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ON THE BLOCKS OF GL(n, q). I

BY

JØRN B. OLSSON

ABSTRACT. A study is made of the distribution of the ordinary irreducible characters of GL(n, q) into p-blocks for primes different from the characteristic. The paper gives a description of all possible defect groups for $p \neq 2$ and their normalizers. Various other results are obtained, including a classification of the blocks of defect 0.

About 20 years ago, J. A. Green determined completely the irreducible characters of the finite general linear groups GL(n, q), n integer, q prime-power, in a long and deep paper [6]. Until now no attempts have been made to try to determine the distribution of the irreducible characters into p-blocks for primes p different from the characteristic. The present paper starts investigations of this problem. There are similarities between general linear groups and symmetric groups, which are illustrated by the analogy of results in this paper to results from the representation theory of symmetric groups. The blocks of Sym(n), the symmetric group of degree n, were determined by Brauer [2] and Robinson [12] in 1947, continuing the fundamental work of Nakayama [10], [11].

In §1 we investigate the possible defect groups for p-blocks of GL(n, q), $p \neq 2$, $p \nmid q$. The Sylow p-subgroups of GL(n, q) are direct products of wreath products of cyclic p-groups (see [14]) and the same is true for the possible defect group. We also determine the normalizers and centralizers of the possible defect groups, so that a theorem of R. Brauer can be applied to determine which of them actually occur. §2 concentrates on studies of the degree formula for the irreducible representations. We give a description of p-blocks of defect 0. After introducing the concept of the (e, p)-series for a partition, it is possible to compute the power of p dividing the degree of a representation. §3 contains a general result on the blocks of GL(n, q).

1. **The defect groups.** Let N be the set of positive integers and q a prime-power. For $n \in N$, GL(n, q) denotes the general linear group of degree n over the finite field GF(q). Let $V(n, q)$ denote an n-dimensional vector space over GF(q), so GL(n, q) is the group of isomorphisms of $V(n, q)$.

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If $p \in N$ is a prime, we say that p is of *degree e w.r.t. q*, if $p \mid (q^e - 1)$ and $p \nmid (q^f - 1)$ for $1 \leqslant f < e$, i.e., if $q + pZ$ is of order e in GF(p). In particular $e \mid p - 1$.

If n is any integer, $\nu(n) \in N$ is defined by $p^{\nu(n)} \mid n$ and $p^{\nu(n)+1} \nmid n$.

In this section we assume that $p \neq 2$ is a prime of degree e w.r.t. q, and that $\nu(q^e - 1) = a\ (> 0)$. We want to study the possible p-defect groups of GL(n, q), and start with a few preliminary lemmas.

LEMMA (1.1). *Let $h \in N$. Then*

$$\nu(q^h - 1) = \begin{cases} 0 & \text{if } e \nmid h, \\ a + \nu(h) & \text{if } e \mid h. \end{cases}$$

The proof is straightforward.

The Sylow p-subgroups of GL(n, q) have been described in [14], and we assume knowledge of this work.

LEMMA (1.2). *Let $s, k \in N$.*

(1) *The Sylow p-subgroups of* GL(s, q^{ek}) *and* GL$(s, q^{ep^{\nu(k)}})$ *are isomorphic.*

(2) $\nu\,|\mathrm{GL}(se, q)| = \nu\,|\mathrm{GL}(s, q^e)| = sa + \nu(s!)$. *In particular the Sylow p-subgroups of these groups are isomorphic.*

PROOF. Use (1.1) and the fact that, for any $s, k \in N$, GL(s, q^k) can be embedded in GL(sk, q).

For $n \in N$, Sym(n) denotes the full symmetric group of degree n. For $j > 0$ let R_j be a Sylow p-subgroup of Sym(p^j) and let $R_0 = 1$.

We define

$$D_{ij} = \mathbf{Z}_{p^{a+i}} \wr R_j \quad \text{for } i, j \geqslant 0,$$

where $\wr$ denotes wreath product. Then D_{ij} can be embedded as a Sylow p-subgroup of GL(p^j, q^{ep^i}).

LEMMA (1.3). *The minimal dimension of a faithful representation of D_{ij} over* GF(q) *is $p^{i+j}e$.*

PROOF. We have to show that $k = p^{i+j}e$ is the smallest integer, such that GL(k, q) contains a subgroup isomorphic to D_{ij}. If A is any group, then for exponents we have $\exp(A \wr \mathbf{Z}_p) = p \cdot \exp(A)$ as is easily seen. Thus if r is the largest integer, such that $p^r e \leqslant n$, then the exponent of a Sylow p-subgroup for GL(n, q) is p^{a+r}. This proves our assertion for $j = 0$.

The centralizer of D_{i0} in GL$(p^i e, q)$ is cyclic by II, 7.3 in [7]. Since D_{ij} contains a direct product of p^j copies of $\mathbf{Z}_{p^{a+i}}$, our result follows for all j.

