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# THE SPECTRAL THEORY OF POSETS AND ITS APPLICATIONS TO $C^*$-ALGEBRAS

BY

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ABSTRACT. This paper uses methods from the spectral theory of partially ordered sets to clarify and extend some recent results concerning approximately finite-dimensional $C^*$-algebras. An extremely explicit description is obtained of the Jacobson topology on the primitive ideal space, and it is shown that this topology has a basis of quasi-compact open sets. In addition, the main results of [4] are proved using only elementary means.

**Introduction.** Bratteli [3] introduced the idea of an approximately finite-dimensional $C^*$-algebra (or AF $C^*$-algebra)—one which is the limit of an inductive system of finite-dimensional $C^*$-algebras. In particular, [3] gives an analysis of the primitive ideals of an AF $C^*$-algebra, relating them to certain subsets of the diagram of $A$—a subset of $N \times N$ equipped with a binary relation, $\downarrow$. In [4], this formalism is used to give some topological results on the space Prim $A$ of primitive ideals of $A$, equipped with the Jacobson topology. Bratteli notices that the diagram may be considered a partially ordered set, but makes no real use of this fact; his topological results are mostly proved by recourse to the ideal theory of $C^*$-algebras.

In this paper, I work entirely within the framework of the theory of partially ordered sets, directly deducing topological results about Prim $A$. Use of a notion of spectrum for a partially ordered set (related to that of [6]) enables one to give an extremely explicit description of the Jacobson topology on Prim $A$. Given this description, I am able to reduce the proofs of the main results on spectral theory in [4] to exercises in elementary set theory. The paper not only adds new methods, but also new results. For example, I show that Prim $A$ has a basis of quasi-compact open sets. Bratteli [5] has since given a different proof of this result based on functional analytic methods.

This seems to be one of the very few classes of $C^*$-algebras where, in the

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absence of Hausdorff separation for Prim $A$, a very explicit description of the Jacobson topology is available. (Another is considered in [1].) The methods of this paper may have applications to algebras which are limits in the category of $C^*$-algebras of systems more general than inductive systems.

The paper is organized as follows. §1 serves to recall some results of [3] and to introduce the Bratteli poset $\mathcal{D}(A)$ of an AF $C^*$-algebra $A$. §2, which contains the bulk of the work, is independent of the other two sections; it deals with the spectral theory of a class of partially ordered sets, a class which includes the Bratteli posets. For each partially ordered set $P$ of this class, I define a topological space Spec $P$ whose topology is explicitly described. A number of results are proved about the topology of Spec $P$. Finally, in §3, I combine the results of the first two sections by means of a theorem which asserts that Prim $A$ is homeomorphic to Spec $\mathcal{D}(A)$, and deduce the main results of the paper.

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CONVENTIONS. If $X$ is a topological space, I shall mean by a *basis* for the topology of $X$ a set $\mathcal{B}$ of open subsets of $X$ such that every open set may be expressed as a union of sets from $\mathcal{B}$.

I shall only ever consider one topology on the $C^*$-algebra $A$—the *norm topology*. Thus a closed ideal of $A$ will mean a norm-closed ideal, and so on.

1. **Bratteli diagrams and Bratteli posets.** I shall begin by recalling some results of [3]. Let $A$ be a $C^*$-algebra (with identity), which is the inductive limit of the system $(A_n, j_n)$, the $A_n$ being finite-dimensional $C^*$-algebras, and $j_n\colon A_{n-1} \longrightarrow A_n$ an embedding. Such an algebra is called an AF $C^*$-algebra. One may identify $A$ with $\bigcup_{n=1}^{\infty} A_n$. Since any finite-dimensional $C^*$-algebra may be uniquely decomposed as a direct sum of $m \times m$ matrix algebras, one can write $A_n = \bigoplus_{k=1}^{k_n} M_{(n,k)}$, where each of the $M_{(n,k)}$ is an $m \times m$ matrix algebra for some $m$. The embedding $j_n$ decomposes into its components $j_{(n-1,k),(n,k')}\colon M_{(n-1,k)} \longrightarrow M_{(n,k')}$. Bratteli denotes by $\mathcal{D}(A)$ the set $\{(n, k) | n \in N, k = 1, \ldots, k_n\}$, and defines a relation "$\downarrow$" on $\mathcal{D}(A)$ by $(n, k) \downarrow (n + 1, k')$ if $j_{(n,k),(n+1,k')}$ is nonzero. I shall call the set $\mathcal{D}(A)$ equipped with the relation $\downarrow$, the *Bratteli diagram* of $A$.

