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NECESSARY AND SUFFICIENT CONDITIONS FOR THE DERIVATION OF INTEGRALS OF L_Ψ-FUNCTIONS

BY

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ABSTRACT. It has been shown recently that a necessary and sufficient condition for a derivation basis to derive the μ-integrals of all functions in $L^{(q)}(\mu)$, where $1 < q < +\infty$, and μ is a σ-finite measure, is that the basis possess Vitali-like covering properties, with covering families having arbitrarily small $L^{(p)}(\mu)$-overlap, where $p^{-1} + q^{-1} = 1$. The corresponding theorem for the case $p = 1$, $q = +\infty$ was established by R. de Possel in 1936.

The present paper extends these results to more general dual Orlicz spaces. Under suitable restrictions on the dual Orlicz functions Φ and Ψ, it is shown that a necessary and sufficient condition for a basis to derive the μ-integrals of all functions in $L_\Psi(\mu)$ is that the basis possess Vitali-like covering families whose $L_\Phi(\mu)$-overlap is arbitrarily small. Certain other conditions relating $L_\Phi(\mu)$-strength and derivability are also discussed.

1. General definitions and terminology. Our universe is a set of points S. We shall agree that if $A \subseteq S$ and $B \subseteq S$, then $A - B = \{x: (x \in A) \wedge (x \notin B)\}$; thus $A - B = A - A \cap B$. If $A \subseteq S$, we shall denote the complement of A in S by $\widetilde{A}$. $\mathcal{M}$ denotes a fixed Boolean σ-algebra of subsets of S, with S as its unit; μ denotes a fixed σ-finite measure defined on $\mathcal{M}$, and μ^* is the completion of μ defined on the class $\mathcal{M}^*$ of subsets of S. We let $\mathcal{N}$ and $\mathcal{N}^*$ denote, respectively, the families of μ- and μ^*-nullsets. We let $\bar{\mu}$ denote the outer measure derived from μ. If $X \subseteq S$, then $\bar{X}$ denotes a measure cover of X; it is well known that $\bar{\mu}(X \cap M) = \mu(\bar{X} \cap M)$ holds for each set $M \in \mathcal{M}$ and each μ-cover $\bar{X}$ of X. For any set $X \subseteq S$, we let χ_X denote the characteristic function of X.

A *derivation basis* $\mathfrak{B}$ is defined as follows. We assume that to each point x of a fixed subset E of X, called the *domain* of $\mathfrak{B}$, there correspond Moore-Smith sequences of $\mathcal{M}$-sets of positive μ-measure, called *constituents*, which are said to *converge* to x, and are denoted generically by $\{M_\iota(x)\}$. We further assume (Fréchet's convergence axiom) that each cofinal subsequence of an x-converging sequence also converges to x. The elements of $\mathfrak{B}$ are thus converging sequences together with corresponding convergence points. We denote by

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$\mathcal{D}$ the family of all $\mathfrak{B}$-constituents; i.e., the family of all sets belonging to one or more of the sequences $\{M_\iota(x)\}$ for some $x \in E$. This family $\mathcal{D}$ is called the *spread* of $\mathfrak{B}$.

If λ is a real-valued function defined on $\mathcal{D}$ and $x \in E$, then we define $D^*\lambda(x)$ and $D_*\lambda(x)$ by

$$D^*\lambda(x) = \sup\left[\limsup \frac{\lambda(M_\iota(x))}{\mu(M_\iota(x))}\right] \quad \text{and} \quad D_*\lambda(x) = \inf\left[\liminf \frac{\lambda(M_\iota(x))}{\mu(M_\iota(x))}\right]$$

where the expressions in brackets mean, respectively, the limit superior and inferior of any fixed x-converging sequence $\{M_\iota(x)\}$, and then the supremum and infimum of these values are taken among all such sequences. $D^*\lambda(x)$ and $D_*\lambda(x)$ are called, respectively, the *upper* and *lower* $\mathfrak{B}$*-derivatives of* λ *at* x. If $D^*\lambda(x) = D_*\lambda(x)$ (whether finite or infinite), then their common value is denoted by $D\lambda(x)$, and is called the $\mathfrak{B}$*-derivative of* λ *at* x.

We say that λ is a *μ-finite μ-integral* iff there exists a μ-measurable function f such that $-\infty < \lambda(M) = \int_M f\,d\mu < +\infty$ whenever $M \in \mathcal{M}$ and $\mu(M)$ is finite. We say that λ is $\mathfrak{B}$-derivable iff $D\lambda(x)$ exists and coincides with $f(x)$ for μ^*-almost $x \in E$.

By a *subbasis* of $\mathfrak{B}$ we mean any basis $\mathfrak{B}^*$ whose associated sequences belong to $\mathfrak{B}$ and which associates with these sequences the same convergence points as does $\mathfrak{B}$. Clearly, the spread $\mathcal{D}^*$ of $\mathfrak{B}^*$ is a subset of $\mathcal{D}$. The domain of $\mathfrak{B}^*$ is the set of its associated points, which is a subset of E.