LEMMA (1.4). (1) *D_{ij} has a characteristic homocyclic subgroup A of* *exponent* p^{a+i} *and order* $p^{(a+i)p^j}$. *A is the only abelian subgroup of* D_{ij} *of this order, and no abelian subgroup has larger order.*

(2) *Consider* $D_{ij} \subseteq \mathrm{GL}(p^{i+j}e, q) = \mathrm{GL}(V)$. *There exist decompositions*

$$V = \sum_{k=1}^{p^j} \oplus V^{(k)}, \qquad A = \prod_{k=1}^{p^j} A^{(k)}$$

such that $\dim V^{(k)} = p^i e$, $A^{(k)} \simeq \mathbf{Z}_{p^{a+i}}$, $A^{(k)}$ *acts faithfully on* $V^{(k)}$ *and trivially on* $V^{(l)}$ *for* $k \neq l$.

PROOF. (1) This is proved by induction on j, using for instance Theorem 2 of [1].

(2) For any $m \in N$, a statement similar to (2) can be formulated for a homocyclic subgroup of $\mathrm{GL}(mp^ie, q)$ of exponent p^{a+i} and order $p^{(a+i)m}$. This statement is proved by induction on m using II, 3.10 in [7] and (1.3) for $j = 0$.

PROPOSITION (1.5). *Let* $e \leqslant n \in N$. *A p-defect group for* $\mathrm{GL}(n, q)$ *is a direct product of p-groups, each isomorphic to some* D_{ij}.

PROOF. Let D be a defect group for the p-block B of $G = \mathrm{GL}(n, q)$.

Case I. $e = 1$.

(a) Assume $\Omega_1(Z(D)) \leqslant Z(\mathrm{GL}(n, q))$. Then $\Omega_1(Z(D))$ is cyclic, so $Z(D)$ is cyclic. Write $|Z(D)| = p^{a+d}$ for some $d \geqslant 0$. This is possible because $Z(G)$ contains a subgroup of order p^a. A basic property for a p-defect group is that it is a Sylow p-subgroup of the centralizer of some p'-element. So assume $D \in \mathrm{Syl}_p(C_G(y))$, where $y \in G$ is a p'-element.

Write $y = y_q y_{q'}$, the product of a q-element and a q'-element. $[y_q, y_{q'}] = 1$.

If $y_q = 1$, let

$$V = \sum_{i=1}^{m} \sum_{j=1}^{s_i} V_{ij}$$

be the decomposition of $V = V(n, q)$ into irreducible $\mathrm{GF}(q)[\langle y \rangle]$-modules, where $V_{ij} \simeq V_{kl}$, if and only if $i = k$. If $\dim V_{ij} = k_i$, then

$$C_G(y) \simeq \prod_{i=1}^{m} \mathrm{GL}(s_i, q^{k_i}). \tag{1}$$

This follows for instance from II, 3.11 in [7].

If $y_q \neq 1$, then $C_G(y) = C_{C_G(y_{q'})}(y_q)$, where $C_G(y_{q'})$ has the form described in (1).

Since $Z(D)$ is cyclic, we conclude that in any case $m = 1$, so $n = s_1 k_1$.

If $y_q = 1$, then s_1 must be a power of p and $\nu(k_1) = d$ in order that the Sylow p-subgroup of $C_G(y) \simeq \mathrm{GL}(s_1, q^{k_1})$ has cyclic center of order p^{a+d}. Thus D has the desired form, as is seen by using (1.2)(1).

If $y_q \neq 1$, then $C_{\mathrm{GL}(s_1, q^{k_1})}(y_q)$ contains a homocyclic subgroup of type $(p^{a_1}, \ldots, p^{a_1})$ (r times) where $a_1 = \nu(q^{k_1} - 1)$ and r is the number of parts in a certain partition of s_1 (corresponding to the Jordan canonical form of y_q). We get $r = 1$. By Lemma 2.1 in [6], D is cyclic and $\nu(k_1) = d$. Thus $D \simeq D_{0d}$ in this case.

(b) If $\Omega_1(Z(D)) \not\trianglelefteq Z(\mathrm{GL}(n, q))$, then there is some $z \in Z(D)$ having at least 2 distinct eigenvalues. Then $C_G(z)$ is a direct product of general linear groups of smaller dimensions. Since $DC_G(D) \leqslant C_G(z)$, D is a defect group for $C_G(z)$, so we are done by induction.

Case II. $e > 1$. Choose $z \in Z(D)$ of order p such that the multiplicty of 1 as an eigenvalue is minimal. We can write

$$C_G(z) \simeq \prod_{i=1}^{t} \mathrm{GL}(s_i, q^{k_i})$$

for some splitting in $n = \Sigma_{i=1}^{t} s_i k_i$, where $k_i = 1$ or e. Now $k_i = 1$ for at most one i. If $k_1 = 1$, then the contribution to D from $\mathrm{GL}(s_1, q)$ is 1 by the choice of z. In $\mathrm{GL}(s_1, q^e)$, p has degree 1 w.r.t. q^e, so we are done by Case I.