In general, let $\mathcal{D}$ be a nonempty subset of $N \times N$ with the property that, for each $n \in N$, $\{m \,|\, (n, m) \in \mathcal{D}\}$ is finite, and $(0, m) \in \mathcal{D}$ if and only if $m = 0$. Suppose $N$ is equipped with a binary relation $\downarrow$ satisfying conditions (i), (ii) and (iii) on p. 201 of [3]. Then $(\mathcal{D}, \downarrow)$ will be said to be a *Bratteli diagram*. In [3] it is shown that every Bratteli diagram arises as the Bratteli diagram of $A$ for some AF-algebra $A$.

(1.1) DEFINITION. Let $(\mathcal{D}, \downarrow)$ be a Bratteli diagram. Define another relation, "$\geqslant$" on $\mathcal{D}$ as follows: for $x, y \in \mathcal{D}$, $x \geqslant y$ if there exists a finite sequence $\{z_i\}_{i=1}^r$ of elements of $\mathcal{D}$ with $z_1 = x$, $z_r = y$, and for each $i = 1, \ldots, r-1$, $z_i \downarrow z_{i+1}$.

It is easily seen that $(\mathcal{D}, \geqslant)$ is a partially ordered set with a maximum element. Let $d\colon \mathcal{D} \longrightarrow N$ be the projection of $\mathcal{D}$ onto its first factor. I will call $(\mathcal{D}, \geqslant)$ the *Bratteli poset of* $(\mathcal{D}, \downarrow)$, $d$ its *degree function.* It is easy to verify the truth of

(1.2) LEMMA. *Let $(\mathcal{D}, \downarrow)$ be a Bratteli diagram, $(D, \geqslant)$ the associated Bratteli poset with degree function $d$. Then*

(i) *If $x \leqslant y$ then $d(x) \geqslant d(y)$.*

(ii) *For all $n \in N$, $\mathcal{D}^n := d^{-1}(n)$ is a finite set.*

(iii) *For all $x, y \in \mathcal{D}$, if $x \geqslant y$ and $x \neq y$, then there exists $z \in \mathcal{D}^{d(x)+1}$ with $x \geqslant z \geqslant y$.*

(iv) *$\mathcal{D}$ has no minimal elements.*

*Conversely, if $(\mathcal{D}, \geqslant)$ is any partially ordered set with a maximum element, equipped with a function $d\colon \mathcal{D} \longrightarrow N$ satisfying* (i)–(iv) *above, there exists a Bratteli diagram whose associated Bratteli poset is isomorphic to $(\mathcal{D}, \geqslant)$.*

Any partially ordered set $(\mathcal{D}, \geqslant)$, equipped with a function $d\colon \mathcal{D} \longrightarrow N$ satisfying (1.2)(i)–(iv) will be called a *Bratteli poset.*

I shall require the following

(1.3) LEMMA. *Let $A$ be an AF $C^*$-algebra, $(\mathcal{D}(A), \downarrow)$ its Bratteli diagram and $(\mathcal{D}(A), \geqslant)$ its Bratteli poset. The following conditions are equivalent*:

(i) *$A$ is abelian.*

(ii) *For all $x \in [\mathcal{D}(A)]^n$, $n \geqslant 1$, there is a unique $y \in \mathcal{D}(A)^{n-1}$ with $y \downarrow x$.*

(iii) *For all $x \in \mathcal{D}(A)$, $\{y \mid y \geqslant x\}$ is a chain.*

PROOF. The equivalence of (i) and (ii) is given in (3.1) of [4]; the equivalence of (ii) and (iii) is simple verification and is left to the reader. □

**2. The spectral theory of certain partially ordered sets.**

(2.1) NOTATION. A poset is a set $P$ together with a partial order, $\leqslant$, and a greatest element, 1.