If $X \subseteq E$ and $\mathfrak{B}^*$ is any subbasis of $\mathfrak{B}$ such that the domain of $\mathfrak{B}^*$ includes X (mod $\mathcal{N}^*$), then the spread $\mathcal{V}$ of $\mathfrak{B}^*$ is called a $\mathfrak{B}$*-fine covering of* X. Sometimes a $\mathfrak{B}$-fine covering of X is defined as any family $\mathcal{V} \subseteq \mathcal{D}$ that contains, for μ^*-almost all $x \in X$, the sets of at least one sequence $\{M_\iota(x)\}$. Although these definitions differ slightly, in their applications they have the same effect, so we may use them interchangeably.

If $\mathcal{H}$ is any finite or countably infinite subfamily of $\mathcal{M}$, then for any $x \in S$, we define $n_{\mathcal{H}}(x)$ as the number of members of $\mathcal{H}$ to which x belongs. We denote the union of the family $\mathcal{H}$ by $\bigcup \mathcal{H}$; it is clear that $n_{\mathcal{H}}(x) = 0$ iff $x \in (S - (\bigcup \mathcal{H}))$. We define $e_{\mathcal{H}}(x) = n_{\mathcal{H}}(x) - 1$ if $x \in \bigcup \mathcal{H}$, $e_{\mathcal{H}}(x) = 0$ for all other $x \in S$. Clearly $e_{\mathcal{H}}(x) > 0$ iff x belongs to at least two members of $\mathcal{H}$. We note that $n_{\mathcal{H}}$ and $e_{\mathcal{H}}$ are μ-measurable functions.

Henceforth, ϕ and ψ will denote real-valued functions on $[0, +\infty)$ subject to the conditions

(a) $\phi(0) = 0$; ϕ is nondecreasing on $[0, +\infty)$;

(b) $\psi(0) = 0$; $\psi(u) = \sup\{x\colon \phi(x) < u\}$ for each u, $0 < u < +\infty$.

We call ψ the function *inverse* to ϕ. If ϕ is strictly increasing, then ψ is the conventional inverse. It follows that ψ is left-continuous on $[0, +\infty)$ and nondecreasing. We find it convenient to extend the domains of definitions of ϕ and ψ to include $+\infty$, by agreeing that $\phi(+\infty) = \lim_{u\to+\infty}\phi(u)$ and $\psi(+\infty) = \lim_{u\to+\infty}\psi(u)$.

Next, we define $\Phi(u) = \int_0^u\phi(t)\,dt$, $\Psi(u) = \int_0^u\psi(t)\,dt$ for each $u \in [0, +\infty]$. Clearly, Φ and Ψ are nondecreasing and continuous on $[0, +\infty)$. It follows easily that if f is a μ-measurable function, then $\phi(|f|)$, $\psi(|f|)$, $\Phi(|f|)$, and $\Psi(|f|)$ are also μ-measurable. Moreover, Young's inequality (cf. [7, pp. 76–78]),

$$uv \leqslant \Phi(u) + \Psi(v),$$

holds for all $u, v \geqslant 0$, with equality iff $v = \phi(u)$ or $u = \psi(v)$.

We define $L_\Phi^*(\mu)$ as the class of all μ-measurable functions f for which $\Phi(|f|)$ is μ-summable over S. For any μ-measurable function f we also define

$$\|f\|_\Phi = \sup\left\{\int_S |fg|\,d\mu\colon \int_S \Psi(|g|)\,d\mu \leqslant 1\right\},$$

and we define $L_\Phi(\mu)$ as the class of all functions f with $\|f\|_\Phi < +\infty$. Analogously, we define the classes $L_\Psi^*(\mu)$ and $L_\Psi(\mu)$. $L_\Phi(\mu)$ and $L_\Psi(\mu)$ are normed linear spaces with respect to the norms $\|\ \|_\Phi$ and $\|\ \|_\Psi$, and are called *(dual) Orlicz spaces*. Young's inequality yields

$$\|f\|_\Phi \leqslant \int_S \Phi(|f|)\,d\mu + 1 \quad \text{and} \quad \|g\|_\Psi \leqslant \int_S \Psi(|g|)\,d\mu + 1,$$

whence $L_\Phi^*(\mu) \subseteq L_\Phi(\mu)$ and $L_\Psi^*(\mu) \subseteq L_\Psi(\mu)$. Moreover, if $f \in L_\Phi(\mu)$ or $f \in L_\Psi(\mu)$, then $|f| < +\infty$ holds μ-almost everywhere in S; if $\|f\|_\Phi = 0$ or $\|f\|_\Psi = 0$, then $f = 0$ holds μ-almost everywhere in S. These properties are used in the work to follow. Proofs may be found in [7, pp. 78–82].

If $\mathfrak{B}$ is a basis with domain $E \subseteq S$, then we say that $\mathfrak{B}$ is *L_Φ-strong* iff for each set $X \subseteq E$ of finite $\bar{\mu}$-measure, each $\mathfrak{B}$-fine covering V of X, and each $\epsilon > 0$, there exists a countable family $H \subseteq V$ such that, setting $H = \bigcup H$, we have

(S1) $\mu(\bar{X} - H) = 0$ (H is an 0-covering of X, or H covers μ^*-almost all of X),

(S2) $\mu(H - \bar{X}) < \epsilon$ (the μ-overflow of H with respect to $\bar{X}$ is less than ϵ),

(S3) $\|e_H\|_\Phi < \epsilon$ (the L_Φ-overlap of H is less than ϵ).