A modification of the above proof gives a slightly stronger result, which can be formulated as follows: Write

$$(*) \qquad n = c + em, \quad m = \sum_{i=1}^{t} m_i p^i, \quad m_i = \sum_{j=1}^{s_i} \alpha_{ij} p^j, \qquad c, m \geqslant 0,$$

where the last sums are arbitrary p-adic splittings. Associated to the splitting $(*)$ of n is the subgroup $G(*)$ of $\mathrm{GL}(n, q)$, which is isomorphic to a direct product of $\mathrm{GL}(c, q)$ and (for each (i, j)) α_{ij} copies of $\mathrm{GL}(p^{i+j}, q) = G_{ij}$. These direct factors are embedded as "diagonal blocks" in $\mathrm{GL}(n, q)$. We use the faithful representation of (1.3) to embed

$$D(*) \cong \prod_{i,j} D_{ij}^{\alpha_{ij}}$$

in $G(*)$. Then $D(*)$ is denoted *the group corresponding to the splitting* $(*)$.

PROPOSITION (1.6). *Let $e \leqslant n \in N$. Any p-defect group for* $\mathrm{GL}(n, q)$ *is conjugate in* $\mathrm{GL}(n, q)$ *to $D(*)$ for some splitting $(*)$ of n.*

In the following we compute $N_G(D)$ and $C_G(D)$, if D is the group corresponding to a splitting, so that Brauer's results (e.g. (5c) in [3]) can be used to determine whether D actually occurs as a defect group in G.

LEMMA (1.7). *Let $\alpha \in N$ and consider the subgroup $D = D_{ij}^{\alpha} \subseteq \mathrm{GL}(\alpha p^{i+j}e, q) = G$. Then*

$$N_G(D) \simeq N_{ij} \wr \mathrm{Sym}(\alpha),$$

where $N_{ij} \simeq N_{\mathrm{GL}(p^{i+j}e,\, q)}(D_{ij})$.

PROOF. Write

$$V = V(\alpha p^{i+j}e, q) = \sum_{k=1}^{\alpha} {}^{\oplus} V^{(k)} \quad \text{and} \quad D = \prod_{k=1}^{\alpha} D^{(k)},$$

where $D^{(k)} \simeq D_{ij}$, $\dim V^{(k)} = p^{i+j}e$ and $D^{(k)} \subseteq \mathrm{GL}(V^{(k)})$.

Let N be the subgroup of $N_G(D)$ in which every element permutes the subspaces $V^{(k)}$. It is easily seen that $N \simeq N_{ij} \wr \mathrm{Sym}(\alpha)$, so we need only show $N = N_G(D)$.

By (1.4)(1) each $D^{(k)}$ contains a homocyclic characteristic subgroup $A^{(k)}$. Let $B = \Pi_k A^{(k)}$. Let

$$V^{(k)} = \sum_{l=1}^{p^j} {}^{\oplus} V_l^{(k)} \quad \text{and} \quad A^{(k)} = \prod_{l=1}^{p^j} A_l^{(k)}, \qquad 1 \leqslant k \leqslant \alpha,$$

be the splittings described in (1.4)(2). The elements of $\bigcup_{k,l}(A_l^{(k)})^{\#}$ are exactly those elements of $B^{\#}$ for which the multiplicity of 1 as eigenvalue is maximal. By (1.4)(1), B char D, so the elements of $N_G(D)$ permute the D-orbits of the subgroups $A_l^{(k)}$ by conjugation, and so they permute the subgroups $A^{(k)}$. Now $V^{(k)}$ are exactly those elements of V, that are fixed element-wise by any element of $A^{(k')}$ for all $k' \neq k$. Therefore any element of $N_G(D)$ permutes the subspaces $V^{(k)}$ of V, so $N_G(D) = N$, proving the lemma.

An application of Schur's lemma (see e.g. [4, 2.1–2.3]) and the above result proves the following:

LEMMA (1.8). *Let D be the subgroup of $G = \mathrm{GL}(n, q)$ corresponding to the splitting*

$$n = c + em, \quad m = \sum m_i p^i, \quad m_i = \sum_j \alpha_{ij} p^j.$$

Then

$$N_G(D) \simeq \mathrm{GL}(c, q) \times \prod_{i,j} N_{ij} \wr \mathrm{Sym}(\alpha_{ij})$$

where the N_{ij}'s are as in (1.7).

Our next step is to consider the structure of the groups N_{ij}. Let us note the following general elementary lemma.

LEMMA (1.9). *Let S be a transitive subgroup of* Sym(n), $n \in N$, *G an arbitrary finite group and G_1 a subgroup of G.*

(1) *An arbitrary element of* $N = N_{G \wr \mathrm{Sym}(n)}(G_1 \wr S)$ *can be written as* $(x_1, \ldots, x_n; \xi)$, *where* $x_i \in N_G(G_1)$, $x_i \equiv x_j \pmod{G_1}$ *for all i, j and* $\xi \in N_{\mathrm{Sym}(n)}(S)$.

(2) *In particular,*

$$N_{G \wr \mathrm{Sym}(n)}(G_1 \wr S)/G_1 \wr S \simeq (N_G(G_1)/G_1) \times (N_{\mathrm{Sym}(n)}(S)/S).$$

(3) *An element of* $C_{G \wr \mathrm{Sym}(n)}(G_1 \wr S)$ *has the form* $(x, \ldots, x; 1)$ *where* $x \in C_G(G_1)$.

PROOF. By an elementary calculation, using the transitivity of S.