If $x \in P$, I set $\uparrow x = \{y \in P \mid y \geqslant x\}$, $\downarrow x = \{y \in P \mid y \leqslant x\}$, $\underline{x} = \{y \in P \mid y$ is maximal in $\downarrow x \setminus \{x\}\}$, $\overline{x} = \{y \in P \mid y$ is minimal in $\uparrow x \setminus \{x\}\}$.

If $x, y \in P$, let $x \wedge y = \{z \in P \mid z$ is maximal in $\downarrow x \cap \downarrow y\}$, $x \vee y = \{z \in P \mid z$ is minimal in $\uparrow x \cap \uparrow y\}$. Some of these sets may be empty.

(2.2) DEFINITION. A poset $P$ is called *co-well ordered* if

(I) Every chain in $P$ has a maximal element.

$P$ is (*weakly*) *well ordered* if

(II) Every chain in $P$ which is bounded below has a minimal element.

A poset $P$ is called graded if there is a poset morphism $d\colon P \longrightarrow N^{\mathrm{op}}$, where $N^{\mathrm{op}}$ is the set of integers with reverse order, such that if $x \leqslant y$ and $d(x) = d(y)$ then $x = y$, and $d(1) = 0$.

Thus, for $x \leqslant y$, $d(x) \geqslant d(y)$. The sets $P^n = d^{-1}(n)$ are called the *layers* of $P$ of *degree n*, and $d$ is called the *degree function associated with* $P$. A graded poset is *laterally finite* if all layers are finite.

The graded poset $P$ with degree function $d$ is said to be *well-graded* if for all $x \in P$, $d(\underline{x}) = \{d(x) + 1\}$.

A poset satisfying (I) will briefly be called a *CW poset*, one satisfying (I) and (II), a *WCW poset.*

(2.3) EXAMPLES. (i) Let $X$ be a set, $P$ the set of all finite subsets under reverse inclusion (i.e. $A \geqslant B$ if $A \subseteq B$). Set $d(F) = \operatorname{card} F$, for $F \in P$. Then $(P, \geqslant)$ is a well-graded WCW poset which is laterally finite if and only if $X$ is finite.

(ii) Let $M$ be a module over a commutative ring $R$ with 1, $P$ the set of finitely-generated submodules under reverse inclusion. If $M$ satisfies the descending chain condition for finitely-generated submodules, then $P$ is a CW poset. If $R$ is a field, then $P$ is a well-graded WCW poset with gradation $d(V) = \dim V$. $P$ is laterally finite if and only if $\dim M < \infty$.

(iii) A partially ordered set $P$ is a Bratteli poset if and only if it is a well-graded laterally finite poset with no minimal elements.

(2.4) REMARKS. (i) Let $S$ be a subset of the CW poset $P$. Then every element of $S$ is majorized by an element which is maximal in $S$. In particular, for $x \in P$, $\underline{x} = \varnothing$ if and only if $\downarrow x = \varnothing$ if and only if $x$ is minimal in $P$; and for $x, y \in P$, $x \wedge y = \varnothing$ if and only if $\downarrow x \cap \downarrow y = \varnothing$.

(ii) Let $S$ be a subset of the WCW poset $P$ which is bounded below. Then every element of $S$ is minimized by an element minimal in $S$. Hence, for any $x \in P$, $\overline{x} = \varnothing$ if and only if $x = 1$; and for all $x, y \in P$, $x \vee y \neq \varnothing$.

[The proofs of (i) and (ii) are straightforward, using Zorn's lemma.]

(iii) Every graded poset is automatically a WCW poset.

(2.5) DEFINITION. A *filter* in a CW poset $P$ is a nonempty subset $F$ satisfying the following:

(i) For all $x \in F$, $\uparrow x \subseteq F$.

(ii) For all $x, y \in F$, $(x \wedge y) \cap F \neq \varnothing$.

A filter is called *recursive* if for all $x \in F$, $\underline{x} = \varnothing$ or $\underline{x} \cap F \neq \varnothing$.

Let $F(P)$ denote the set of all filters in $P$, Spec $\bar{P} \subseteq F(P)$ the set of all recursive filters. A filter which is maximal with respect to inclusion is called an *ultrafilter*.

The proof of the following lemma is an easy exercise in elementary set theory, using (2.4)(i) and Definition (2.5).

LEMMA. *Let $F \in \mathcal{F}(\mathcal{P})$. Then*

(i) $1 \in F$.

