

SPHERICAL DISTRIBUTIONS ON LIE GROUPS AND C^∞ VECTORS

BY

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ABSTRACT. Given a Lie group G (not necessarily unimodular) and a subgroup K of G (not necessarily compact), it is shown how to associate with every finite-dimensional unitary irreducible representation δ of K a class of distributions analogous to the class of spherical functions of height δ familiar from the unimodular-maximal compact case. The two concepts agree as nearly as possible. A number of familiar theorems are generalized to our situation. As an application we obtain a generalization of the Frobenius reciprocity theorem and of Plancherel's theorem to arbitrary induced representations of Lie groups.

I. Introduction. Let G be a Lie group (not necessarily unimodular) and K a closed (but *not* necessarily compact!) subgroup. In this paper we show how to associate with every finite-dimensional irreducible unitary representation δ of K a class of distributions which is analogous to the class of spherical trace functions of height δ on a unimodular group having a large compact subgroup (see [1]). We obtain generalizations of a large number of results familiar from the large compact case. For example, we obtain an essentially one-to-one correspondence between spherical distributions and certain representations of an algebra of K central distributions.

In the unimodular-large compact case our concept agrees with the usual concept of spherical trace function [1] to the extent that the distributions we study are given via integration against spherical trace *functions*. However, we do not, in general, obtain all spherical trace functions in this manner. The reason for this is our insistence upon unitarity. It can be shown (although we will not do so here) that a development of spherical functions entirely analogous to the one adopted here can be carried out in the large compact unimodular case utilizing nonunitary induced representations, which does yield all spherical tract functions.

It is interesting (although perhaps not surprising) that in the nonunimodular case the nonunimodular trace operators Q_π associated with the nonunimodular Plancherel formula play a significant role.

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To motivate our generalization, suppose that G is compact. Let U be the unitary representation of G induced by δ acting via right translation in the space $H = \{F \in L^2(G, H_\delta) \mid F(kx) = \delta(k)F(x), k \in K, x \in G\}$. Let Π be a projection onto a maximal *primary* subspace of U . Then one can show (as in the proof of Theorem 1 below) that there is a unique function $\mathcal{D}_\Pi: G \rightarrow \text{Hom}_c(H_\delta, H_\delta)$ such that $\mathcal{D}_\Pi(kxk') = \delta(k')^{-1} \mathcal{D}_\Pi(x) \delta(k)^{-1}$ ($x \in G, k, k' \in K$) for which $\pi F(y) = \int_G \mathcal{D}_\Pi(x) F(xy) dx$. One can also show that $\text{tr } \mathcal{D}_\Pi(\cdot) = D_\Pi(\cdot)$ is a spherical trace function of height δ and that $\mathcal{D}_\Pi(\cdot) = \int_K D_\Pi(k \cdot) \delta(k^{-1}) dk$. Furthermore, every spherical trace function arises in this manner. Thus, spherical trace functions are simply kernels (in the sense of integral operators) of certain intertwining operators.

It is this point of view we adopt in this paper. There are a large number of problems involved. If G is not compact, intertwining operators for U need no longer be integral operators. However, it turns out that they are always convolution against distributions. Furthermore, U need have no primary subspaces. This causes problems, but they are solvable due to the direct integral theory of unitary representations.

Our techniques are C^∞ vector techniques.

II. C^∞ vectors. In this section we gather together the results from C^∞ theory we shall need later.

Let G be a Lie group and let U be a continuous unitary representation of G in a separable Hilbert space H . A vector $v \in H$ is said to be a C^∞ vector for U if the map $g \rightarrow U(g)v$ is a C^∞ map of G into H . If X is an element of the Lie algebra L of G , then $\partial U(X)v$ is defined to be $d/dt|_{t=0} U(\exp tX)v$ for $v \in C^\infty(U)$. The mapping $X \rightarrow \partial U(X)$ extends to a representation map of the enveloping algebra \mathcal{O} into the operators on $C^\infty(U)$. We topologize $C^\infty(U)$ via the seminorms $\|\partial U(X) \cdot\|, X \in \mathcal{O}$. C^∞ is an invariant subspace of H . The restriction U^∞ of U to C^∞ is a differentiable representation in the C^∞ topology. In text we will use without specific reference many standard results on C^∞ vectors. We refer the reader to Poulsen [6] for such results.

In general, if T is a topological vector space we denote by T^\wedge the space of *conjugate* linear continuous functionals on T . If $A: T_1 \rightarrow T_2$ is a continuous linear mapping of two such spaces we denote by A^\wedge its adjoint: $A^\wedge: T_2^\wedge \rightarrow T_1^\wedge$. If U is a representation of G in T_1 , we denote by U^\wedge the representation $U^\wedge(x) = U(x^{-1})^\wedge$. If T has a continuous skew symmetric innerproduct (\cdot, \cdot) defined on it, T is isomorphic with a subspace of T^\wedge which, if T is a Hilbert space, is all of T^\wedge .

We shall consistently identify T with this subspace. These comments apply in particular to H and to $C^\infty(U)$ relative to the restriction of the scalar product on H to $C^\infty(U)$. We shall denote $C^\infty(U)^\wedge$ by $C^{-\infty}(U)$ and $(U^\infty)^\wedge$ by $U^{-\infty}$. Note that under this identification $U^\wedge = U$.

The following theorems are the main facts about C^∞ vectors we shall need [2], [5].

THEOREM A (GOODMAN). *Let U be decomposed into a direct integral $\int_M \otimes U^\alpha du(\alpha)$ where U^α are unitary representations of G realized in spaces H^α . Then $v \in C^\infty(U)$ if $v^\alpha \in C^\infty(U^\alpha)$ for a.e. α and $\alpha \rightarrow \|\partial U^\alpha(X)v^\alpha\|^2$ is integrable for all $X \in \mathfrak{g}$ and for a.e. α . In this case $(\partial U(X)v)^\alpha = \partial U^\alpha(X)v^\alpha$ for a.e. α .*

THEOREM B. *In the above notation, if φ is a continuous functional on $C^\infty(U)$ then there exist a.e. unique functionals φ^α on $C^\infty(U^\alpha)$ such that $\varphi(v) = \int_M \varphi^\alpha(v^\alpha) du(\alpha)$ for all $v \in C^\infty(U)$ where the integral converges absolutely.*

Now, if V is another such representation let $I(U, V)$ be the space of all continuous intertwining operators from U to V . If $A \in I(U, V)$, it is easily seen that A maps $C^\infty(U)$ into $C^\infty(V)$ continuously. We shall denote this mapping by A^∞ . Also $A^* \in I(V, U)$. Let $A^{-\infty} = ((A^*)^\infty)^*$, $A^{-\infty}: C^{-\infty}(U) \rightarrow C^{-\infty}(V)$. Notice that the scalar product allows us to identify H with a subspace of $C^{-\infty}(U)$ and that under this identification $A^{-\infty}$ extends A .

Now, if U is irreducible, there is a natural Hilbert space structure on $I(U, V)$. In fact, if $A, B \in I(U, V)$, $B^*A \in I(U, U)$ and hence $B^*A = C(A, B)I$ by Shure's lemma where $C(A, B) \in \mathbb{C}$. $C(\cdot, \cdot)$ defines a Hermitian scalar product on $I(U, V)$ which is obviously complete (in fact $C(B, B)$ is the square of the operator norm of B).

III. Spherical functions. Let G be a (not necessarily unimodular) Lie group and K any closed subgroup. Let Ω and ω be the respective modular functions for right Haar measure (i.e. $\int_G f(yx) dx = \Omega(y^{-1}) \int_G f(x) dx$). Let $\Delta = (\omega/\Omega)^{1/2}$. Any locally integrable function f on G which satisfies $f(kx) = \Delta^2(k)f(x)$ defines a Radon measure $\mu(f)$ on $K \backslash G$ via the formula $\mu(f)(\tilde{\varphi}) = \int_G f(x)\varphi(x) dx$ where $\varphi \in C_c(G)$, $\tilde{\varphi}(x) = \int_K \varphi(kx) dk$.