It can be shown by an exhaustion process that an equivalent formulation of this definition results if (S1) is replaced by

(S1)′ $\mu(\bar{X} - H) < \epsilon$ (H is an ϵ-covering of $\bar{X}$).

2. **Derivability implies $L_\Phi(\mu)$-strength.** Throughout this section, in addition to the general restrictions imposed on ϕ and ψ in §1, we shall assume:

(I) ϕ is continuous on $(0, +\infty)$ with $\lim_{u\to+\infty}\phi(u) = +\infty$. This implies that $\phi(u)$ and $\psi(u)$ are finite for each u, $0 \leqslant u < +\infty$ and $\lim_{u \to +\infty} \psi(u) = +\infty$; also, by the definition adopted in §1, $\phi(+\infty) = \psi(+\infty) = +\infty$. Consequently, $f \in L^*_\Phi(\mu) \subseteq L_\Phi(\mu)$ and $\phi(f) \in L^*_\Psi(\mu) \subseteq L_\Psi(\mu)$ whenever f is a bounded μ-measurable function vanishing outside a set of finite μ-measure.

(II) There exists a positive number M such that, for each $u \geqslant 0$, $\Phi(2u) \leqslant M\Phi(u)$. This implies $\phi(u) > 0$ for each $u > 0$; in particular, $\phi(1) > 0$. Moreover, it can be shown that (a) $L^*_\Phi(\mu) = L_\Phi(\mu)$; (b) given $\epsilon > 0$ there exists $\eta > 0$ such that $\|f\|_\Phi < \epsilon$ whenever $\int_S \Phi(|f|)\, d\mu < \eta$ (cf. [7, pp. 81, 83]).

We further assume that:

(III) $\mathfrak{B}$ is a derivation basis with domain $E \subseteq S$ that derives the μ-integrals of all functions in $L_\Psi(\mu)$ that vanish outside a set of finite μ-measure. This tacitly requires that if $g \in L_\Psi(\mu)$ $(= L^*_\Psi(\mu))$ and g vanishes outside a set of finite μ-measure, then $\int_S |g|\, d\mu < +\infty$; i.e., g has a μ-finite μ-integral.

2.1. Lemma. *If $\|f\|_\Phi < 1$, then $\phi(|f|) \in L_\Psi$.*

Proof. We first consider a function f, bounded, nonnegative, and vanishing outside a set of finite μ-measure. From (I), we have $f \in L_\Phi(\mu)$ and $\phi(f) \in L_\Psi(\mu)$. From Young's inequality in the special case $u = f$, $v = \phi(f)$ we obtain

$$\int_S \Psi(\phi(f))\, d\mu \leqslant \int_S \Psi(\phi(f))\, d\mu + \int_S \Phi(f)\, d\mu = \int_S f\phi(f)\, d\mu,$$

whence we see (recall §1) that

$$\|\phi(f)\|_\Psi \leqslant \int_S \Psi(\phi(f))\, d\mu + 1 \leqslant \int_S f\phi(f)\, d\mu + 1 \leqslant \|f\|_\Phi \cdot \|\phi(f)\|_\Psi + 1.$$

By hypothesis, $\|f\|_\Phi = k < 1$, so that the preceding inequality yields $\|\phi(f)\|_\Psi \leqslant 1/(1-k) < +\infty$.

In the general case, we may represent $|f|$ as a limit of a nondecreasing sequence $\{f_n\}$ of nonnegative functions, each of which vanishes outside a set of finite μ-measure. Because $f_n \uparrow |f|$ on S, we see that $\|f_n\|_\Phi \leqslant \|f\|_\Phi = k < 1$ and so, by what was just proved, $\|\phi(f_n)\|_\Psi \leqslant 1/(1-k)$ for $n = 1, 2, \ldots$. Using the facts that $\phi(0) = 0$, ϕ is continuous on $(0, +\infty)$, and $f_n \uparrow f$ as $n \to +\infty$, we infer that $\phi(f_n) \uparrow \phi(f)$ on S. Judiciously using the monotone convergence theorem in conjunction with the definition of $\|\ \|_\Psi$, it is essentially routine now to infer that $\|\phi(|f|)\|_\Psi = \lim_{n \to +\infty} \|\phi(f_n)\|_\Psi \leqslant 1/(1-k)$; hence $\phi(|f|) \in L_\Psi(\mu)$.

2.2. Lemma. *If A is an M-set of finite μ-measure, then $\mathfrak{B}$ derives the μ-integrals of χ_A and $\chi_{\tilde{A}}$.*

Proof. It is clearly sufficient to show that $\mathfrak{B}$ derives the μ-integral of χ_A. From (I) we see that $\phi(\chi_A) \in L_\Psi(\mu)$; thus $\mathfrak{B}$ derives the μ-integral of

$\phi(\chi_A)$. However, $\phi(\chi_A) = \phi(1)\chi_A$ and $\phi(1) > 0$; hence $\mathfrak{B}$ derives the μ-integral of χ_A.