LEMMA (1.10). (1) *Let* $G_{ij} = \mathrm{GL}(p^{i+j}e, q)$. *Then*

$$N_{ij} = N_{G_{ij}}(D_{ij}) \simeq N_{M \wr \mathrm{Sym}(p^j)}(\mathbf{Z}_{p^{a+i}} \wr R_j).$$

Here $\mathbf{Z}_{p^{a+i}}$ *is considered a subgroup of (the Singer-cycle of)* $\mathrm{GL}(p^ie, q)$ *and* $M = N_{\mathrm{GL}(p^ie,q)}(\mathbf{Z}_{p^{a+i}})$.

(2) $D_{ij} \cdot C_{G_{ij}}(D_{ij}) \simeq D_{ij} \times \mathbf{Z}_r$, *where* $q^{p^ie} - 1 = p^{a+i} \cdot r$.

PROOF. We apply (1.4)(2). An argument similar to that in the proof of (1.7) (about the multiplicity of 1 as eigenvalue) shows that every element of $N_{G_{ij}}(A)$ and therefore $N_{G_{ij}}(D_{ij})$ permutes the subgroups $A^{(k)}$ by conjugation (in the notation of (1.4)). It follows that $N_{G_{ij}}(A) \simeq M \wr \mathrm{Sym}(p^j)$. By induction on j it is easily seen that R_j is a transitive subgroup of $\mathrm{Sym}(p^j)$. (1.9)(3) and Lemma 2.1 in [6] proves (2).

Let us note that (1.9)(1) gives a description of the elements of N_{ij}. M is an extension of $\mathbf{Z}_{(q^{ep^i}-1)}$ by a cyclic group of order p^ie (by II, 7.3 in [7]).

If $n = c + em$, $m = \Sigma m_ip^i$, $m_i = \Sigma \alpha_{ij}p^j$ is a splitting of n and D the corresponding group, the results above imply that if $G = \mathrm{GL}(n, q)$, then

$$D \cdot C_G(D) \simeq \mathrm{GL}(c, q) \times D \times C,$$

where C is an abelian p'-group. By (5C) in [3], D occurs as a p-defect group for G if and only if $\mathrm{GL}(c, q)$ has a p-block of defect 0, and there exists an $N_G(D)$-conjugacy class of irreducible characters for C, satisfying a certain inertial condition. $\mathrm{GL}(c, q)$ has p-blocks of defect 0, if and only if, $e > 1$, as we shall see in the next section. In many cases it is possible to determine $N_G(D)$-conjugacy classes of characters of C, satisfying the inertial condition, e.g. if $\alpha_{ij} < p$ for all i, j.

2. **The degree formula and blocks of defect 0.** We study the degree formula for the irreducible characters of GL(n, q), as given on p. 44 in [6].

Par(n) is the set of partitions of the positive integer n. To $\lambda = (l_1, \ldots, l_r) \in \mathrm{Par}(n)$, $l_1 \geqslant \cdots \geqslant l_r$, we associate the *Young-diagram* $Y(\lambda)$ of n nodes (boxes) ([8], [13]). $H(\lambda)$ denotes the collection of n integers, which are the *hook-lengths* of $Y(\lambda)$ (see [13, p. 44]).

EXAMPLE. If $\lambda = (5, 3, 2) \in \mathrm{Par}(10)$, then

$$Y(\lambda) = \begin{array}{l} \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot \end{array} \qquad \text{and}$$

$$H(\lambda) = \{1, 1, 1, 2, 2, 3, 4, 4, 6, 7\}.$$

If k is a positive integer define

$$\phi_k(t) = (1 - t)(1 - t^2) \cdots (1 - t^k),$$

$$\phi_0(t) = 1 \quad \text{and} \quad \psi_k(t) = (-1)^k \phi_k(t).$$

If $\lambda \in \mathrm{Par}(n)$ is as above and q is a prime power, let

$$(\lambda : q) = \prod_{i=1}^{r} \phi_{l_i + r - i}(q) \prod_{1 \leqslant i < j \leqslant r} (1 - q^{l_i - l_j - i + j}).$$

Let us note:

PROPOSITION (2.1). $(\lambda : q) = (-1)^n \Pi_{h \in H(\lambda)}(q^h - 1)$.

PROOF. This is proved exactly as the degree formula 2.37 in Robinson's book. Robinson's proof does not give any details, but these can be found in Frame, Robinson and Thrall's work on hook-lengths [5]. Their Lemma 1 can be formulated as follows: Let $H_i(\lambda)$ be the set of hook-lengths for the nodes in the ith row of $Y(\lambda)$. (So $H(\lambda) = \bigcup_{i=1}^{r} H_i(\lambda)$.) Then $\{1, 2, \ldots, l_i + r - i\} = H_i(\lambda) \cup \{l_i - l_j - i + j \mid i + 1 \leqslant j \leqslant r\}$ $(h_{i1} = h_i + r = i)$.

Using this, the proposition is immediate from the definition of $(\lambda : q)$.

This proposition is the main tool in the classification of the p-blocks of defect 0. First we need some more definitions.