If $\mu(f)$ is finite we shall say that f is integrable and we shall write $\mu(f)(K \backslash G) = S f(x) dx$. We remark that f is integrable iff $\mu(|f|)$ is a finite measure. We also remark that if g say belongs to $C_c(G)$, then

$$S \left(\int_K \Delta^{-2}(k) g(kx) dk \right) dx = \int_G g(x) dx.$$

If f is a locally integrable Hilbert space valued function which satisfies the above transformation property, such that $\|f(\cdot)\|$ is integrable, we may define $S f(x) dx$ by the requirement $(S f(x) dx, w) = S(f(x), w) dx$ for all w in the space.

Now, let δ be a finite N -dimensional unitary irreducible representation of K in a Hilbert space H^δ . Let U be the representation of G which acts via right translation in the space H of all locally integrable H^δ -valued functions F which satisfy

$$(1) \quad F(kx) = \Delta(k)\delta(k)F(x),$$

$$(2) \quad S(\|F\|^2) < \infty.$$

U is of course the unitary representation induced by δ .

Now, let A be a self-intertwining operator for U . We shall now associate with A a matrix-valued distribution and a scalar-valued distribution in much the same way as we associated matrix-valued functions with intertwining operators in the compact case. Let us recall that if $f \in L^1(G)$, the operator $U(f)$ is defined by $\int_G \tilde{f}(x)U(x) dx$ where $\tilde{f}(x)$ is the function $f(x^{-1})\Omega(x^{-1})$. Also $f^*(x)$ is $\tilde{f}(x)^{-}$, $f \cdot x = f(\cdot x)$ and $x \cdot f = f(x^{-1} \cdot)\Omega(x^{-1})$.

THEOREM 1. *Let $A: H \rightarrow H$ be an intertwining operator for U . Then there is a unique continuous linear map $\mathcal{D}_A: C_c^\infty(G) \rightarrow \text{Hom}_C(H^\delta, H^\delta)$ satisfying $\mathcal{D}_A(k \cdot f) = \Delta(k)\mathcal{D}_A(f)\delta(k^{-1})$ for which*

$$AU(f)F(e) = S(\mathcal{D}_A(x \cdot f)F(x)) dx.$$

(Note that the integral makes sense.)

PROOF. For $f \in C_c^\infty(G)$, let $\tau(f): H^\delta \rightarrow H$ be

$$\tau(f) = \int_K f(kx)\Delta(k^{-1})\delta(k^{-1}) dk.$$

It is clear that $\tau(f)$ maps into $C^\infty(U)$. Hence $A\tau(f)$ maps into $C^\infty(U)$. Since $C^\infty(U)$ consists of C^∞ -, H^δ -valued functions, point evaluation is continuous so we may define a linear mapping $\mathcal{D}_A(f)$ of H^δ into H^δ by $v \rightarrow A\tau(f)v(e)$.

LEMMA. $\mathcal{D}_A(f)^\wedge = \mathcal{D}_A(f^*)$.

PROOF. Let $f, g \in C_c^\infty(G)$. Then, for $v, w \in H^\delta$,

$$\begin{aligned} (A\tau(f)v, \tau(g)w) &= S\left(\int_K (A\tau(f)v(x), \delta(k^{-1})w)\Delta(k^{-1})\bar{g}(kx) dk\right) dx \\ (*) \quad &= S\left(\int_K (A\tau(f)v(kx), w)\bar{g}(kx)\Delta^{-2}(k) dk\right) dx \\ &= S\left(\int_G \bar{g}(x)A\tau(f)v(x) dx, w\right). \end{aligned}$$

But

$$\begin{aligned} A\tau(f)v(x) &= (U(x)A\tau(f))(e) = (AU(x)\tau(f))(e) \\ &= A\tau(f \cdot x)(e) = \mathcal{D}_A(f \cdot x). \end{aligned}$$

Hence (*) equals $(\mathcal{D}_A(f \cdot g^*)^\wedge v, w)$. Taking adjoints, (*) also equals $(v, \mathcal{D}_A(g \cdot f^*)w)$. Letting g go through a partition of unity the result follows. Q.E.D.

Now, from the lemma and its proof

$$\mathcal{D}_A(x \cdot f)^\wedge = \mathcal{D}_A^\wedge(f^* \cdot x) = A^\wedge \tau(f^*)(x).$$

Hence

$$\begin{aligned} (S(\mathcal{D}_A(x \cdot f)F(x))dx, v) &= S(F(x), A^\wedge \tau(f^*)v(x))dx \\ &= (F, A^\wedge \tau(f^*)v) = (AF, \tau(f^*)v) \\ &= S\left(\int_K (Af(x), f^*(kx)\Delta^{-1}(k)\delta^{-1}(k)v) dk\right) dx \\ &= S\left(\int_K (\tilde{f}(kx)\Delta^{-1}(k)\delta(k)AF(x), v) dk\right) dx \\ &= S\left(\int_K (\tilde{f}(kx)\Delta^{-2}(k)AF(kx), v) dk\right) dx \\ &= \left(\int_G \tilde{f}(x)AF(x) dx, v\right) \\ &= (U(f)AF(e), v). \end{aligned}$$

This shows existence. Uniqueness follows by reversing the chain of equalities.

The continuity of $f \rightarrow \mathcal{D}_A(f)$ is clear. Q.E.D.

DEFINITION. \mathcal{D}_A is called the matrix kernel of A . Its trace is denoted D_A and is called the scalar kernel.

The lemma used in the above proof has the following simple but important corollary.

COROLLARY. For all $k \in K$,

- (a) $\mathcal{D}_A(f \cdot k) = \Delta(k)\delta(k)\mathcal{D}_A(f)$,
- (b) $D_A(k \cdot f \cdot k) = \Delta^2(k)D_A(f)$.

DEFINITION. Property (b) above is called K -centrality.

Let us also remark that if there were a locally integrable function $\mathcal{D}_A(x)$ valued in operators on H^δ which satisfied $\mathcal{D}_A(k_1 x k_2) = \Delta(k_1 k_2^{-1})\delta(k_2^{-1})\mathcal{D}_A(x)\delta(k_1^{-1})$ for which

$$AF(y) = \int_G \mathcal{D}_A(x)F(xy) dx$$

for a dense set of F in \mathcal{H} , then (as the reader may easily verify) $\mathcal{D}_A(f) = \int_G \mathcal{D}_A(x)f(x) dx$ for all $f \in C_c^\infty(G)$. Hence, we are justified in considering \mathcal{D}_A as an analogue of the kernel of A in the compact case.

D_A determines A . We may in fact write A as a "convolution" against D_A , a fact that will be important later. To show this we need some notation. Let μ and η be measures on G for which either μ or η has compact support or both are finite. Then there exists a unique measure ρ such that for $f \in C_c(G)$, $\iint f(xy) d\mu(x) d\eta(y) = \int f(z) d\rho(z)$. We write $\rho = \mu * \eta$. We shall identify functions g on K with the measure on G supported in K given by gdk on K . Thus

$$f * g(x) = \int_K f(xk^{-1})g(k) dk,$$

$$g * f(x) = \int_K \Omega(k^{-1})g(k)f(k^{-1}x) dk.$$

Also let $\chi_\delta(k) = N^{-1} \operatorname{tr} \delta(k)$ and let $\lambda_\delta(k) = \Delta^{-1}(k)\chi_\delta(k)$. A distribution φ on G is said to transform on the left according to δ if $\lambda_\delta * f = 0$ implies $\varphi(f) = 0$ for $f \in C_c^\infty(G)$. Notice that we cannot say $\varphi(\lambda_\delta * f) = \varphi(f)$ since $\lambda_\delta * f$ will not usually have compact support.

LEMMA. D_A transforms on the left according to δ .

PROOF. Let $g \in C_c^\infty(G)$. If $f \in C_c^\infty(G)$ satisfies $\lambda_\delta * f = 0$, then

$$\begin{aligned} D_A(f * g) &= \int f(x)D_A(x \cdot g) dx \\ &= S\left(\int_K f(kx)\Delta^{-2}(k)D_A(kx \cdot g) dk dx\right) \\ &= S\left(\int_K f(kx)\Delta(k^{-1})\operatorname{tr} \mathcal{D}_A(x \cdot g)\delta(k^{-1}) dk dx\right). \end{aligned}$$

But $\lambda_\delta * f = 0$ implies, by irreducibility,

$$\int_K f(kx)\Delta(k^{-1})\delta(k^{-1}) dk = 0.$$

Hence $D_A(f * g) = 0$ for all g . Therefore $D_A(f) = 0$. Q.E.D.