2.3. LEMMA. *If H is any finite or countably infinite subfamily of M, then*

$$\int_S \Phi(n_{\mathsf{H}})\, d\mu \leqslant M \int_S \Phi(e_{\mathsf{H}})\, d\mu + \Phi(1)\mu(\bigcup \mathsf{H}).$$

PROOF. Let $A = \{x: n_{\mathsf{H}}(x) \geqslant 2\}$, $B = \{x: n_{\mathsf{H}}(x) = 1\}$ and note that for $x \in A$, $2 \leqslant n_{\mathsf{H}}(x) = e_{\mathsf{H}}(x) + 1 \leqslant 2e_{\mathsf{H}}(x)$. Also, $B \subseteq \bigcup \mathsf{H}$, so that using (II) we obtain

$$\int_S \Phi(n_{\mathsf{H}})\, d\mu = \int_A \Phi(n_{\mathsf{H}})\, d\mu + \int_B \Phi(n_{\mathsf{H}})\, d\mu \leqslant M \int_S \Phi(e_{\mathsf{H}})\, d\mu + \Phi(1)\mu(\bigcup \mathsf{H}).$$

2.4. LEMMA. *Let H denote any finite or countably infinite subfamily of M for which $\int_S \Phi(n_{\mathsf{H}})\, d\mu$ is finite. If W is any M-set and $\mathsf{G} = \mathsf{H} \cup \{W\}$, then*

$$0 \leqslant \int_S \Phi(e_{\mathsf{G}})\, d\mu \leqslant \int_S \Phi(e_{\mathsf{H}})\, d\mu + \int_W \phi(n_{\mathsf{H}})\, d\mu.$$

PROOF. Let $H = \bigcup \mathsf{H}$. We note that $e_{\mathsf{G}}(x) = e_{\mathsf{H}}(x)$ if $x \in (H - W)$; $e_{\mathsf{G}}(x) = 0$ if $x \in (W - H)$; and $e_{\mathsf{G}}(x) = n_{\mathsf{H}}(x)$ if $x \in W \cap H$. Then, because all the following integrals are finite by virtue of our hypotheses, we have

$$\begin{aligned} 0 &\leqslant \int_S \Phi(e_{\mathsf{G}})\, d\mu = \int_{H-W} \Phi(e_{\mathsf{G}})\, d\mu + \int_{W\cap H} \Phi(e_{\mathsf{G}})\, d\mu \\ &= \int_{H-W} \Phi(e_{\mathsf{H}})\, d\mu + \int_{W\cap H} \Phi(n_{\mathsf{H}})\, d\mu \\ &= \int_H \Phi(e_{\mathsf{H}})\, d\mu - \int_{W\cap H} \Phi(e_{\mathsf{H}})\, d\mu + \int_{W\cap H} \Phi(n_{\mathsf{H}})\, d\mu \\ &= \int_S \Phi(e_{\mathsf{H}})\, d\mu + \int_{W\cap H} (\Phi(n_{\mathsf{H}}) - \Phi(e_{\mathsf{H}}))\, d\mu. \end{aligned} \tag{1}$$

Now $\int_S \Phi(n_{\mathsf{H}})\, d\mu$ is finite, so that n_{H} and e_{H} are finite μ-almost everywhere in S. Hence, for μ-almost all $x \in W \cap H$, $n_{\mathsf{H}}(x)$ and $e_{\mathsf{H}}(x)$ are positive integers differing by 1. Applying the mean-value theorem to Φ yields $0 \leqslant \Phi(n_{\mathsf{H}}(x)) - \Phi(e_{\mathsf{H}}(x)) = \phi(\xi)$, where $e_{\mathsf{H}}(x) < \xi < n_{\mathsf{H}}(x)$. Thus $\phi(\xi) \leqslant \phi(n_{\mathsf{H}}(x))$, and therefore

$$0 \leqslant \Phi(n_{\mathsf{H}}) - \Phi(e_{\mathsf{H}}) \leqslant \phi(n_{\mathsf{H}}) \tag{2}$$

holds μ-almost everywhere in $W \cap H$. The desired result is obtained by substituting (2) into the final term of (1) and then observing that $\int_{W\cap H} \phi(n_{\mathsf{H}})\, d\mu = \int_W \phi(n_{\mathsf{H}})\, d\mu$.

2.5. LEMMA. *Suppose that $X \subseteq E$, $\bar{X}$ is any μ-cover of X, $0 < \bar{\mu}(X) = \mu(\bar{X}) < +\infty$, and V is any $\mathfrak{B}$-fine covering of X. Suppose also that $0 < \alpha < 1$ and H is a finite or countably infinite subfamily of M subject to the conditions:*

(i) $\int_S \Phi(e_{\mathsf{H}})\, d\mu \leqslant \alpha\mu(\bar{X} \cap H)$, *where* $H = \bigcup \mathsf{H}$;

(ii) $(1-\alpha)\Sigma_{V\in H}\mu(V) \leqslant \mu(\bar{X} \cap H)$;
(iii) $\mu(\bar{X} - H) > 0$;
(iv) $\phi(n_H) \in L_\Psi$.