(ii) *If $x \in F$ is such that $\underline{x} \cap F = \emptyset$, then $x$ is the smallest element of $F$.*

(iii) *If $F$ is an ultrafilter, $F$ is recursive.*

*Topology on* Spec $\mathcal{P}$. I will always assume that $\mathcal{P}$ satisfies at least (I).

For $a \in \mathcal{P}$, define $S(a) \subseteq \text{Spec } \mathcal{P}$ by $S(a) = \{F \in \text{Spec } \mathcal{P} \mid a \in F\}$.

(2.7) REMARKS. (i) Let $a, b \in \mathcal{P}$. Then $S(a) \cap S(b) = \bigcup_{c \in a \wedge b} S(c)$. [Let $F \in \text{Spec } \mathcal{P}$. By 2.5(i) and (ii), $a \in F$ and $b \in F$ if and only if there exists $c \in a \wedge b$ such that $c \in F$.]

(ii) Let $F, G \in \text{Spec } \mathcal{P}$. If $F \not\subseteq G$, then there is an $a \in \mathcal{P}$ such that $F \in S(a)$ but $G \notin S(a)$.

Thus, the sets $S(a)$, $a \in \mathcal{P}$, form a basis for the open sets of a $T_0$-topology on Spec $\mathcal{P}$. This topology will be called the *natural topology* and denoted by $o$.

(2.8) Let $X$ be a topological space. A subset $C$ of $X$ is called *irreducible* if it is closed and not contained in the union of two proper closed subsets of itself.

LEMMA. *Let $C \subseteq \text{Spec } \mathcal{P}$ be o-closed, and set $F_C = \bigcup_{F \in C} F$. The following statements are equivalent*:

(i) *$C$ is irreducible.*

(ii) *Let $a, b \in \mathcal{P}$. Then $C \cap S(a) \neq \emptyset \neq C \cap S(b)$ implies that there exists $c \in a \wedge b$ such that $S(c) \cap C \neq \emptyset$.*

(iii) *Let $a, b \in \mathcal{P}$. Then if there exist $F, G \in C$ with $a \in F$, $b \in G$ then there exists $H \in C$ with $a \wedge b \cap H \neq \emptyset$.*

(iv) $F_C \in \text{Spec } \mathcal{P}$.

*If these conditions are satisfied, then for all $a \in \mathcal{P}$, $F_C \in S(a)$ if and only if $S(a) \cap C \neq \emptyset$.*

PROOF. (i) $\Rightarrow$ (ii) If $C$ does not satisfy (ii), one may choose $a, b$ with $C \cap S(a) \neq \emptyset \neq C \cap S(b)$, but for all $c \in a \wedge b$, $S(c) \cap C = \emptyset$. By (2.7)(i), the latter statement implies that $[S(a) \cap S(b)] \cap C = \emptyset$. Hence $C \cap [S(a)]^c \neq \emptyset \neq C \cap [S(b)]^c$ and $([S(a)]^c \cap C) \cup ([S(b)]^c \cap C) = C$. Thus $C$ is not irreducible.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) is trivial.

(iv) $\Rightarrow$ (v) Suppose $F_C \in \text{Spec } \mathcal{P}$. If $C$ is not irreducible, there exist $a, b \in \mathcal{P}$ with $([S(a)]^c \cap C) \cup ([S(b)]^c \cap C) = C$ but $[S(a)]^c \cap C \neq \emptyset \neq [S(b)]^c \cap C$. By the former statement (via 2.7(i)), $S(c) \cap C = \emptyset$ for all $c \in a \wedge b$; by the latter $a \in F_C$ and $b \in F_C$ which implies $(a \wedge b) \cap F_C \neq \emptyset$. This is a contradiction.

The last statement of the lemma is clear. □

THEOREM. *For every CW poset* $\mathcal{P}$, *the natural topology on* Spec $\mathcal{P}$ *has the property that every irreducible set is a singleton closure.*

PROOF. Let $C$ be an irreducible set, and define $F_C$ as in the lemma. The lemma assures us that $C$ is in fact the closure of $\{F_C\}$. □

*Acyclic CW posets.*

(2.9) PROPOSITION. *Let* $\mathcal{P}$ *be a CW poset. The following statements are equivalent*:

(i) *For all* $x, y \in \mathcal{P}$ *such that* $x \wedge y \neq \emptyset$ *either* $x \leqslant y$ *or* $y \leqslant x$.