Now, let $C_\delta^\infty(G) = \lambda_\delta * C_c^\infty(G)$. If φ transforms according to λ_δ on the left, then φ defines a functional φ' on $C_\delta^\infty(G)$ via $\varphi'(\lambda_\delta * f) = \varphi(f)$.

Let $v, w \in H^\delta$ and let $v \otimes w$ be the operator $(\cdot, w)v$. Since δ is irreducible, it is completely irreducible. Thus there are finite sequences $k_i \in K$ and $c_i \in C$ such that $N^{-1} \sum c_i \Delta^{-1}(k_i)\delta(k_i) = v \otimes w$. Applying $\Delta^{-1}(k)\delta(k)^{-1}$ to both sides and taking the trace we conclude that

$$\Delta(k^{-1})(\delta(k^{-1})v, w) = \sum c_i \lambda_\delta(k^{-1}k_i)\Delta^{-2}(k).$$

It follows then from the definition of τ that

$$(\tau(f)v(\cdot), w) = \sum c_i (\lambda_\delta \cdot k_i) * f = \lambda_\delta * \left(\sum c_i k_i^{-1} \cdot f\right).$$

This is an element of $C_\delta^\infty(G)$ and

$$\begin{aligned} D'_A((\tau(f)v(\cdot), w)) &= D'_A\left(\lambda_\delta * \sum c_i k_i^{-1} \cdot f\right) \\ &= D_A\left(\sum c_i k_i^{-1} \cdot f\right) = \operatorname{tr} \sum c_i \Delta(k_i^{-1})\mathcal{D}_A(f)\delta(k_i) \\ &= \operatorname{tr} \mathcal{D}_A(f)(v \otimes w) = (\mathcal{D}_A(f)v, w). \end{aligned}$$

Hence $\mathcal{D}_A(f)$ is, in a sense, the integral of $\tau(f)$ against D_A .

Furthermore $\mathcal{D}_A(f) = A\tau(f)(e)$. Hence

$$\begin{aligned}(A\tau(f)v(x), w) &= (U(x)A\tau(f)v(e), w) = (A\tau(f \cdot x)v(e), w) \\ &= D'_A((\tau(f \cdot x)v(\cdot), w)) = D'_A((\tau(f)v(\cdot x), w)).\end{aligned}$$

Hence $A\tau(f)v(x)$ is the "integral" of $\tau(f)v(\cdot x)$ against D_A . This determines A at the dense set of elements of the form $\tau(f)v(\cdot)$ and hence determines A .

The significance of these observations is that they may be turned into a characterization of the distributions D_A . We shall need to consider the norm defined on $C(G)$ by $\|f\|_K = S(\sup_K |f(kx)|^2 \Delta^{-2}(k)) dx$. This is finite on $C_\delta^\infty(G)$. Now let φ be a K central distribution which transforms according to δ on the left. For $f \in C_\delta^\infty(G)$, let $[\varphi : f](x) = \varphi'(f \cdot x)$. We shall say that φ is *finite* provided the mapping of $C_\delta^\infty(G)$ into $C(G)$ given via $f \rightarrow [\varphi : f]$ is bounded in $\|\cdot\|_K$. Let F_δ be the space of such functionals.

THEOREM 2. *A distribution φ is of the form D_A for some $A \in I(U, U)$ iff $\varphi \in F_\delta$.*

PROOF. Let us note that if $F \in H$, then $\Delta^{-2}(k)\|F(kx)\|_{H\delta}^2 = \|F(x)\|_{H\delta}^2$. Hence $\|\|F(\cdot)\|_{H\delta}\|_K = \|F\|_H$.

Now, consider $|D'_A(\lambda_\delta * f) \cdot x| = |D_A(f \cdot x)|$. This is bounded by a sum of terms of the form

$$\begin{aligned}|(\mathcal{D}_A(f \cdot x)v, w)| &= |(A\tau(f)v(x), w)| \\ &\leq \|A\tau(f)v(x)\|_{H\delta} \|w\|.\end{aligned}$$

Taking K norms, we see that $\|D'_A : \lambda_\delta * f\|_K$ is bounded by a sum of terms $C\|A\tau(f)v\|_H \leq C'\|\tau(f)v\|_H$.

Let w_1, \dots, w_d be an orthonormal basis for H^δ and fix $w = w_i$. Let c_i and k_i be chosen as above. Then

$$(\tau(f)v(x), w) = \lambda_\delta * \left(\sum c_i k_i^{-1} \cdot f \right)(x).$$

Now if in the integral defining $\lambda_\delta * (k_i^{-1} \cdot f)$ we replace k by $k_i^{-1}kk_i$ and note $\lambda_\delta(k_i^{-1}kk_i) = \lambda_\delta(k)$, we discover that $\lambda_\delta * (k_i^{-1} \cdot f) = k_i^{-1} \cdot (\lambda_\delta * f)\omega(k_i)$. It follows that $\|(\tau(f)v(\cdot), w)\|_K \leq C_w \|\lambda_\delta * f\|_K$. Since $\|\tau(f)v(x)\|_{H\delta} \leq \Sigma |(\tau(f)v(x), w_i)|$, we have shown that D'_A is finite.

Conversely, if $\varphi \in F_\delta$, the map $(v, w) \rightarrow \varphi'((\tau(f)v(\cdot), w))$ is continuous on $H^\delta \times H^\delta$ and hence there is an operator $\mathcal{D}_\varphi(f)$ which represents it. Of course, if $\tau(f)v(\cdot) = \tau(g)w(\cdot)$, then for all x , $\mathcal{D}_\varphi(f \cdot x)v = \mathcal{D}_\varphi(g \cdot x)w$. This allows us to define an operator A on the set of $\tau(f)v(\cdot)$ by $A(\tau(f)v)(x) = \mathcal{D}_\varphi(f \cdot x)v$. It is

trivial that $A(\tau(f)\nu(\cdot))$ satisfies the transformation property of H . It follows from the definition of φ' that, for all $w \in \mathcal{H}^\delta$,

$$\|A(\tau(f)\nu(\cdot), w)\|_K \leq c \|\tau(f)\nu(\cdot, w)\|_K$$

where C is the bound on $[\varphi : \cdot]$.

By reasoning similar to that done in the first part of the proof it follows that A is bounded in H norm and hence extends uniquely to a bounded operator on H which clearly intertwines U . It is obvious from the uniqueness of D_A that $D_A = \varphi$. Q.E.D.

REMARKS. We could, if we so desired, define intrinsically an algebra structure on F_δ that would make $A \rightarrow D_A$ an algebra isomorphism of $I(U, U)$ onto F_δ . However, since F_δ is in one-to-one correspondence with $I(U, U)$, we could also define our algebra structure by requiring this map to be an isomorphism. We take this latter course since we shall not need the intrinsic definition. The intrinsic definition is closely associated with convolution and is convolution when this makes sense.

Now, if A were to be a projection onto a primary subspace, it would be logical, in view of the comments in §I, to call D_A a spherical distribution. However, this would yield a rather trivial theory in the cases where U had a continuous spectrum. We shall take a more general approach.

To motivate our discussion we prove a lemma. First some notation. Let V be a unitary representation of G realized in L . Let $f \in C_c^\infty(G)$ and consider $V(f)$. Since $V(x)V(f) = V(f \cdot x)$, it is easily verified that $V(f): L \rightarrow C^\infty(V)$. Hence $V(f)^\wedge: C^{-\infty}(V) \rightarrow L$. Also, $V(x)V(f)^\wedge = V^\wedge(x \cdot f)$ so as before $V(f)^\wedge$ in fact maps $C^{-\infty}(V)$ into $C^\infty(V)$. We shall denote $V(f^*)^\wedge$ by $V^{-\infty}(f)$ (it is in fact the integral of $V^{-\infty}$ against f although we shall not need this fact). Then if $v, w \in C^{-\infty}(V)$ such expressions as $\langle V^{-\infty}(f)v, w \rangle$ make sense. If $v \in C^\infty(V)$ and $w \in C^{-\infty}(V)$, we define $\langle w, v \rangle$ to be the conjugate of $\langle v, w \rangle$ so that $\langle w, V^{-\infty}(f)v \rangle$ is also meaningful. Also let ϵ denote evaluation at $e \in C^\infty(V)$.