Then there exists a set W such that

$$\text{(v)} \quad W \in \mathcal{V} \quad \textit{and} \quad \frac{1}{\phi(1)} \int_W \phi(n_H)\, d\mu + \mu(W - \bar{X}) \leqslant \frac{\alpha}{2(1+\phi(1))}\mu(W).$$

Moreover, if W is any set satisfying (v) *and if we set* $\mathcal{G} = \mathcal{H} \cup \{W\}$, $G = \bigcup \mathcal{G}$, *then*

(vi) $\int_S \Phi(e_G)\, d\mu \leqslant \alpha\mu(\bar{X} \cap G)$ *and*
(vii) $(1-\alpha)\Sigma_{V\in G}\mu(V) \leqslant \mu(\bar{X} \cap G)$.

Proof. From (ii) and the fact that $\mu(\bar{X}) < +\infty$, we see that $\mu(H)$ is finite; (iv) and Lemma 2.2 ensure that $\mathfrak{B}$ derives the μ-integrals of both $\phi(n_H)$ and $\chi_{\tilde{\bar{X}}}$. Thus, if we define

$$\lambda(M) = \frac{1}{\phi(1)} \int_M \phi(n_H)\, d\mu + \mu(M - \bar{X})$$

for each set $M \in \mathcal{M}$, then $\mathfrak{B}$ derives λ. Consequently, because of (iii) and the fact that $\mathcal{V}$ is a $\mathfrak{B}$-fine covering of X, there must exist a point $z \in (X - H)$ with $D\lambda(z) = 0$ and a set W associated with z satisfying (v).

Now suppose that W is an arbitrary set satisfying (v). Then

$$\text{(1)} \qquad \mu(W - (\bar{X} - H)) = \mu(W \cap (\tilde{\bar{X}} \cup H)) \leqslant \mu(W - \bar{X}) + \mu(W \cap H);$$

also $\phi(1)\chi_{W\cap H} = \phi(\chi_{W\cap H}) \leqslant \phi(n_H \cdot \chi_W)$, and therefore

$$\phi(1)\mu(W \cap H) \leqslant \int_W \phi(n_H)\, d\mu.$$

Substituting this last inequality into (1) yields

$$\text{(2)} \qquad \begin{aligned} \mu(W - (\bar{X} - H)) &\leqslant \mu(W - \bar{X}) + \frac{1}{\phi(1)} \int_W \phi(n_H)\, d\mu \\ &\leqslant \frac{\alpha}{2(1+\phi(1))}\mu(W) \leqslant \frac{\alpha}{2}\mu(W), \end{aligned}$$

which easily yields in turn

$$\text{(3)} \quad (1-\alpha/2)\mu(W) \leqslant \mu(W \cap (\bar{X} - H)) \quad \text{and} \quad \mu(W) \leqslant 2\mu(W \cap (\bar{X} - H)).$$

From (3) and (v) we see that

$$\text{(4)} \qquad \int_W \phi(n_H)\, d\mu \leqslant \frac{\alpha \cdot \phi(1)}{2(\phi(1)+1)}\mu(W) \leqslant \alpha\mu(W \cap (\bar{X} - H)).$$

We have seen that $\mu(H)$ is finite; and $\int_S \Phi(e_H)\, d\mu$ is finite by (i); therefore $\int_S \Phi(n_H)\, d\mu$ is finite by Lemma 2.3. From (i), (4), and Lemma 2.4 we obtain

$$\int_S \Phi(e_G)\,d\mu \leqslant \int_S \Phi(e_H)\,d\mu + \int_W \phi(n_H)\,d\mu$$

$$\leqslant \alpha[\mu(\bar{X}\cap H) + \mu(W\cap(\bar{X}-H))] = \alpha\mu(\bar{X}\cap G),$$

which establishes (vi). Finally, from (ii) and (3) we have

$$(1-\alpha)\sum_{V\in G}\mu(V) = (1-\alpha)\sum_{V\in H}\mu(V) + (1-\alpha)\mu(W)$$

$$\leqslant \mu(\bar{X}\cap H) + \left(1-\frac{\alpha}{2}\right)\mu(W)$$

$$\leqslant \mu(\bar{X}\cap H) + \mu(W\cap(\bar{X}-H)) = \mu(\bar{X}\cap G),$$

which confirms (vii).

2.6. THEOREM. $\mathfrak{B}$ *is* L_Φ*-strong.*

PROOF. We choose an arbitrary set $X \subseteq E$ with $0 < \bar{\mu}(X) < +\infty$, select any μ-cover $\bar{X}$ of X, let V denote an arbitrary $\mathfrak{B}$-fine covering of X, and suppose given $\epsilon > 0$. We may and do assume $\epsilon < 1$.

Next, we determine $\eta > 0$ so that, in accordance with (II) (b), $\|f\|_\Phi < \epsilon/2 < 1/2$ whenever $\int_S \Phi(|f|)\,d\mu < \eta$. We may and do suppose that $\eta < \epsilon$. Finally we choose α so that $0 < \alpha < 1$, $\alpha\mu(\bar{X}) < \eta$ and $[\alpha/(1-\alpha)]\mu(\bar{X}) < \eta$.