(ii) *For all* $x \in \mathcal{P}$, $\uparrow x$ *is totally ordered.*

(iii) *Every filter is totally ordered.*

(iv) *Every recursive filter is totally ordered.*

PROOF. (iii) ⇒ (ii) ⇒ (i) ⇒ (iv) is straightforward using (2.5). For (iv) ⇒ (iii), note that every filter is contained in an ultrafilter and apply (2.6)(iii). □

The CW poset $\mathcal{P}$ is called *acyclic* if it satisfies the equivalent conditions of the above proposition.

(2.10) LEMMA. *In an acyclic CW poset* $\mathcal{P}$ *every maximal chain is a recursive filter. If* $\mathcal{P}$ *is further a WCW poset, the converse is also true and* Spec $\mathcal{P}$ *is precisely equal to the set of maximal chains in* $\mathcal{P}$.

PROOF. The first statement follows from (2.6)(iii) and (2.9). For the second, suppose $\mathcal{P}$ is an acyclic WCW poset, $F \in$ Spec $\mathcal{P}$. Then $F$ is a chain. If $F$ is not maximal, choose $y \in \mathcal{P} \backslash F$ such that $F \cup \{y\}$ is a chain. (2.5)(i) shows that $y$ is a lower bound for $F$, and hence by (II), $F$ has a minimum element $x_0 \geqslant y$, $x_0 \neq y$. By (2.4)(i) and (2.5), $\underline{x}_0 \cap F \neq \emptyset$, a contradiction. □

(2.11) LEMMA. *A CW poset* $\mathcal{P}$ *is acyclic if*

(v) *For all* $a \in \mathcal{P}$, $[S(a)]^c = \bigcup_{b \in [\downarrow a \cup \uparrow a]^c} S(b)$.

*If* $\mathcal{P}$ *is further a WCW poset, the converse is also true*; *viz. if* $\mathcal{P}$ *is acyclic,* $\mathcal{P}$ *satisfies* (v).

PROOF. (v) ⇒ (2.9)(iv) Let $F \in$ Spec $\mathcal{P}$. Then for $x, y \in F$, $S(x) \cap S(y) \neq \emptyset$, so by (v), $\bigcup\{S(b) | b \in [\uparrow y \cup \downarrow y]^c\} \not\supseteq S(x)$. Hence $x \in \uparrow y \cup \downarrow y$. The second statement follows from (2.10). □

THEOREM. *Let* $\mathcal{P}$ *be an acyclic WCW poset. Then the natural topology is a zero-dimensional Hausdorff topology.*

*Duality theorems for well-graded acyclic posets.*

(2.12) Let $X$ be a set. A function $D$: $X \times X \longrightarrow R$ is called an *ultrametric* if

(i) For all $x, y \in X$, $D(x, y) \geqslant 0$; if $D(x, y) = 0$ then $x = y$.

(ii) Let $x, y, z \in X$. Then $D(x, z) \leqslant \max(D(x, y), D(y, z))$.

Notice that every ultrametric is a metric.

(2.13) LEMMA. *Let $P$ be an acyclic well-graded poset with degree function $d$. There exists an ultrametric $D$ on* Spec $P$ *with range* $\{0\} \cup \{2^{-n} \mid n \in N\}$ *such that the topology induced by $D$ is the o-topology*; *indeed the sets $S(a)$ with $d(a) = n$ are precisely the $2^{-n}$ balls. Moreover,* Spec $P$ *equipped with $D$ is a complete metric space.*

PROOF. For $F, G \in \operatorname{Spec} P$, let $S_{F,G} = \{n \in N \mid \forall k \leqslant n, P^k \cap F = P^k \cap G\}$. Since $1 \in F \cap G$, $S_{F,G} \neq \emptyset$. If $S_{F,G}$ is unbounded then $F = G$. If $F \neq G$, let $m_{F,G} = \max[S_{F,G}]$. Define $D$: Spec $P \times$ Spec $P \longrightarrow \{0\} \cup \{2^{-n} \mid n \in N\}$ by

$$D(F, G) = 0, \qquad \text{if } F = G,$$

$$= 2^{-m_{F,G}}, \quad \text{if } F \neq G.$$

It is easily checked that $D$ is an ultrametric. Using this definition, one sees that for $F \in \operatorname{Spec} P$ such that

(1) $F \cap P^n = \{a\}$, $\quad B(F, 2^{-n}) := \{G \in \operatorname{Spec} P \mid D(F, G) \leqslant 2^{-n}\} = S(a)$.