LEMMA. Let π be an orthogonal projection onto an invariant subspace H_π of H . Let $V = U|_{H_\pi}$ and let $\epsilon_\pi = \epsilon|_{C^\infty(V)} (= C^\infty(U) \cap H_\pi)$. Then for all $f \in C_c^\infty(G)$, $\mathcal{D}_\pi(f) = \epsilon_\pi V^{-\infty}(f)\epsilon_\pi^\wedge$.

PROOF. By definition, $\mathcal{D}_\pi(f)v = \epsilon\pi\tau(f)v$. From the computation done at the end of Theorem I, $(\epsilon U(f)\pi)^\wedge = \pi^\wedge\tau(f^*) = \pi\tau(f^*)$. Hence by definition,

$$\begin{aligned} \mathcal{D}_\pi(f) &= \epsilon\pi\tau(f) = \epsilon_\pi(\epsilon U(f^*)\pi)^\wedge = \epsilon_\pi(\epsilon_\pi U(f^*)\pi)^\wedge \\ &= \epsilon_\pi\pi^\wedge U(f^*)^\wedge \epsilon_\pi^\wedge = \epsilon_\pi V^{-\infty}(f)\epsilon_\pi^\wedge, \end{aligned}$$

since $\pi^\wedge = \pi$. Q.E.D.

Suppose U is type I. Let \mathcal{U} be the C^* -algebra generated by U and let $M = \mathcal{U}^\wedge$ (the space of equivalence classes of irreducible representations of \mathcal{U} given the Mackey-Borel structure; see [3]). Then there is a Borel measure μ on M such that $U = \int_M \bigoplus U^\alpha d\mu(\alpha)$ where each U^α is a primary representation which generates a von Neumann algebra of class $\alpha \in \mathcal{U}^\wedge$. Let H^α be the representation space of U^α .

By the finite dimensionality of H^δ and Theorem B of §II, ϵ is a direct integral of continuous maps $\epsilon^\alpha: C^\infty(U^\alpha) \rightarrow H^\alpha$.

DEFINITION. Let $\mathcal{D}^\alpha(f) = \epsilon^\alpha(U^\alpha)^{-\infty}(f)(\epsilon^\alpha)^\wedge$ and let $D^\alpha(f) = \text{tr } \mathcal{D}^\alpha(f)$. \mathcal{D}^α and D^α are called, respectively, the matrix-valued and scalar-valued spherical distributions of height δ associated with the element $\alpha \in \mathcal{U}^\wedge$. (Notice that this is really defined only a.e. with respect to μ . Note also that if we replace μ by a measure in the same measure class we change D^α and \mathcal{D}^α by constant multiples. The \mathcal{D}^α are, however, otherwise well defined.)

It is obvious from the uniqueness of the ϵ^α that they satisfy similar transformation laws as does ϵ . Hence, it is easily seen that \mathcal{D}^α and D^α satisfy the same transformation properties as the \mathcal{D}_A and D_A did above.

It is natural to ask if D^α has an expression as a trace of the restriction of an irreducible representation of G to a K -invariant subspace analogous to the usual definition (see [1]). This is the case, although it will take some work to show it. Consider for $\alpha \in M$ the map $(\epsilon^\alpha)^\wedge: H^\alpha \rightarrow C^{-\infty}(U^\alpha)$. Let $\pi^\alpha = (\epsilon^\alpha)^\wedge \epsilon^\alpha$ and let K^α be the image of $(\epsilon^\alpha)^\wedge$. K^α is invariant under $(U^\alpha)^{-\infty}|K$ and $(\epsilon^\alpha)^\wedge$ intertwines $\Delta U^\alpha|K$ and δ (this is just a restatement of the transformation property of ϵ^α). ϵ has the property that the only U invariant subspace on which it is zero is the zero subspace. Hence the ϵ^α are nonzero for a.e. α . It follows from Shure's lemma that $(\epsilon^\alpha)^\wedge$ is invertible as a map of H^δ into K^α . Let j^α be its inverse. Then

$$\mathcal{D}^\alpha(f) = \epsilon^\alpha(U^\alpha)^{-\infty}(f)(\epsilon^\alpha)^\wedge = j^\alpha \pi^\alpha (U^\alpha)^{-\infty}(f)(j^\alpha)^{-1}.$$

Hence $D^\alpha(f) = \text{tr } \pi^\alpha (U^\alpha)^{-\infty}(f)|K^\alpha$. Since π^α maps onto K^α and intertwines $\Delta^2(U^\alpha)^{-\infty}|K$, this is in form analogous to the usual definition of spherical function. The problem, however, is that U^α is only primary and not irreducible. The fact that one can compute the spherical functions using any one of the irreducible constituents of U^α seems to be a deep fact and is our first main result.

THEOREM 3. Let $\alpha \in M$. Let T^α be any irreducible unitary representation quasi-equivalent to U^α realized in L^α . Then for a.e. α there is a map $\sigma^\alpha: C^\infty(T^\alpha) \rightarrow C^{-\infty}(T^\alpha)$ which intertwines $T^\alpha|K$ and $\Delta(T^\alpha)^{-\infty}|K$ and which represents D^α in the following sense.

There is a canonical Hilbert space structure on the image L^α_δ of σ^α which

makes $\Delta T^{\alpha-\infty}|K|L_\delta^\alpha = T^0$ unitary for which the map $\sigma^\alpha T^\alpha(f)|L_\delta^\alpha$ is trace class for all $f \in C_c^\infty(G)$. Then $D^\alpha(f) = \text{tr } \sigma^\alpha T^\alpha(f)^{-\infty}|L_\delta^\alpha$.

Furthermore, T^0 is the orthogonal direct sum in L_δ^α of n copies of δ where n is the (possibly infinite) multiplicity of T in U^α .

PROOF. Let

$$L_\delta^\alpha = \text{span} \{A^{-\infty} \epsilon^\alpha(v) | A \in I(U^\alpha, T^\alpha), v \in H^\delta\}.$$

Before describing the Hilbert structure on L_δ^α we need a lemma.

LEMMA. Suppose $A_1, \dots, A_K \in I(U^\alpha, T^\alpha)$ are linearly independent and suppose $v_1, \dots, v_K \in H^\delta$ satisfy $\sum_{i=1}^K A_i^{-\infty} (\epsilon^\alpha)^\wedge(v_i) = 0$. Then $v_1 = v_2 = \dots = v_K = 0$.

PROOF. We proceed by induction on K . We omit the $K = 1$ case since it is analogous to the induction step. Since the set of W such that $-A_{K+1}^{-\infty} \epsilon^\alpha(W) = \sum_{i=1}^K A_i^{-\infty} \epsilon^\alpha(W_i)$ for some $W_i \in H^\delta$ is δ -invariant and nonzero (v_{K+1} belongs to it!), it is all of H^δ . For all W there are elements $W_i = \tau_i(W)$ which satisfy the above. By the induction hypothesis the W_i are unique. It follows that the τ_i are linear and intertwine δ and hence $\tau_i \equiv c_i I$. Hence $-A_{K+1}^{-\infty} (\epsilon^\alpha)^\wedge(v) = \sum_{i=1}^K c_i A_i^{-\infty} (\epsilon^\alpha)^\wedge(v)$ for all v , i.e. $-\epsilon^\alpha A_{K+1}^\wedge = \epsilon^\alpha (\sum c_i A_i^\wedge)$. Now ϵ^α has the property that $\epsilon^\alpha(U^\alpha(g)v) = 0$ for all g implies $v = 0$ ($v \in C^\infty(U^\alpha)$) (Theorem IV of [5]). Applying $U^\alpha(g)$ on the right in the above equation and using the commuting property of the A_i , we see that $A_{K+1} = \sum c_i A_i$, contradicting independence and thus the fact that $v_{K+1} \neq 0$. Q.E.D.