We define $\lambda(M) = \mu(M - \bar{X})$ for each set $M \in M$. From Lemma 2.2 we know that $\mathfrak{B}$ derives λ. Thus, because $\mu(\bar{X}) > 0$ and V is a $\mathfrak{B}$-fine covering of X, there must exist a point $z \in X$ with $D\lambda(z) = 0$ and a set W associated with z for which

$$(1) \qquad W \in V \quad \text{and} \quad \mu(W - \bar{X}) \leqslant \alpha\mu(W)/2.$$

We let F_1 denote the family of all sets W for which (1) holds. Then $F_1 \neq \emptyset$; also, it follows from (1) that $\mu(W) < 2\mu(\bar{X})$ whenever $W \in F_1$. Hence, if we set $\zeta_1 = \sup_{W\in F_1}\mu(W)$, then $0 < \zeta_1 < +\infty$. We choose a member V_1 of F_1 with $\mu(V_1) > \tfrac{1}{2}\zeta_1$ and set $H_1 = \{V_1\}, H_1 = \bigcup H_1 = V_1$. From (1) and the nature of H_1, it follows readily that H_1 satisfies (i), (ii) and (iv) of Lemma 2.5.

We proceed inductively. We suppose $k \geqslant 1$ and that the family $H_k = \{V_1, V_2, \ldots, V_k\}$ satisfies conditions (i), (ii), and (iv) of Lemma 2.5 with $H_k = \bigcup H_k$. If $\mu(\bar{X} - H_k) = 0$, then we define $H_{k+1} = H_k, H_{k+1} = H_k$, so that H_{k+1} also satisfies (i), (ii), and (iv) of Lemma 2.5.

If $\mu(\bar{X} - H_k) > 0$, we use Lemma 2.5 to see that the family F_{k+1}, consisting of those sets W satisfying the relation

$$(2) \quad W \in V \quad \text{and} \quad \frac{1}{\phi(1)}\int_W \phi(n_{H_k})\,d\mu + \mu(W - \bar{X}) \leqslant \frac{\alpha}{2(1+\phi(1))}\mu(W),$$

is nonempty. Using (2) and following the line of proof of Lemma 2.5 down to (3) of that lemma, we find that $\mu(W) \leqslant 2\mu(W \cap \bar{X}) < +\infty$ whenever $W \in \mathcal{F}_{k+1}$. Thus, setting $\zeta_{k+1} = \sup_{W \in \mathcal{F}_{k+1}} \mu(W)$, we see that $0 < \zeta_{k+1} < +\infty$. We select a member V_{k+1} of $\mathcal{F}_{k+1}$ such that $\mu(V_{k+1}) > \frac{1}{2}\zeta_{k+1}$, and we define $\mathcal{H}_{k+1} = \mathcal{H}_k \cup \{V_{k+1}\}$, $H_{k+1} = \bigcup \mathcal{H}_{k+1}$. It follows that if we put $\mathcal{H}_{k+1} = \mathcal{G}$, then $\mathcal{H}_{k+1}$ satisfies (vi) and (vii) of Lemma 2.5. Also, because $\phi(n_{\mathcal{H}_{k+1}})$ is bounded, $\mu(H_{k+1}) \leqslant \mu(H_k) + \mu(V_{k+1}) < +\infty$, and $n_{\mathcal{H}_{k+1}}$ vanishes outside of H_{k+1}, we see that $\phi(n_{\mathcal{H}_{k+1}}) \in L_\Psi(\mu)$. Thus $\mathcal{H}_{k+1}$ satisfies (i), (ii), and (iv) of Lemma 2.5; and this is true regardless of whether $\mu(\bar{X} - H_k) = 0$ or $\mu(\bar{X} - H_k) > 0$.

We thus obtain inductively a nested sequence $\{\mathcal{H}_k\}$ of finite subfamilies of $\mathcal{V}$ each satisfying (i), (ii), and (iv) of Lemma 2.5. We let $\mathcal{H} = \bigcup_{k=1}^{\infty} \mathcal{H}_k$, $H = \bigcup \mathcal{H}$. Applying the monotone convergence theorem to (i) and (ii) yields

$$\int_S \Phi(e_{\mathcal{H}})\, d\mu \leqslant \alpha\mu(\bar{X} \cap H) \leqslant \alpha\mu(\bar{X}) < \eta \quad \text{and}$$

(3)

$$(1-\alpha)\mu(H) \leqslant (1-\alpha) \sum_{V \in \mathcal{H}} \mu(V) \leqslant \mu(\bar{X} \cap H) \leqslant \mu(\bar{X}) < +\infty.$$

Recalling our conditions on α and η, (3) implies

$$\|e_{\mathcal{H}}\|_\Phi \leqslant 2\|e_{\mathcal{H}}\|_\Phi = \|2e_{\mathcal{H}}\|_\Phi < \epsilon < 1 \quad \text{and}$$

(4)

$$\mu(H - \bar{X}) \leqslant \alpha\mu(H) \leqslant \frac{\alpha}{1-\alpha}\mu(\bar{X}) < \eta < \epsilon.$$

Thus $\mathcal{H}$ satisfies conditions (S2) and (S3) of L_Φ-strength (cf. §1). It remains to be shown that $\mathcal{H}$ covers μ^*-almost all of X. Suppose, on the contrary, $\mu(\bar{X} - H) = \bar{\mu}(X - H) > 0$. Then $\mu(\bar{X} - H_k) \geqslant \mu(\bar{X} - H) > 0$ for $k = 1, 2, \ldots$, which means that the inductive process does not stop producing new sets, and so $\mathcal{H}$ is a countably infinite family of sets $\{V_1, V_2, \ldots, V_k, \ldots\}$ chosen from $\mathcal{V}$, satisfying (3); i.e., (i) and (ii), as well as (iii), of Lemma 2.5.