Conversely, for any point $a \in P^n$, $S(a)$ is a $2^{-n}$-ball. [Choose a maximal chain $F$ with $a \in F$. Then $S(a) = B(F, 2^{-n})$.]

Finally, let $\{F_n\}$ be a $D$-Cauchy sequence in Spec $P$. To define $F \in$ Spec $P$ such that $F_n \longrightarrow F$, notice that for all $k \in N$, there exists $N \in N$ such that if $m, n \geqslant N$, $D(F_n, F_m) \leqslant 2^{-k}$. Let $N_k$ be the minimum such $N$, and define the subset $F$ of $P$ by $F \cap P^k = F_{N_k} \cap P^k$. It follows that, for all $k \in N$, $r \leqslant k$ implies $F \cap P^r = F_{N_k} \cap P^r$. Using this fact, one verifies that $F \in$ Spec $P$, and further that, for $k \in N$, $D(F_{N_k}, F) \leqslant 2^{-k}$. Hence $F_n \longrightarrow F$. □

We deduce

THEOREM. *Let $P$ be an acyclic, laterally countable well-graded poset. Then* Spec $P$, *equipped with the above metric is a zero-dimensional Polish space.*

(2.14) I will now show that, under certain conditions, the natural topology is a locally compact topology.

Let $P$ be a well-graded poset. $x \in P$ is said to be *finitely based* if, for all $y \in \downarrow x$, $\underline{y}$ is finite. $P$ is said to be (*spectrally*) *finitely based* if every recursive filter contains a finitely based element.

Notice that for any well-graded poset $P$, 1 is finitely based if and only if $P$ has finite layers.

The proof of the following lemma is left to the reader.

LEMMA. *Let* $\mathcal{P}$ *be an acyclic well-graded poset.* $a \in \mathcal{P}$ *is finitely based if and only if* $S(a)$ *is a compact set in the natural topology.*

THEOREM. *Let* $\mathcal{P}$ *be an acyclic well-graded poset. Then* Spec $\mathcal{P}$ *is a locally compact space if and only if* $\mathcal{P}$ *is spectrally finitely based.* Spec $\mathcal{P}$ *is compact if and only if* $\mathcal{P}$ *has finite layers.*

(2.15) In this paragraph, we prove a converse to Theorem (2.13).

THEOREM. *Let* $X$ *be a zero-dimensional Polish space. Then there exists a countable acyclic well-graded poset* $\mathcal{P}$ *with no minimal elements, such that* Spec $\mathcal{P}$ *is homeomorphic to* $X$. *If* $X$ *is locally compact (compact),* $\mathcal{P}$ *is finitely based (has finite layers).*

PROOF. Let $X$ be a zero-dimensional Polish space. I shall assume the reader is familiar with the construction of a sifting, as given in [2, IX, §6.5]. I call a sifting $(C_n, p_n, \varphi_n)$ *open* if for any $c \in C_n$, $\varphi_n(c)$ is open. A slight modification of the construction of [2, IX, §6.5, Lemma 3] proves that $X$ has a strict open sifting $(C_n, p_n, \varphi_n)$. It is then clear that the function $f$: $L(C) \longrightarrow X$ constructed in [2, §6.5], is open and hence is a homeomorphism. Let $\mathcal{C} = \bigcup_{n=0}^{\infty} C_n$. Defining, for $c \in C_n$ and $c' \in C_m$, $c \geqslant c'$ if $n \geqslant m$ and $P_{nm}(c) = c'$ makes $\mathcal{C}$ into a poset; if we set $d(c) = n$ for $c \in C_n$, $\mathcal{C}$ becomes a countable well-graded poset with no minimal elements. It is not too hard to see that Spec $\mathcal{C}$ is homeomorphic to $L(C)$. This proves the theorem. □

(2.16) Let $Y$ be the topological space of the irrational numbers, and let $K$ be the Cantor space constructed on the interval $[0, 1]$. By the information in [7, §33, I], $K$ is the compactification of $Y$. Using this fact, together with Theorems 1 and 2 of [7, §32, II] and (2.15), one deduces

PROPOSITION. (i) *Let* $\mathcal{P}$ *be an acyclic well-graded poset. Then* Spec $\mathcal{P}$ *is homeomorphic to a closed subspace of* $Y$. *If* $\mathcal{P}$ *is in addition laterally finite, then* Spec $\mathcal{P}$ *is homeomorphic to a closed subspace of* $K$.