Index the set $F = \{f_{ij}\}$ of irreducible polynomials over GF(q) (omitting the linear polynomial with root 0) in such a way that $\{f_{i1}, \ldots, f_{is_i}\}$ is the set of polynomials in F which are of degree i, $i = 1, 2, \ldots$. An *index for n* is a list of partitions (ν_{ij}), $i = 1, 2, \ldots, j = 1, 2, \ldots, s_i$, satisfying $\Sigma_{i,j} |\nu_{ij}| i = n$, where $|\nu_{ij}|$ is the integer of which ν_{ij} is a partition.

The set of indices for n is in a canonical one-to-one correspondence with the set of characters and the set of conjugacy classes of GL(n, q).

Two indices (ν_{ij}) and (μ_{ij}) are called *equivalent*, if for each partition λ and each $i \geq 1$, $|\{j|\nu_{ij} = \lambda\}| = |\{j|\mu_{ij} = \lambda\}|$ (the cardinalities coincide). The equivalence classes are called *types* for n, and we denote by $\{\nu_{ij}\}$ the type containing the index (ν_{ij}).

An irreducible character (or a conjugacy class) for $GL(n, q)$ is said to be of *type* $\{\nu_{ij}\}$, if the corresponding index belongs to that type.

Characters of the same type have the same degree, and the class number for conjugacy classes of the same type is also constant.

The degree for an irreducible character χ of type T is

$$\chi(1) = \psi_n(q) \prod_{\nu_{ij} \in T} (-1)^{|\nu_{ij}|} \{\nu_{ij} : q\}, \tag{2.2}$$

where, if $\lambda = (l_1, \ldots, l_r)$, $l_1 \geq l_2 \geq \cdots \geq l_r$,

$$\{\lambda : q\} = q^{l_2 + 2l_3 + \cdots} / (\lambda : q)$$

and $(\lambda : q)$ is defined above.

If $\lambda \in \mathrm{Par}(n)$ and $k_1 \geq \cdots \geq k_s > 0$ are the parts of the conjugate partition, let

$$n_\lambda = \sum_{i=1}^{s} \binom{k_i}{2}$$

and

$$a_\lambda(q) = q^{|\lambda| + 2n_\lambda} \prod_{i=1}^{s} \phi_{k_i - k_{i+1}}\left(\frac{1}{q}\right)$$

$(k_{s+1} = 0)$. If $x \in c$, a conjugacy class of $G = GL(n, q)$ of type $T = \{\nu_{ij}\}$, then

$$|C_G(x)| = \prod_{\nu_{ij} \in T} a_{\nu_{ij}}(q^i). \tag{2.3}$$

If $\lambda \in \mathrm{Par}(n)$, $k \in N$, we say that λ is *k-irreducible* if $H(\lambda)$ does not contain an integer which is a multiple of k. We have in fact that λ is k-irreducible if and only if $k \notin H(\lambda)$ by §5 in [10].

From (2.1) and (2.2) we get

THEOREM (2.4). *Let $T = \{\nu_{ij}\}$ be a type for n, p a prime of degree $e \leq n$ w.r.t. q. The following statements are equivalent*:

(i) *Characters of type T are in p-blocks of defect* 0.

(ii) *For all i, j, ν_{ij} is $e/(e, i)$-irreducible.*

PROOF. By (2.2), (i) is equivalent to

(ii)′ $p \nmid (\nu_{ij} : q^i)$ for all i, j.

So we need only show $p \nmid (\nu_{ij} : q^i)$ if and only if ν_{ij} is $e/(e, i)$-irreducible. By (2.1)

$$(\nu_{ij} : q^i) = \pm \prod_{h \in H(\nu_{ij})} (q^{hi} - 1).$$

Since p is of degree e w.r.t. q we have for all $k \in N$ that $p \mid (q^k - 1)$ if and only if $e \mid k$. Thus

$$p \mid (\nu_{ij} : q^i) \iff e \mid hi \text{ for some } h \in H(\nu_{ij})$$

$$\iff \frac{e}{(e, i)} \mid h \text{ for some } h \in H(\nu_{ij})$$

$$\iff \nu_{ij} \text{ is } \frac{e}{(e, i)}\text{-reducible.}$$

The similarity between the above result and an analogous result for the symmetric group is obvious: If $\lambda \in \mathrm{Par}(n)$ and X_λ is the corresponding character for Sym(n), then X_λ is a p-block of defect 0 if and only if λ is p-irreducible. So, in this sense, the degree of a prime divisor in $|\mathrm{GL}(n, q)|$ behaves as the prime divisors of $|\mathrm{Sym}(n)|$, and types for GL(n, q) correspond to partitions for Sym(n).

Let us note the following:

COROLLARY (2.5). *If $p \mid |\mathrm{GL}(n, q)|$ is of degree e, then* GL(n, q) *has p-blocks of defect* 0, *if and only if,* $e > 1$.

PROOF. The "only if" part is clear, since $|Z(\mathrm{GL}(n, q))| = q - 1$ and p-defect groups always contain p-elements in the center of a group.

To prove the "if"-part, we consider 2 cases.

(1) $e \nmid n$. Let $T = \{\nu_{n1}\}, \nu_{n1} = (1)$.

(2) $e \mid n$. Let $T = \{\nu_{11}, \nu_{n-11}\}, \nu_{11} = \nu_{n-11} = (1)$.