We wish to show that $\mathcal{H}$ also satisfies (iv) of that lemma. To this end, we set $A = \{x: n_{\mathcal{H}}(x) = 1\}$, $B = \{x: n_{\mathcal{H}}(x) \geqslant 2\}$ and note that $n_{\mathcal{H}} = \chi_A + n_{\mathcal{H}}\chi_B \leqslant \chi_A + 2e_{\mathcal{H}}$, so that $\phi(n_{\mathcal{H}}) \leqslant \phi(\chi_A + 2e_{\mathcal{H}}) = \phi(\chi_A) + \phi(2e_{\mathcal{H}})$. Now χ_A is bounded, $A \subseteq H$, and $\mu(A) \leqslant \mu(H) < +\infty$, and therefore $\chi_A \in L_\Psi$. Also, from (4) and Lemma 2.1, we conclude that $\phi(2e_{\mathcal{H}}) \in L_\Psi$. Accordingly $\phi(\chi_A) + \phi(2e_{\mathcal{H}}) \in L_\Psi$ and therefore $\phi(n_{\mathcal{H}}) \in L_\Psi$.

We are now free to apply Lemma 2.5 to produce a set $W \in \mathcal{V}$ such that

$$\frac{1}{\phi(1)}\int \phi(n_{\mathcal{H}})\, d\mu + \mu(W - \bar{X}) \leqslant \frac{\alpha}{2(1+\phi(1))}\mu(W). \tag{5}$$

From (5) and the fact that $n_{\mathcal{H}_k} \uparrow n_{\mathcal{H}}$ as $k \longrightarrow +\infty$, it follows that the relation

$$\frac{1}{\phi(1)}\int \phi(n_{\mathcal{H}_k})\, d\mu + \mu(W - \bar{X}) \leqslant \frac{\alpha}{2(1+\phi(1))}\mu(W)$$

holds for $k = 1, 2, \ldots$, and therefore $W \in \mathfrak{F}_{k+1}$ for each such k. Hence $0 < \mu(W) \leq \zeta_{k+1} < 2\mu(V_{k+1})$ for $k = 1, 2, \ldots$. However, from (3) we infer that $\mu(V_k) \longrightarrow 0$ as $k \longrightarrow +\infty$. This contradiction forces us to conclude that $\mu(\bar{X} - H) = 0$ and completes the proof of the theorem.

Theorem 2.6 can be applied in many situations; in particular, if $\phi(u) = u^{p-1}$ for all $u \geq 0$, where $p > 1$, we find that $\Phi(u) = u^p$ and $\Psi(u) = u^q$ (to within multiplicative constants) for all $u \geq 0$, and q satisfies the relation $p^{-1} + q^{-1} = 1$. We can assert that if $\mathfrak{B}$ derives the μ-integrals of all functions in $L^{(q)}(\mu)$, then $\mathfrak{B}$ is $L^{(p)}(\mu)$-strong. (Cf. [1], [2], [4] and [5, pp. 35–40] for results on this and related problems.)

Unfortunately, the theorem is inapplicable in the classic case $\Psi(u) = u(\log^+ u)^{n-1}$ that arises in connection with the interval basis in Euclidean n-space, $n \geq 2$. Here, it turns out that $\Phi(u)$ is an exponential function for u sufficiently large, and so fails to satisfy (II). Attempts by the writer to circumvent this difficulty have been unsuccessful. A. Cordoba [1] has some results in this connection.

3. Some additional conditions related to L_Φ-strength and derivability.

As in §2, $\mathfrak{B}$ denotes a derivation basis with domain $E \subseteq S$.

3.1. Definition. If $X \subseteq S$ then a point $x \in S$ is said to be *totally interior to* X (with respect to $\mathfrak{B}$) iff for each x-converging sequence $\{M_\iota(x)\}$ there exists some index ι_0 such that $M_\iota(x) \subseteq X$ whenever $\iota \succ \iota_0$. We let $I(X)$ denote the set of points that are totally interior to X. If G is such a subset of S that $E \cap G \subseteq I(G)$ (mod $\mathfrak{N}^*$), then G is called a *D-open* set (named after A. Denjoy). We let $\mathfrak{G}$ denote the family of all such sets.

3.2. Definition. We say that condition (G_σ) holds iff S is the union of a nondecreasing sequence $\{G_n^0\}$ of $\mathfrak{G}$-sets such that $G_n^0 \in \mathfrak{M}$ and $\mu(G_n^0) < +\infty$ for $n = 1, 2, \ldots$.

In what follows we shall quote, without proof, several theorems taken from [3]. These were proved under a definition of (G_σ) slightly more restrictive than the one given in 3.2; however, those theorems are valid under the slightly weaker form of (G_σ) above.