(ii) *Let* $X$ *be any Polish space. Then there exists a laterally countable well-graded poset* $\mathcal{P}$, *with no minimal elements such that* $X$ *is a quotient space of* Spec $\mathcal{P}$. *If* $X$ *is, in addition, compact, we may suppose that* $\mathcal{P}$ *is laterally finite.*

*Acyclic covers.*

(2.17) Let $\mathcal{P}$ be a CW poset. Denote by $\widetilde{\mathcal{P}}$ the set of all totally ordered subsets $T$ of $\mathcal{P}$ such that

(i) $T$ has a minimum element $x_T$.

(ii) $T$ is maximal amongst the totally ordered subsets with minimum element $x_T$.

Define $p$: $\widetilde{\mathcal{P}} \longrightarrow \mathcal{P}$ by $p(T) = x_T = \min T$. For $T_1, T_2 \in \widetilde{\mathcal{P}}$, let $T_1 \geqslant T_2$ if $T_1 \subseteq T_2$. One then has

PROPOSITION. *Let $\mathcal{P}$ be a CW poset. $\widetilde{\mathcal{P}}$, equipped with $\geqslant$ is an acyclic CW poset and $p$ is a surjective poset map. If $\mathcal{P}$ is WCW so is $\widetilde{\mathcal{P}}$; if $\mathcal{P}$ is graded (well-graded), so is $\widetilde{\mathcal{P}}$. Further, if $\mathcal{P}$ is graded with finite layers, so is $\widetilde{\mathcal{P}}$.*

The proof of this proposition is elementary, given the following lemma.

LEMMA. *Let $\mathcal{P}$ be a CW poset and let $T \in \widetilde{\mathcal{P}}$. Then $\underline{T} = \{T \cup \{y\} \mid y \in \underline{p(T)}\}$.*

PROOF. First notice that any set of the form $T \cup \{y\}$, $y \in \underline{p(T)}$ is in $\widetilde{\mathcal{P}}$. Thus, we must check that any element of $\downarrow T \backslash \{T\}$ is majorized by an element of $\widetilde{\mathcal{P}}$ of this form. Let $S \in \downarrow T \backslash \{T\}$. Then $p(T) \in S$, and $S \cap \downarrow p(T) \backslash \{p(T)\}$ is a nonempty chain. Hence, by (I) it has a maximum, $y$ say. By the maximality of $S$, $y \in \underline{p(T)}$, and so $T \cup \{y\} \subseteq S$, as required. □

*Auxiliary topology on* Spec $\widetilde{\mathcal{P}}$.

(2.18) Since $\widetilde{\mathcal{P}}$ is a CW poset, one may of course define the $o$-topology on Spec $\widetilde{\mathcal{P}}$. In this paragraph, I will define a second, coarser topology on Spec $\widetilde{\mathcal{P}}$.

For $a \in \widetilde{\mathcal{P}}$, let $\mathcal{W}(a) \subseteq$ Spec $\widetilde{\mathcal{P}}$ be defined by $\mathcal{W}(a) = \bigcup_{p(T) \leqslant a} S(T)$. It is easily seen that $\mathcal{W}(a) = \{F \in \text{Spec } \widetilde{\mathcal{P}} \mid \text{for some } T \in F, T \cap \downarrow a \neq \varnothing\}$. Using this fact, it is easy to prove that for $a, b \in \mathcal{P}$, $\mathcal{W}(a) \cap \mathcal{W}(b) = \bigcup_{c \in a \wedge b} \mathcal{W}(c)$. Thus $\{\mathcal{W}(a) \mid a \in \mathcal{P}\}$ is a basis for a topology $\tau$ on Spec $\widetilde{\mathcal{P}}$. $\tau$ is clearly coarser than $o$.