We will use the lemma in the form of the following corollary.

COROLLARY. If e_1, \dots, e_d are linearly independent elements of H^δ and $A_1, \dots, A_d \in I(U^\alpha, T^\alpha)$ are such that $\sum A_i^{-\infty} (\epsilon^\alpha)^\wedge(e_i) = 0$, then $A_i = 0$, $i = 1, \dots, d$.

PROOF. Let B_1, \dots, B_K be a basis for the vector space spanned by the $\{A_i\}$. Write $A_i^{-\infty} = \sum c_{ij} B_j^{-\infty}$. Then

$$0 = \sum_{ij} c_{ij} B_j^{-\infty} (\epsilon^\alpha)^\wedge(e_i) = \sum_j B_j (\epsilon^\alpha)^\wedge \left(\sum_i c_{ij} e_i \right).$$

Hence, by the above $\sum_i c_{ij} e_i = 0$, i.e., $c_{ij} = 0$. Hence $A_i = 0$ for all i . Q.E.D.

Now, let e_1, \dots, e_d be an orthonormal basis for H^δ . Let $L_\delta^i = \{A^{-\infty} \epsilon^\alpha(e_i) | A \in I(U^\alpha, T^\alpha)\}$. By the above corollary L_δ^α is the direct sum of the L_δ^i . From the same corollary, the map $A \rightarrow A^{-\infty} \epsilon^\alpha(e_i)$ is 1-1. Hence we may define an innerproduct $(\cdot, \cdot)_\delta^\alpha$ on L_δ^α by setting

$$\left(\sum A_i^{-\infty} \epsilon^\alpha(e_i), \sum B_i^{-\infty} \epsilon^\alpha(e_i) \right)_\delta^\alpha = \sum C(A_i^\wedge, B_i^\wedge)$$

where C is the innerproduct $I(T^\alpha, U^\alpha)$ defined in §II. Notice that under this innerproduct $L_\delta^\alpha \approx I(T^\alpha, U^\alpha)^d$. Hence completeness follows. Notice also that for $A_i \in I(U^\alpha, T^\alpha)$, $v_i \in H^\delta$, $i = 1, 2$, we have

$$(A_1^{-\infty} \epsilon^\alpha \wedge (v_1), A_2^{-\infty} \epsilon^\alpha \wedge (v_2))_\delta^\alpha = C(A_2^\wedge, A_1^\wedge)(v_1, v_2).$$

Hence the scalar product does not depend on the choice of basis for H^δ . The above equality and the fact that $A_i^{-\infty} \epsilon^\alpha \wedge$ intertwines $\Delta(T^\alpha)^{-\infty}|K$ and δ show that $\Delta(T^\alpha)^{-\infty}|K$ is unitary on L_δ^α . The injection of L_δ^α into $C^{-\infty}(T^\alpha)$ is continuous since $A_n^{-\infty} \epsilon^\alpha \wedge (e_i)$ converges to $A^{-\infty} \epsilon^\alpha \wedge (e_i)$ iff A_n^\wedge converges to A^\wedge in $\mathcal{C}(\cdot, \cdot)$ which is equivalent to A_n^\wedge tending to A^\wedge uniformly. This implies that $A_n^{-\infty} \rightarrow A^{-\infty}$ uniformly on bounded subsets of $C^{-\infty}(T^\alpha)$ and hence that, in $C^{-\infty}(T^\alpha)$, $A_n^{-\infty} \epsilon^\alpha \wedge (e_i) \rightarrow A^{-\infty} \epsilon^\alpha \wedge (e_i)$. Thus the natural injection of L_δ^α into $C^{-\infty}(T^\alpha)$ is continuous. Let ρ^α denote the injection and let $\sigma^\alpha: C^\infty(T^\alpha) \rightarrow L_\delta^\alpha$ be its adjoint. (C^∞ is reflexive!)

Now, we are finally in a position to prove the trace formula. Let E_j , $j = 1, \dots$, be an orthonormal basis of $I(T^\alpha, U^\alpha)$. It is easily seen that E_i are isometries and that H^α is the orthogonal direct sum of the images of the E_i . It follows that the maps $\pi_i = E_i E_i^\wedge$ are orthogonal projections onto invariant irreducible subspaces of H^α and $\sum_i \pi_i = I$. Hence, for $f \in C_c^\infty(G)$,

$$\begin{aligned} D^\alpha(f) &= \text{tr } \mathcal{D}^\alpha(f) = \text{tr } \epsilon^\alpha (U^\alpha)^{-\infty}(f)(\epsilon^\alpha)^\wedge \\ &= \sum_i \text{tr } \epsilon^\alpha \pi_i (U^\alpha)^{-\infty}(f)(\epsilon^\alpha)^\wedge \\ &= \sum_i \text{tr } \epsilon^\alpha E_i E_i^\wedge (U^\alpha)^{-\infty}(f)(\epsilon^\alpha)^\wedge \\ &= \sum_i \text{tr } \epsilon^\alpha E_i (T^\alpha)^{-\infty}(f)(E_i^\wedge)^{-\infty}(\epsilon^\alpha)^\wedge. \end{aligned}$$

But

$$\begin{aligned} &(\epsilon^\alpha E_i (T^\alpha)^{-\infty}(f)(E_i^\wedge)^{-\infty}(\epsilon^\alpha)^\wedge e_j, e_j) \\ &= \langle (T^\alpha)^{-\infty}(f)(E_i^\wedge)^{-\infty}(\epsilon^\alpha)^\wedge e_j, (E_i^\wedge)^{-\infty}(\epsilon^\alpha)^\wedge e_j \rangle = (\sigma^\alpha(T^\alpha)^{-\infty}(f) F_{ij}, F_{ij})_\delta^\alpha, \end{aligned}$$

where $F_{ij} = (E_i^\wedge)^{-\infty}(\epsilon^\alpha)^\wedge(e_j)$.

Note that the series of operators on H^δ , $\sum_i \epsilon^\alpha \pi_i (U^\alpha)^{-\infty}(f)(\epsilon^\alpha)^\wedge$, converges pointwise since $\sum \pi_i$ converges in $C^\infty(U^\alpha)$ and hence it converges in any norm by finite dimensionality. Hence

$$D^\alpha(f) = \sum_{ij} (\sigma^\alpha(T^\alpha)^{-\infty}(f) F_{ij}, F_{ij})_\delta^\alpha = \text{tr } \sigma^\alpha(T^\alpha)^{-\infty}(\tilde{f})|L_\delta^\alpha,$$

as claimed.

To finish the proof we need only compute the multiplicity of δ in T^0 .

However, if E_i are as above $v \rightarrow (E_i^\wedge)^{-\infty}(\epsilon^\alpha)^\wedge(v)$ intertwines δ and T^0 . Since the ranges of these maps are orthogonal and clearly span L_δ^α topologically, the multiplicity of δ in T^0 is the dimension of the span of the E_i which is the multiplicity of T^α in U^α . Q.E.D.

Now, if K is compact, the above can be made considerably more precise. Notice that in this case $\Omega|K = \omega \equiv 1$. Let $N = \dim H_\delta$.

THEOREM 4. *Let K be compact and let notation be as above. Let $T^\alpha(\chi_\delta) = \int_K \chi_\delta(k^{-1})T^\alpha(k)dk$. Let L_0^α be the image of L^α under $T^\alpha(\chi_\delta)$ (so L^α is the space of vectors which transform according to δ under T^α). Then there is a continuous map $Q^\alpha: C^\infty(T^\alpha) \rightarrow C^\infty(T^\alpha)$ such that $L_\delta^\alpha = (Q^\alpha)^\wedge(L_0^\alpha)$ and such that $N^{-1/2}Q^\alpha$ is a unitary isomorphism of L_0^α onto L_δ^α . Furthermore*

$$\sigma^\alpha = N^{-1}(Q^\alpha)^\wedge T^\alpha(\chi_\delta)Q^\alpha$$

and $D^\alpha(f) = N^{-1} \operatorname{tr} Q^\alpha T^\alpha(\chi_\delta * f)Q^\alpha|L_0^\alpha$. Q^α can be chosen so that it is formally selfadjoint on $C^\infty(T^\alpha)$ and satisfies $Q^\alpha(U^\alpha)^\infty(g) = (\Omega(g))^{1/2}(U^\alpha)^\infty(g)Q^\alpha$ for all $g \in G$. This uniquely determines Q^α . In fact Q^α is the nonunimodular weight function of Moore. In particular, if G is unimodular $Q^\alpha = I$ and $D^\alpha(f) = \operatorname{tr} N^{-1}T^\alpha(\chi_\delta)T^\alpha(f)|L_0^\alpha$.