In both cases it follows from (2.4) that characters of type T are in p-blocks of defect 0.

Lemma 2 in [5] can be reformulated as follows:

LEMMA (2.6). *Let* $\lambda \in \mathrm{Par}(k)$, $k \in N$, $h \in H(\lambda)$. *Suppose that the hook H in $Y(\lambda)$ has length $h = n \cdot m$, $n \geq 1$, $m \geq 1$. Then exactly n of the h hook-lengths of nodes in H are divisible by m.*

In the final part of this section we describe the power of a prime dividing the degree of a character of GL(n, q). This is done by modifying Nakayama's highly original approach in the case of the symmetric group (§6–§8 in [10]).

DEFINITION. Suppose that the prime p is of degree e w.r.t. q. Let $\lambda \in \mathrm{Par}(n)$, $n \in N$. The *(e, p)-series for* λ is defined as follows. Determine the largest integer μ_1, such that $p^{\mu_1} e \in H(\lambda)$. Remove a hook of length $p^{\mu_1} e$ from λ to get $\tilde{\lambda}$. Let μ_2 be the largest integer $\leq \mu_1$, such that $p^{\mu_2} e \in H(\tilde{\lambda})$ and remove a hook of length $p^{\mu_2} e$ from $\tilde{\lambda}$. Continuing this process as long as possible gives the (e, p)-series $p^{\mu_1} e, p^{\mu_2} e, \ldots, p^{\mu_k} e$ $(\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k \geq 0)$ for λ. By

§4 in [10], this series is independent of the choice of hooks.

PROPOSITION (2.7). *Let p be a prime of degree e w.r.t. q, $a = \nu(q^e - 1)$. Let $\lambda \in \mathrm{Par}(n)$ and let $p^{\mu_1}e, \ldots, p^{\mu_k}e$ be the (e, p)-series for λ. Then $\nu((\lambda : q)) = b$, where*

$$b = \sum_{r=1}^{k} [p^{\mu_r}b + (p^{\mu_r - 1} + p^{\mu_r - 2} + \cdots + 1)].$$

PROOF. Sketched. (Missing details can easily be obtained by studying §7–§8 in [10].) Let $\lambda = (\alpha_1, \ldots, \alpha_t)$, $\alpha_1 \geqslant \cdots \geqslant \alpha_t$.

If $\lambda' = (\gamma_1, \ldots, \gamma_{t'})$ is the dual partition to λ, define $\beta_i = \alpha_i + (t - i)$ for $1 \leqslant i \leqslant t$ and $\delta_j = \gamma_j + (t' - j)$ for $1 \leqslant j \leqslant t'$.

Let H_{ij} be the hook in $Y(\lambda)$ with the (i, j)th node as corner and let h_{ij} be the hook-length of H_{ij}. Assume $h_{ij} = p^{\mu_1}e$, and pick sequences of integers

$$i_0 = i > i_1 > \cdots > i_u, \qquad j_0 = j > j_1 > \cdots > j_v$$

of maximal length such that

$$\beta_{i_\rho} - \beta_i = \rho p^{\mu_1 + 1}e, \qquad \rho = 0, 1, \ldots, u,$$

$$\delta_{j_\rho} - \delta_j = \rho p^{\mu_1 + 1}e, \qquad \rho = 0, 1, \ldots, v.$$

An argument of Nakayama shows that there exist no i' or j' such that

$$\beta_{i'} - \beta_i = u'p^{\mu_1 + 1}e \quad \text{for } u' > u,$$

$$\delta_{j'} - \delta_j = v'p^{\mu_1 + 1}e \quad \text{for } v' > v.$$

For $1 \leqslant u' \leqslant u$, $1 \leqslant v' \leqslant v$

$$h_{i_{u'}j_{v'}} = e(p^{\mu_1} + (u' + v')p^{\mu_1 + 1})$$

(the hook-length for the $(i_{u'}, j_{v'})$-hook of λ). The (e, p)-series for λ is $p(u + v) + 1$ times $p^{\mu_1}e$ followed by the (e, p)-series of $\tilde{\tilde{\lambda}}$, where $\tilde{\tilde{\lambda}}$ is obtained by removing the (i_u, j_v)-hook from λ.

On the other hand, if

$$u = \sum_{i=0}^{l} a_i p^i \quad \text{and} \quad v = \sum_{i=0}^{m} b_i p^i$$

are the p-adic decompositions of u and v, then in a suitable ordering $p^{l+\mu_1+1}e, \ldots, p^{l+\mu_1+1}e$ (a_l times), $\ldots, p^{\mu_1+1}e, \ldots, p^{\mu_1+1}e$ (a_0 times) $p^{m+\mu_1+1}e, \ldots, p^{m+\mu_1+1}e$ (b_m times), $\ldots, p^{\mu_1+1}e, \ldots, p^{\mu_1+1}e$ (b_0 times) are the first terms in the (e, p)-series for $\tilde{\lambda}$, where $\tilde{\lambda}$ is obtained by removing H_{ij} from λ. The remainder of the (e, p)-series for $\tilde{\lambda}$ coincides with the (e, p)-series for $\tilde{\tilde{\lambda}}$.

