|Q_p$ is a holomorphic homeomorphism of Q_p onto the open subset $\varphi(Q_p)$ of V_p.

Now let q be a point of Q_p and carry through the above analysis for q, obtaining the variety V_q, the neighborhood Q_q, etc. If Q is an open subset of $Q_p \cap Q_q$, then $\varphi(Q)$ is an open subset of V_p and of V_q. It follows that V_p and V_q have the same germ at $\varphi(q)$; hence $\ker(\gamma_q \circ T) = \{f: f|V_p \equiv 0 \text{ near } \varphi(q)\}$.

The open sets Q_p cover L; let $Q_1, \ldots, Q_s$ be a finite subcover and $V_1, \ldots, V_s$ the corresponding varieties. Set $W_i = \varphi(Q_i)$ so that W_i is an open subset of V_i. We assert that φ is a biholomorphism of Q_i with W_i. To see this, use Theorem 3.2 to choose functions $h_1, \ldots, h_t$ holomorphic near L such that $h = (h_1, \ldots, h_t)$ maps a neighborhood of a point y in Q_i biholomorphically onto a subvariety of a polydisk in $\mathbf{C}^t$. The map $(\varphi|Q_i)^{-1}$ is continuous and so is holomorphic at $\varphi(y)$ if $h_j \circ (\varphi|Q_i)^{-1}$ is holomorphic at $\varphi(y)$ for each j. But if $\tilde{h}_j \in \Gamma(K, \mathcal{O}_X)$ with $T(\tilde{h}_j) = h_j$, then $h_j \circ (\varphi|Q_i)^{-1} = \tilde{h}_j|W_i$. This establishes our assertion that $\varphi|Q_i$ is a biholomorphism.

We have now constructed a collection of "local varieties." The next step is to show that they fit together properly. If $x \in Q_i \cap Q_j$ then the varieties V_x, V_i and V_j all have the same germ at $\varphi(X)$. It follows that W_i is a neighborhood of $\varphi(x)$ in $W = \bigcup W_i$. If we now shrink each Q_i slightly, we can be assured that each W_i is an open subset of W. Let $Q = \bigcup Q_i$; then φ is a biholomorphism of Q onto W. Observe that: W is a subvariety of some open subset of X; $W \supset \varphi(L)$; each W_i is an open subset of V_i and of W; and

$$\ker(T) = \{f: (f|W) \equiv 0 \text{ near } \varphi(L)\}.$$

Unfortunately, W need not be an open subset of $\bigcup V_i$ (since V_1, for example, may have extraneous branches which meet V_2) and need not itself be a variety near K.

In order to remedy this deficiency, we proceed as follows. If $x \in K$, choose functions $f_1^x, \ldots f_t^x$ in $\ker(T)$ which generate the ideal of a variety, call it Z_x, near K, and also generate $\ker(T)_x$. Then Z_x and W have the same germ at x. Moreover the ideal in ${}_y\mathcal{O}_X$ generated by $f_1^x, \ldots, f_t^x$ is a subset of $\ker(T)_y$ for each y in $\varphi(L)$. It follows that the germ of Z_x at y contains the germ of W at y for each y in $\varphi(L)$. Choose a neighborhood W'_x of x in W which is also an open subset of Z_x. Then the family $\{W'_x\}$ is an open cover of $\varphi(L)$ in W, so we can extract a finite subcover $\{W'_1, W'_2, \ldots, W'_m\}$; let $Z_1, \ldots, Z_m$ be the corresponding varieties near K, and set $Z = \bigcap Z_i$, so that Z is a variety near K which contains $\varphi(L)$. We see that the germ of Z at y agrees with the germ of W at y for each y in $\varphi(L)$. Hence there is a set W_0 which is a neighborhood of $\varphi(L)$ in both W and Z; certainly $\varphi^{-1}(W_0)$ is a neighborhood of L in Y and φ is a biholomorphism of $\varphi^{-1}(W_0)$ onto W_0; hence $\varphi(L)$ is holomorphically convex in Z. Finally $\ker(T) = \{f: (f|Z) \equiv 0 \text{ near } \varphi(L)\}$.

Now φ induces an isomorphism $T_3: \Gamma(\varphi(L), \mathcal{O}_Z) \longrightarrow \Gamma(L, \mathcal{O}_X)$ and we have the natural maps

$$T_1: \Gamma(K, \mathcal{O}_X) \longrightarrow \Gamma(K \cap Z, \mathcal{O}_Z), \quad T_2: \Gamma(K \cap Z, \mathcal{O}_Z) \longrightarrow \Gamma(\varphi(L), \mathcal{O}_Z).$$

It is clear that $T = T_3 \circ T_2 \circ T_1$; since T_3 is an isomorphism it follows that T_2 is onto, and this completes the proof.

We remark that, because the proof shows that φ is a biholomorphism of a neighborhood of L onto a neighborhood of $\varphi(L)$ in Z, the decomposition of T is unique. We also arrive at the following simple characterization of those ideals which are kernels of surjections.

COROLLARY 5.7. *Let J be a radical ideal in $\Gamma(K, \mathcal{O}_X)$. Then the following statements are equivalent*:

(i) *there is an analytic space $(Y, \mathcal{O}_Y)$, a compact holomorphically convex set $L \subset Y$ and a surjection $T: \Gamma(K, \mathcal{O}_X) \longrightarrow \Gamma(L, \mathcal{O}_Y)$ with* $\ker(T) = J$;

(ii) *the ideal generated by the image of J under the natural map $\rho: \Gamma(K, \mathcal{O}_X) \longrightarrow \Gamma(Z(J), \mathcal{O}_X)$ is finitely generated*;

(iii) *the restriction of the sheaf $S(J)$ to $Z(J)$ is coherent*;

(iv) *for each x in $Z(J)$ there is a variety V near K such that $J_y = \{\alpha: (\alpha|V) = 0\}$ for each y in $Z(J)$ near x.*

PROOF. It was shown in the proof of Theorem 5.6 that (i) implies (iv). The equivalence of (ii) and (iii) follows from Theorem 4.1, while it follows from Theorem 4.2 that (iv) implies (ii). To see that (iii) implies (i), choose functions $g_1, \ldots, g_s$ in J which generate the ideal of a variety, call it W, and such that ${}_yg_1, \ldots, {}_yg_s$ generate J_y for each y in $Z(J)$. It is easy to see that $J = \{h: (h|W) \equiv 0 \text{ near } Z(J)\}$. Let

$$\rho_1: \Gamma(K, \mathcal{O}_X) \longrightarrow \Gamma(K \cap W, \mathcal{O}_W), \quad \rho_2: \Gamma(K \cap W, \mathcal{O}_W) \longrightarrow \Gamma(Z(J), \mathcal{O}_W)$$

be the natural maps. It is easy to see that $Z(J)$ is the intersection of a descending family of sets of the form $K \cap V$, where V is a variety near K. By Theorem 3.6, each of these sets is holomorphically convex, and it is easy to see that $Z(J)$ is then holomorphically convex also. The homomorphism ρ_1 is surjective by Theorem 3.6, ρ_2 is surjective by Lemma 5.4 and $\ker(\rho_2 \circ \rho_1) = J$, which completes the proof.

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