PROOF. Before proceeding, let us recall some facts about tensor products of Hilbert spaces. If H is a Hilbert space let H^* denote the conjugate space (i.e. $|i\rangle$ with scalar multiplication given by $c \cdot v = \bar{c}v$). Then $H^{**} = H$. If H_1 and H_2 are Hilbert spaces, $H_1 \otimes H_2$ is the space of Hilbert-Schmidt mappings of H_1^* into H_2 . If $v \in H_1$ and $w \in H_2$, then $v \otimes w \in H_1 \otimes H_2$ is the map $(v, \cdot)w$. The tensor product is associative. In fact, by definition $(H_1 \otimes H_2)^* = H_1^* \otimes H_2^*$. If $F: H_1^* \otimes H_2^* \rightarrow H_3$ belongs to $(H_1 \otimes H_2) \otimes H_3$ and $v \in H_1^*$ we obtain from F a map $F_v: H_2^* \rightarrow H_3$ given by $F_v(w) = F(v \otimes w)$. Then $F_v \in H_2 \otimes H_3$ and $v \rightarrow F_v$ defines an element $\tilde{F} \in H_1 \otimes (H_2 \otimes H_3)$. The mapping $F \rightarrow \tilde{F}$ is a unitary isomorphism.

If $A_i: H_i \rightarrow H_i$ are continuous linear operators, $i = 1, 2$, then $A_1 \otimes A_2$ is the operator on $H_1 \otimes H_2$ given by $F \rightarrow A_2 F A_1^*$ (A_1^* is A_1 considered as an operator on H_1^*). If V_i are representations of G in H_i , $V_1 \otimes V_2(b) = V_1(b^{-1}) \otimes V_2(b)$.

We shall need the following lemma, which is known.

LEMMA. *Let T be an irreducible unitary representation of G in H and let I be the identity representation in H . Then $I(T, I \otimes T)$ is the space of operators $v \rightarrow w \otimes v$, $w \in H$.*

Now, let $\tilde{H} = L^2(G, dx, H^\delta)$ and let \tilde{R} and \tilde{L} be, respectively, the left and right regular unitary representations of G in \tilde{H} (i.e. $\tilde{L}(y)F(x) = \Omega^{-1/2}(y)F(y^{-1}x)$

and $\tilde{R}(y)F(x) = F(xy)$ for all $y, x \in G, F \in \tilde{H}$. Then, by compactness of K , \tilde{H} is set theoretically and norm-wise just the space of all functions F in \tilde{H} which satisfy $\tilde{L}(k^{-1})F = \delta(k)F$ for all $k \in K$. The map $\Gamma = \int_K \delta(k)\tilde{L}(k)dk$ is easily seen to be a projection onto \tilde{H} which commutes with \tilde{R} . Also $U = \tilde{R}|_H$.

The harmonic analysis of \tilde{R} is essentially equivalent to that of the L^2 regular representations. Let R and L denote, respectively, the L^2 right and left unitary representations of G in $L^2(G, dx)$. The map $v \otimes f \rightarrow v\tilde{f}(\cdot)$ sets up a unitary isomorphism between $H^\delta \otimes L^2$ and \tilde{H} . Its inverse is given by considering $F \in \tilde{H}$ to be a map of $(H^\delta)^*$ into L^2 given via $v \rightarrow \langle F(\cdot), v \rangle$. Under this isomorphism \tilde{R} becomes $I \otimes R$, \tilde{L} becomes $I \otimes L$ and $\Gamma = \int_K \delta(k) \otimes L(k)dk$. Also $C^\infty(\tilde{R}) = H^\delta \otimes C^\infty(R)$. If $F \in H^\delta \otimes C^\infty(R)$ and ρ represents evaluation at e in $C^\infty(R)$ (which is continuous) then $v \rightarrow \rho(F(v))$ defines an element $\tilde{e}(F)$ of $(H^\delta)^\wedge = H^\delta$. The functional $F \rightarrow \tilde{e}(F)$ is easily seen to correspond to evaluation at e in \tilde{H} .

In [5] we obtained the following description of the harmonic analysis of R .

THEOREM C. *There is a measure space M , a Borel measure μ on M and a measurable family T^α of irreducible unitary representations of G in spaces L^α indexed by M such that*

$$L^2(G) \approx \int_M \oplus (L^\alpha)^* \otimes L^\alpha d\mu(\alpha).$$

Under this isomorphism

$$R = \int_M \oplus I^{\alpha*} \otimes T^\alpha d\mu(\alpha),$$

$$L = \int_M \oplus (T^\alpha)^* \otimes I^\alpha d\mu(\alpha) \quad (I^\alpha = \text{identity on } L^\alpha).$$

Furthermore there is an essentially selfadjoint operator $Q^\alpha: C^\infty(T^\alpha) \rightarrow C^\infty(T^\alpha)$ continuously such that for all $F \in C^\infty(I^{\alpha*} \otimes T^\alpha)$, $Q^\alpha F$ is bounded and of trace class. In this case $\rho = \int \oplus \rho^\alpha d\mu(\alpha)$ where ρ^α is the functional $F \rightarrow \text{tr } Q^\alpha F$ on $C^\infty(I^{\alpha*} \otimes T^\alpha)$.

If $f \in C_c^\infty(G)$, then $T^\alpha(f)Q^\alpha$ is a bounded Hilbert-Schmidt operator of L^α into L^α for which $f \rightarrow T^\alpha(f)Q^\alpha$ extends to a unitary isomorphism of L^2 onto the above direct integral which intertwines $(I^\alpha)^* \otimes T^\alpha$ and \tilde{R} and $(T^\alpha)^* \otimes I^\alpha$ and \tilde{L} .

Furthermore, Q^α satisfies $T^\alpha(g)Q^\alpha = \Omega^{-1/2}(g)Q^\alpha T^\alpha(g)$ on $C^\infty(T^\alpha)$ and is uniquely determined up to scalars by this property.

Now, from the general theory of direct integrals, tensor products factor through direct integrals. Hence

$$\begin{aligned} \tilde{H} &\approx \int_M \oplus H^\delta \otimes ((L^\alpha)^* \otimes L^\alpha) d\mu(\alpha) \\ &\approx \int_M \oplus (H^\delta \otimes (L^\alpha)^*) \otimes L^\alpha d\mu(\alpha). \end{aligned}$$

Let $\tilde{H}^\alpha = (H^\delta \otimes (L^\alpha)^*) \otimes L^\alpha$, $\tilde{R}^\alpha = (I \otimes I^{\alpha*}) \otimes T^\alpha$, $\tilde{I}^\alpha = (I \otimes (T^\alpha)^*) \otimes I^\alpha$. Let $\Gamma^\alpha = \int_K \delta(k) \otimes T^{\alpha*}(k) dk$ and let $\tilde{\Gamma}^\alpha = \Gamma^\alpha \otimes I^\alpha$ (so that Γ is the direct integral of $\tilde{\Gamma}^\alpha$). $\tilde{\epsilon}$ also factors as a direct integral of maps $\tilde{\epsilon}^\alpha: C^\infty(\tilde{R}^\alpha) \rightarrow H^\delta$ given via $(\tilde{\epsilon}^\alpha F, w) = \rho^\alpha(Fw)$ for $F \in C^\infty(\tilde{R})$, $w \in H_\delta$. Here we are considering $F \in H^\delta \otimes ((L^\alpha)^* \otimes L^\alpha)$. (This follows from the uniqueness in Theorem B of the introduction.)