3.3. Theorem. *If* (G_σ) *holds and* $\mathfrak{B}$ *is* $L_\Phi(\mu)$*-strong, then* $\mathfrak{B}$ *derives the* μ*-integrals of all functions in* $L_\Psi(\mu)$*, whose* μ*-integrals are* μ*-finite.*

From Theorems 2.6 and 3.3, we obtain

3.4. Corollary. *If* ϕ *and* Φ *satisfy the conditions of* §2 *and* (G_σ) *holds, then* $L_\Phi(\mu)$*-strength of* $\mathfrak{B}$ *is equivalent to the* $\mathfrak{B}$*-derivability of all functions in* $L_\Psi(\mu)$ *whose* μ*-integrals are* μ*-finite.*

3.5. DEFINITION. If $\mathcal{H}$ is any countable subfamily of $\mathcal{M}$ and $0 < \alpha < +\infty$, then we define $\mathcal{H}(\alpha)$ as the family of those members V of $\mathcal{H}$ for which $\int_V \phi(e_{\mathcal{H}})\, d\mu \leq \alpha\mu(V)$; also, we define $\mathcal{H}'(\alpha) = \mathcal{H} - \mathcal{H}(\alpha)$.

3.6. DEFINITION. *Condition* (C1). To each $\epsilon > 0$, each $\alpha > 0$, each set $X \subseteq E$ of finite $\overline{\mu}$-measure, each $z > \mu(\overline{X})$ and each $\mathfrak{B}$-fine covering $\mathcal{V}$ of X, there exists a finite family $\mathcal{H} \subseteq \mathcal{V}$ for which, setting $H = \bigcup \mathcal{H}$, we have

$$\mu(\overline{X} - H) < \epsilon; \qquad \sum_{V \in \mathcal{H}} \mu(V) < z; \qquad \mu(\bigcup \mathcal{H}'(\alpha)) < \epsilon.$$

3.7. DEFINITION. *Condition* (C2). To each $\epsilon > 0$, each set $X \subseteq E$ of finite $\overline{\mu}$-measure, and each $\mathfrak{B}$-fine covering $\mathcal{V}$ of X, there exists a finite family $\mathcal{H} \subseteq \mathcal{V}$ for which, putting $H = \bigcup \mathcal{H}$, we have

$$\mu(\overline{X} - H) < \epsilon; \qquad \mu(H - \overline{X}) < \epsilon; \qquad \int_S e_{\mathcal{H}} \phi(e_{\mathcal{H}})\, d\mu < \epsilon.$$

3.8. THEOREM. *If ϕ and Φ satisfy the conditions of* §2 *and* (G_σ) *holds, then* (C1) $\longrightarrow$ (C2) $\longrightarrow$ $\mathfrak{B}$ *is $L_\Phi(\mu)$-strong.*

3.9. DEFINITION. Let $\mathcal{U}$ be a nonempty subfamily of $\mathcal{M}$ whose members are of positive μ-measure, and suppose that δ is a positive real-valued function on $\mathcal{U}$. Let E denote the set of points x in S for which there exists at least one ordinary sequence $\{V_n\}$ with $x \in V_n$, $V_n \in \mathcal{U}$, $n = 1, 2, \ldots,$ and $\lim_{n \to +\infty} \delta(V_n) = 0$. We define a basis $\mathfrak{B}$ by associating with each $x \in E$ the totality of sequences just described. The domain of $\mathfrak{B}$ is clearly E and its spread is a subset of $\mathcal{U}$. We call such a basis $\mathfrak{B}$ a $[\mathcal{U}, \delta]$*-basis* [5, p. 8].

3.10. THEOREM. *If $\mathfrak{B}$ is a $[\mathcal{U}, \delta]$-basis,* (G_σ) *holds, ϕ satisfies the conditions of* §2, *and both Φ and Ψ satisfy condition* (II) *of* §2, *then* (C1) $\longleftrightarrow$ (C2) $\longleftrightarrow$ $\mathfrak{B}$ *is $L_\Phi(\mu)$-strong.*

As a result of Corollary 3.4, we obtain the following:

3.11. COROLLARY. *Under the assumptions of Theorem* 3.10, (C1) $\longleftrightarrow$ (C2) $\longleftrightarrow$ $\mathfrak{B}$ *is $L_\Phi(\mu)$-strong* $\longleftrightarrow$ $\mathfrak{B}$ *derives the μ-integrals of all functions in $L_\Psi(\mu)$ whose μ-integrals are μ-finite.*

We note that Corollary 3.4 establishes the equivalence of $L^{(p)}(\mu)$-strength of $\mathfrak{B}$ and the $\mathfrak{B}$-derivability of the μ-integrals of all $L^{(q)}(\mu)$-functions, where $p^{-1} + q^{-1} = 1$. Also, because in this case $\Phi(u) = u^p$, $\Psi(u) = u^q$ (to within constant multipliers) and both Φ and Ψ satisfy (II) of §1, it follows from Corollary 3.11 that if $\mathfrak{B}$ is a $[\mathcal{U}, \delta]$-basis as well, then all four conditions named in that corollary are equivalent.

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