Let us prove

$$\nu(\lambda : q) - \nu(\tilde{\lambda} : q)$$

(1) $$= p^{\mu_1}a + [p^{\mu_1 - 1} + p^{\mu_1 - 2} + \cdots + 1] - (u + \nu(u!)) - (v + \nu(v!)).$$

To do this we compare the hook-lengths for λ and $\tilde{\tilde{\lambda}}$ (and use (2.1)). This has been analyzed by Nakayama [10, §3].

We get that, if H_{ij} are the hook-lengths in H_{ij}, then

$$\nu(\lambda : q) - \nu(\tilde{\lambda} : q) = \sum_{h \in H_{ij}} \nu(q^h - 1) + \sum_{k=1}^{i-1} \nu\left(\frac{q^{h_{ik}} - 1}{q^{h_{ik} - h_{ij}} - 1}\right)$$

$$+ \sum_{l=1}^{j-1} \nu\left(\frac{q^{h_{lj}} - 1}{q^{h_{lj} - h_{ij}} - 1}\right).$$

Using (1.1) it follows from (2.6) [with $h = p^{\mu_1}e$ and $m = p^r e, r \leqslant \mu_1$] that

(2) $$\sum_{h \in H_{ij}} \nu(q^h - 1) = p^{\mu_1}a + \sum_{k=0}^{\mu_1 - 1} p^k.$$

Since $h_{ij} = p^{\mu_1}e$, it follows that $\nu(q^{h_{ik}} - 1) = \nu(q^{h_{ik} - h_{ij}} - 1)$ unless $p^{\mu_1 + 1}e \mid h_{ik} - h_{ij}$, in which case $\nu(q^{h_{ik} - h_{ij}} - 1) = a + \nu(h_{ik} - h_{ij})$. Now $h_{ik} - h_{ij} = \delta_k - \delta_j$, so $p^{\mu_1 + 1}e \mid h_{ik} - h_{ij}$ happens exactly v times. It follows that

$$\sum_{k=1}^{i-1} \nu\left(\frac{q^{h_{ik}} - 1}{q^{h_{ik} - h_{ij}} - 1}\right) = -[v + \nu(v!)]$$

and similarly we get

$$\sum_{l=1}^{j-1} \nu\left(\frac{q^{h_{lj}} - 1}{q^{h_{lj} - h_{ij}} - 1}\right) = -[u + \nu(u!)]$$

so (1) is proved, using (2).

Now it is a fairly straightforward calculation to finish the proof of (2.7). By induction we can assume (2.7) true for $\tilde{\lambda}$ and $\tilde{\tilde{\lambda}}$, so we know $\nu(\tilde{\lambda} : q)$. Since we also know $\nu(\lambda : q) - \nu(\tilde{\lambda} : q)$, it is readily computed that

$$\nu(\lambda : q) - \nu(\tilde{\tilde{\lambda}} : q) = (p(u + v) + 1)\left[p^{\mu_1}a + \sum_{k=0}^{\mu_1 - 1} p^k\right].$$

$\nu(\tilde{\tilde{\lambda}} : q)$ is known by induction, and we are done.

Using (2.7) and (2.2) we get a description of the power of p dividing the degree of an irreducible character χ of type T. We note that if p is of degree

e w.r.t. q, then p is of degree $e/(e, i)$ w.r.t. q^i for any $i \in N$. Thus (2.7) also gives an alternative proof of Theorem (2.4).

If $\lambda \in \mathrm{Par}(n)$ and $r \in N$, let

$$H(\lambda)_r = \{h' \in N \mid \exists h \in H(\lambda): h = rh'\}.$$

If $|H(\lambda)_r| = b$, one can define a (reducible) representation for Sym(b), called the *r-quotient for* λ, which is of degree $f_\lambda^{(r)} = b!/(\Pi_{h \in H(\lambda)_r} h)$. (See [13, §4.4] or [5].)

A type $T = \{\nu_{ij}\}$ for n is called *primary*, if $|\{\nu_{ij} | \nu_{ij} \neq 0\}| = 1$, and an irreducible character for GL(n, q) of primary type is called *primary irreducible.*

Let χ be a primary irreducible character for GL(n, q) of type T, where, if λ is the nonzero partition of T, $n = |\lambda|d$. Moreover, let p be a prime of degree $e \leqslant n$ w.r.t. q and $a = \nu(q^e - 1)$. Put $e_1 = e/(e, d)$, and assume $|H(\lambda)_{e_1}| = b$.

LEMMA (2.8). *In the above notation*

$$\nu(\chi(1)) = \nu\,|\mathrm{GL}(n, q)| - \nu\,|\mathrm{GL}(b, q^{ep^{\nu(d)}})| + \nu(f_\lambda^{(e_1)}).$$

PROOF. By (2.2) we need only show

$$\nu(\lambda : q^d) = \nu\,|\mathrm{GL}(b, q^{ep^{\nu(d)}})| - \nu(f_\lambda^{(e_1)}).$$

By (2.1) and (1.1)

$$\nu(\lambda : q^d) = \sum_{h \in H(\lambda)} \nu(q^{hd} - 1) = b(a + \nu(d)) + \sum_{h' \in H(\lambda)_{e_1}} \nu(h').$$

The last sum in this equation is equal to $\nu(b!) - \nu(f_\lambda^{(e_1)})$, so

$$\nu(\lambda : q^d) = b(a + \nu(d)) + \nu(b!) - \nu(f_\lambda^{(e_1)}).$$

Now the result follows from (1.2) with $s = b$ and $e = ep^{\nu(d)}$.