Under the above isomorphism of $H^\delta \otimes ((L^\alpha)^* \otimes L^\alpha)$ with $(H^\delta \otimes (L^\alpha)^*) \otimes L^\alpha$, this becomes $\tilde{\epsilon}^\alpha(W \otimes v) = W^\wedge(Q^\alpha v)$. (It is clear that $v \in C^\infty(T^\alpha)$ if $W \otimes v \in C^\infty(\tilde{R}^\alpha)$.) In fact, as a map of $(H^\delta)^*$ into $(L^\alpha)^* \otimes L^\alpha$, $W \otimes v$ becomes $\varphi \rightarrow W(\varphi) \otimes v$. Now for $\varphi \in (H^\delta)^*$, $\rho^\alpha(W(\varphi) \otimes v) = \text{tr } Q^\alpha \circ (W(\varphi) \otimes v) = (Q^\alpha v, W(\varphi)) = (W^\wedge Q^\alpha v, \varphi)$. This shows our claim.

Let $H^\alpha = \tilde{\Gamma}^\alpha(\tilde{H}^\alpha)$ and $U^\alpha = \tilde{R}^\alpha|_{H^\alpha}$. Then $H = \int_M \oplus H^\alpha d\mu(\alpha)$. This is the primary decomposition of U and each U^α is quasi-equivalent to T^α . Now, let $A \in I(T^\alpha, \tilde{R}^\alpha)$. Since $I \otimes I^\alpha = I$ by the above Lemma, A is of the form $v \rightarrow W \otimes v$ for some $W \in H^\delta \otimes (L^\alpha)^*$.

Now, L_δ^α is by the proof of Theorem 3 the space of functionals $A^{\wedge-\infty}(\epsilon^\alpha)^\wedge(w)$, $w \in H^\delta$, $A \in I(T^\alpha, U^\alpha)$, i.e. the functionals $x \rightarrow (\epsilon^\alpha Ax, w) = (\tilde{\epsilon}^\alpha(W \otimes x), w) = (W^\wedge Q^\alpha x, w) = \langle x, (Q^\alpha)^\wedge W^* w \rangle$ ($x \in C^\infty(T^\alpha)$). (Here $(Q^\alpha)^\wedge$ maps $C^{-\infty}(T^\alpha)$ into $C^{-\infty}(T^\alpha)$ and $W^* w$ is thought of as an element of $C^{-\infty}(T^\alpha)$.)

$A \in I(T^\alpha, U^\alpha)$ iff $\tilde{\Gamma}^\alpha A = A$, i.e. iff $\Gamma^\alpha W = W$. This is equivalent to W intertwining δ^* and $(T^\alpha)^*|_K$ which is equivalent to W^* intertwining δ and $T^\alpha|_K$. Hence L_δ^α is the image under $(Q^\alpha)^\wedge$ of the space of vectors $W^* w$ (w and W varying) which is just the image under $(Q^\alpha)^\wedge$ of the space $T^\alpha(\chi_\delta)(L^\alpha)$. This proves the first part of our claim.

To compute $\|(Q^\alpha)^\wedge W^* w\|_\delta^\alpha$ note that by definition it is $c(A, A)\|w\|^2$. However, A^\wedge is the map of $(H^\delta \otimes (L^\alpha)^*) \otimes L^\alpha$ into L^α given by $F \otimes v \rightarrow v(F, W)$. Hence $A^\wedge A(x) = (W, W)x$. Thus $c(A, A) = (W, W) = \text{tr } W^\wedge W$. But $W^\wedge W$ intertwines δ^* so $W^\wedge W = dI$. Furthermore $dN = \text{tr } W^\wedge W = (W, W)$ so $c(A, A)\|w\|^2 = Nd\|w\|^2$. Hence $d\|w\|^2 = (W^* w, W^* w) = \|W^* w\|^2$. Thus $\|(Q^\alpha)^\wedge W^* w\|_\delta^\alpha = N^{1/2} \|W^* w\|$, proving the unitarity.

To compute D^α we need σ^α . σ^α is the adjoint of the injection i of L_δ^α into $C^{-\infty}(T^\alpha)$. If $(Q^\alpha)^\wedge(x) \in L_\delta^\alpha$, $x \in T(\chi_\delta)L^\alpha$, and if $v \in C^\infty(T^\alpha)$, then

$$\begin{aligned} \langle v, i((Q^\alpha)^\wedge(x)) \rangle &= (Q^\alpha v, x) = (T^\alpha(\chi_\delta)Q^\alpha v, x) \\ &= ((Q^\alpha)^\wedge T^\alpha(\chi_\delta)Q^\alpha v, (Q^\alpha)^\wedge x)_\delta^\alpha N^{-1}. \end{aligned}$$

Hence $\sigma^\alpha = N^{-1}(Q^\alpha)^\wedge T^\alpha(\chi_\delta)Q^\alpha$.

It follows that

$$\begin{aligned} ND^\alpha(f) &= \text{tr}(Q^\alpha)^\wedge T^\alpha(\chi_\delta)Q^\alpha T^\alpha(f)|_{L_\delta^\alpha} \\ &= \text{tr } T^\alpha(\chi_\delta)Q^\alpha (T^\alpha)^{-\infty}(f)(Q^\alpha)^\wedge|_{L_0^\alpha} \end{aligned}$$

by unitary invariance of the trace. Also, on $C^\infty(T^\alpha)$, $T^\alpha(f)Q^\alpha = (T^\alpha)^{-\infty}(f)(Q^\alpha)^\wedge$ by the selfadjointness of Q^α . Hence $T^\alpha(f)Q^\alpha$ has a bounded extension. Furthermore $\Omega \equiv 1$ on K and hence Q^α commutes with $T^\alpha|_K$. Thus $D^\alpha = N^{-1}Q^\alpha T^\alpha(\chi^\delta)T^\alpha(f)Q^\alpha|_{L_0^\alpha}$. This finishes the theorem. Q.E.D.

REMARKS. Theorem 3 can be considered both a generalization of the Frobenius reciprocity theorem and a generalization of the Plancherel theorem. In fact, the first statement is obvious. To see the second, note that by definition $\mathcal{D}^\alpha(f) = \epsilon^\alpha(U^\alpha)^{-\infty}(f)(\epsilon^\alpha)^\wedge$ so

$$\int_M \mathcal{D}^\alpha(f) d\mu(\alpha) = \epsilon U^{-\infty}(f) \epsilon^\wedge = \tau(f)(e).$$

Theorem 3 then says that corresponding to a.e. irreducible representation T^α occurring in the decomposition of U there is a subspace L_δ^α of $C^{-\infty}(U)$ and a Hilbert structure defined on L_δ^α for which $\text{tr } \tau(f)(e) = \int_M \text{tr } \sigma^\alpha T^\alpha(f)|_{L_\delta^\alpha} d\mu(\alpha)$. Note also that although the Hilbert structure on L_δ^α is difficult to describe theoretically, the definition of L_δ^α given in the proof of Theorem 3 depends only on the unitary equivalence class of T^α in the sense that if T^α is unitarily equivalent to S^α , then the unitary equivalence gives rise to a unitary isomorphism of the corresponding L_δ^α , as the reader may easily verify. L_δ^α depends on the measure μ to the extent that changing μ multiplies the scalar product by a positive constant. It would of course be desirable to have an intrinsic description of L_δ^α such as is available in the K -compact case. We do not know of such a description except in certain specific cases which will be described in a later work.

From Theorem 4 we obtain the consistency of our concept of spherical function and Godement's.