3. **A general result.** In §2 we determined the types of blocks of defect 0. It still remains to determine the distribution of the other characters into p-blocks. As a step towards this, the result below may be useful.

If χ_i is an irreducible character of GL(n_i, q), $i = 1, 2$, then one can define an irreducible character $\chi_1 \circ \chi_2$ of GL($n_1 + n_2$, q). (See [6, p. 410].) A character χ of type $T = \{\nu_{ij}\}$ can be written as a $\circ$-product of primary irreducible characters of the smaller dimensional linear groups (GL($|\nu_{ij}|i$, q)).

We prove:

PROPOSITION (3.1). *Let p be a prime, $p \nmid q$. Let α and α' be irreducible characters of $G_1 = \mathrm{GL}(n, q)$ in the same p-block and let β be an irreducible char-*

acter for $G_2 = \mathrm{GL}(m, q)$. *Then* $\alpha \circ \beta$ *and* $\alpha' \circ \beta$ *belong to the same p-block for* $G = \mathrm{GL}(n + m, q)$.

PROOF. We use Green's notation. By assumption, if $x_1 \in c_1$, where c_1 is a conjugacy class of G_1, then

$$(1) \qquad \frac{|G_1|}{|C_{G_1}(x_1)|}\frac{\alpha(x_1)}{\alpha(1)} \equiv \frac{|G_1|}{|C_{G_1}(x_1)|}\frac{\alpha'(x_1)}{\alpha'(1)} \pmod P$$

where P is a suitable prime-ideal.

If $x \in c$, where $c = (\cdots f^{\nu(f)} \cdots)$ is a conjugacy class for G, then by (2.3)

$$|C_G(x)| = \prod_{f \in F} a_{\nu(f)}(q^{d(f)}),$$

where $d(f)$ is the degree of f. We also have

$$(\alpha \circ \beta)(1) = \frac{\psi_{n+m}(q)}{\psi_n(q)\psi_m(q)}\,\alpha(1)\beta(1)$$

by Lemma 2.7 in [6].

Thus

$$\frac{|G|}{|C_G(x)|(\alpha \circ \beta)(1)} = q^r\,\frac{|G_1|}{\alpha(1)}\cdot\frac{|G_2|}{\beta(1)}\cdot\frac{1}{\Pi_f a_{\lambda(f)}(q^{d(f)})}$$

where r is some nonnegative integer.

So by Theorem 2 and Lemma 2.6 in [6],

$$(2) \qquad \begin{aligned} &\frac{|G|}{|C_G(x)|}\frac{1}{(\alpha \circ \beta)(1)}(\alpha \circ \beta)(x) \\ &\quad = q^r \sum_{c_1, c_2} \frac{|G_1|}{\alpha(1)}\,\alpha(x_1)\,\frac{|G_2|}{\beta(1)}\,\beta(x_2) \prod_{f \in F} \frac{g^{\lambda(f)}_{\lambda_1(f),\lambda_2(f)}(q^{d(f)})}{a_{\lambda(f)}(q^{d(f)})} \end{aligned}$$

where $c_i = (\cdots f^{\lambda_i(f)} \cdots)$ runs through the conjugacy classes of G_i, $x_i \in c_i$, $i = 1, 2$ and $g^{\lambda}_{\lambda_1,\lambda_2}(q)$ is a Hall-polynomial.

We exploit a remark of Morris in [9]. What he denotes $F^{\lambda}_{\alpha\beta}$ is $q^{-n_\lambda + n_\alpha + n_\beta} g^{\lambda}_{\alpha\beta}(1/q)$. We get in fact:

If $\{\lambda\} \subseteq \{\alpha\}\{\beta\}$ (multiplication of Schur functions), then there exists a polynomial $r^{\lambda}_{\alpha\beta}(q)$, such that

$$q^l g^{\lambda}_{\alpha\beta}(q) = \frac{a_\lambda(q)}{a_\alpha(q)a_\beta(q)}\,r^{\lambda}_{\alpha\beta}(q)$$

where l is a suitable integer.

It follows that the right hand side of (2) becomes

$$\sum_{c_1,c_2} q^b \cdot \frac{|G_1|\alpha(x_1)}{|C_{G_1}(x_1)|\alpha(1)} \cdot \frac{|G_2|\beta(\alpha_2)}{|C_{G_2}(x_2)|\beta(1)} \cdot s_{c_1,c_2}(q)$$

where b is some integer (depending on c_1, c_2) and the s_{c_1,c_2}'s are polynomials. Using (1) our result follows.

It is still an open question, to which extent a converse to (3.1) is valid.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, ILLINOIS 60637

DEPARTMENT OF MATHEMATICS, AARHUS UNIVERSITET, AARHUS, DENMARK

Current address: Abteilung Mathematik, Universität Dortmund, D-46 Do-Hombruch, Postfach 500, West Germany