COROLLARY. *If K is a large compact subgroup of G and G is unimodular then almost every D^α is given via integration against a spherical function in the sense of Godement [1].*

Now, still assuming K is compact, let $I_{c,\delta}$ be the space of all functions in $C_c(G)$ which satisfy (i) $f(kxk^{-1}) = f(x)$ and (ii) $\bar{\chi}_\delta * f = f$. It is well known that $I_{c,\delta}$ plays an important role in the study of spherical functions. To see why this is so first note that every such f , when identified with the distribution $f dx$ transforms on the left according to $\lambda_\delta (= \chi_\delta$ in this case) and is K central. f is also finite. In fact, A_f is the operator $F \rightarrow \int_G f(x)F(x \cdot) dx$. Hence $I_{c,\delta} \subset F_\delta$. The significant fact about $I_{c,\delta}$ is the following

LEMMA. *The set of $A_f, f \in I_{c,\delta}$, is strong operator dense in $I(U, U)$ (if K is compact).*

PROOF. Let notation be as in the previous proof. Since $\tilde{R} = I \otimes R$, the space $I(\tilde{R}, \tilde{R})$ is the set of operators of the form $B \otimes A$ where $A \in I(R, R)$ and

$B \in \text{Hom}_c(H^\delta, H^\delta)$. From the general results of von Neumann algebras, $I(R, R)$ is the von Neumann algebra generated by $\{L(x) | x \in G\}$. Hence, $\{L(f) | f \in C_c^\infty(G)\}$ is strong operator dense in $I(R, R)$, and thus the operators $B \otimes L(f)$ are strong operator dense in $I(\tilde{R}, \tilde{R})$. In terms of \tilde{H} these are just the operators $F \rightarrow \int_G g(x)BF(x \cdot) dx$.

Composing with Γ we conclude that the operators on H given by

$$\begin{aligned} A: F &\rightarrow \int_G \int_K g(x)\delta(k)BF(xk^{-1} \cdot) dk dx \\ &= \int_G \int_K g(xk)\delta(k)B dk F(x \cdot) dx \\ &= \int_{K \setminus G} \int_K \int_K g(k' xk)\delta(k)BF(k' x \cdot) dk dk' dx \\ &= \int_{K \setminus G} \int_K \int_K g(k' xk)\delta(k)B\delta(k')F(x \cdot) dk dk' dx \end{aligned}$$

are dense in $I(U, U)$. Let $H(x) = \int_K \int_K g(k' xk)\delta(k)B\delta(k') dk dk'$ and let $h(x) = \text{tr } H(x)$. It is easily seen that h is K central, compactly supported and transforms on the left according to λ_δ .

Furthermore,

$$\begin{aligned} \int_K h(kx)\delta(k) dk &= \int_K \text{tr } H(kx)\delta(k) dk \\ &= \int_K \text{tr}(H(x)\delta(k^{-1}))\delta(k) dk = H(x). \end{aligned}$$

Thus

$$\begin{aligned} \int_G h(x)F(x) dx &= \int_{K \setminus G} \int_K h(kx)F(kx) dk \\ &= \int_{K \setminus G} \int_K h(kx)\delta(k)F(x) dx = \int_{K \setminus G} H(x)F(x \cdot) dx \\ &= A(F). \end{aligned}$$

This proves our claim. Q.E.D.

As an almost immediate corollary to this we have the following.

THEOREM 5. *There is an almost everywhere defined one-to-one correspondence between spherical distributions of height δ and certain irreducible representations of $I_{c,\delta}$.*

PROOF. Let notation be as in the proof of Theorem 3. Since $I(U, U)$ is the commutator of U and we are dealing with the central decomposition, $I(U, U)$ decomposes as the direct integral of commutators of the U^α . Furthermore, this decomposition has the property that for a.e. α the mapping $\pi^\alpha: A \rightarrow A^\alpha$ is a

* representation of $I(U, U)$. Also, any countable dense subset of $I(U, U)$ has a strong operator dense image under π^α for a.e. α . Since $I_{c,\delta}$ contains a countable dense set, its image under π^α is, for a.e. α , dense. Since U^α is primary, the algebra $I(U^\alpha, U^\alpha)$ has a faithful irreducible representation which is continuous in the strong operator topology. In fact the left action of $I(U^\alpha, U^\alpha)$ on the Hilbert space $I(T^\alpha, U^\alpha)$ given via composition is easily seen to define such a representation. Call this representation Λ^α . Then $\Lambda^\alpha \pi^\alpha$ is an irreducible representation of $I(U, U)$ and the image of $I_{c,\delta}$ under it is strongly dense in its image. Hence $\Lambda^\alpha \pi^\alpha|_{I_{c,\delta}}$ is still irreducible. Note also that $\Lambda^\alpha \pi^\alpha$ acts on $I(T^\alpha, U^\alpha)$ and hence its dimension is $\dim I(T^\alpha, U^\alpha)$ which by Theorem 3 is the number of times δ occurs in L_δ^α . Q.E.D.

REMARKS. If K is not compact we can, with considerably less work, prove the same theorem with F_δ in place of $I_{c,\delta}$, since F_δ is isomorphic with $I(U, U)$.

However, what we cannot prove for K general is the following theorem.

THEOREM 6. *If G is unimodular, D^α is a spherical distribution of height δ which corresponds to the representation Λ^α of $I(U, U)$, then for all $f \in I_{c,\delta}(G)$, $\Lambda^\alpha(f)$ is trace class and $D^\alpha(f) = C^\alpha \operatorname{tr} \Lambda^\alpha(f)$ for some $C^\alpha \in \mathbb{C}$.*

PROOF. Let us not initially assume G unimodular. Then for $f \in C_c^\infty(G)$, by Theorem 4,

$$\begin{aligned} D^\alpha(f) &= N^{-1} \operatorname{tr} Q^\alpha T^\alpha(\chi_\delta * f) Q^\alpha | L_0^\alpha \\ &= N^{-1} \operatorname{tr} Q^\alpha T^\alpha(\chi_\delta * f) Q^\alpha T^\alpha(\chi_\delta) = N^{-1} \operatorname{tr} Q^\alpha T^\alpha(\chi_\delta * f^* \chi_\delta) Q^\alpha. \end{aligned}$$

If $f \in I_{c,\delta}$ this becomes $N^{-1} \operatorname{tr} Q^\alpha T^\alpha(f) Q^\alpha$. If G is unimodular we may take the $Q^\alpha = I^\alpha$.

Now, consider $H \subset \tilde{H} = H^\delta \otimes L^2$ as in the proof of Theorem 4. Then, by the proof of Theorem 4 and the lemma, the restriction to H^α of the space of operators $T^\alpha(f)^* \otimes I^\alpha$, $f \in I_{c,\delta}$ is strongly dense in $I(U^\alpha, U^\alpha)$. Hence the representation π^α of the above proof, when restricted to $I_{c,\delta}$ is the map $f \rightarrow T^\alpha(f)^* \otimes I^\alpha | H^\alpha$. Now, for $f \in I_{c,\delta}$, the map

$$T^\alpha(f)^* \otimes I \rightarrow T^\alpha(f)^* \otimes I | H^\alpha$$

is 1-1 since $T^\alpha(f)^*$ commutes with Γ^α . The von Neumann algebra generated by $T^\alpha(h)^* \otimes I$ ($h \in C_c(G)$) has a semifinite trace τ given by $\tau(T^\alpha(h)^* \otimes I) = \operatorname{tr} T^\alpha(h)^*$ which is (for a.e. α) finite for $h \in C_c^\infty(G)$. Hence the algebra generated by $T^\alpha(f)^* \otimes I | H^\alpha$ has a semifinite trace. Since this algebra is type I and is a factor (it is the commutator of the algebra generated by U^α), we conclude that if Λ^α is any irreducible * representation of it, then $\operatorname{tr} \Lambda^\alpha(T^\alpha(f)^* \otimes I)$ is finite for $f \in I_{c,\delta}$ and is a multiple of $\tau(T^\alpha(f)^* \otimes I) = \operatorname{tr} T^\alpha(f)^*$. Since this is just $\pi^\alpha(f)$, we are done. Q.E.D.

REMARKS. In the K compact case one could almost certainly prove results similar to the above without the assumption that G is a Lie group. However, in the noncompact case one cannot so easily do without C^∞ vector techniques, even in the unimodular case. In fact, there exist examples of nilpotent Lie groups G and *cocompact* discrete subgroups Γ for which no unitary infinite-dimensional irreducible representations contain vectors invariant under Γ . In particular, if δ is the identity representation of Γ , then for "most" α , $L_\delta^\alpha \subset C^{-\infty}(U^\alpha) \sim C^\infty(U^\alpha)$. In a subsequent paper we will give "integral formulas" for the spherical functions in this case.

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