PROPOSITION. *Let $\mathcal{P}$ be a CW poset. The following statements are equivalent:*

(i) *$\mathcal{P}$ is acyclic.*
(ii) *$p$: $\widetilde{\mathcal{P}} \longrightarrow \mathcal{P}$ is bijective.*
(iii) *For any $T \in \widetilde{\mathcal{P}}$, $S(T) = \mathcal{W}(p(T))$.*

PROOF. (i) ⇒ (ii) ⇒ (iii) is trivial. Suppose $\mathcal{P}$ is not acyclic. By (2.9)(ii), we may find $a \in \mathcal{P}$ and two chains $T_1, T_2$, maximal in $\uparrow a$ with $p(T_1) = p(T_2) = a$, $T_1 \neq T_2$. Choose ultrafilters $F_i$ in $\widetilde{\mathcal{P}}$ with $T_i \in F_i$. One then has $F_1 \in \mathcal{W}(p(T_2))$ but $F_1 \notin S(T_2)$. □

(2.19) LEMMA. *On any topological space $X$, there is an equivalence relation $R_X$, given by $xR_Xy$ if and only if $\overline{\{x\}} = \overline{\{y\}}$. $X/R_X$ is a $T_0$ space, in fact it is the left reflection of $X$ into the category of $T_0$ spaces. The front adjunction of this reflection is the quotient map $q_X$: $X \longrightarrow X/R_X$.*

Since this lemma is well known and easily proved, I shall omit its proof.

Let $\mathcal{P}$ be a CW poset and let $R$ be the equivalence relation of the lemma applied to the topological space $(\operatorname{Spec}\widetilde{\mathcal{P}}, \tau)$. Set $\operatorname{Spec}_0 \widetilde{\mathcal{P}} = (\operatorname{Spec}\widetilde{\mathcal{P}}, \tau)/R$. Define $m$: $\operatorname{Spec}\widetilde{\mathcal{P}} \longrightarrow 2^{\mathcal{P}}$ by $m(F) = \bigcup_{T\in F}[\uparrow p(T)] = \uparrow\bigcup_{T\in F} p(T) = \uparrow\bigcup_{T\in F} T$.

(2.20) LEMMA. *Let $F, G \in \operatorname{Spec}\widetilde{\mathcal{P}}$. The following statements are equivalent.*

(i) *For all $a \in \mathcal{P}$, $F \in \mathcal{W}(a)$ if and only if $G \in \mathcal{W}(a)$.*

(i)$^1$ *For all $a \in \mathcal{P}$, $\bigcup_{T\in F} T \cap \downarrow a \neq \varnothing$ if and only if $\bigcup_{T\in G} T \cap \downarrow a \neq \varnothing$.*

(ii) *For all $T_F \in F$ and for all $T_G \in G$, there exist $T'_F \in F$ and $T'_G \in G$ such that $p(T'_G) \leqslant p(T_F)$, and $p(T'_F) \leqslant p(T_G)$.*

(iii) *The $\tau$-closure of $\{F\}$ is equal to the $\tau$-closure of $\{G\}$.*

The proof, an easy exercise in elementary set theory, is omitted.

PROPOSITION. *With the notation of* (2.19), *for $F, G \in \operatorname{Spec}\mathcal{P}$, $m(F) \in \operatorname{Spec}\mathcal{P}$, and if $F R G$ then $m(F) = m(G)$.*

PROOF. It is clear from the definition of $m$ that $m(F)$ satisfies (2.5)(i) and (ii). To show that $m(F)$ is recursive, let $x \in m(F)$, $\underline{x} \neq \varnothing$. Choose $T \in F$ such that $x \geqslant p(T)$. If $x \neq p(T)$, then $\underline{x} \cap \uparrow p(T) \neq \varnothing$ and we have finished. If $x = p(T)$, Lemma (2.17) implies that $\underline{x} \cap m(F) \neq \varnothing$. The second statement follows from the lemma. □

Let $m_0$: $\operatorname{Spec}_0 \widetilde{\mathcal{P}} \longrightarrow \operatorname{Spec}\widetilde{\mathcal{P}}$ be the mapping naturally induced by $m$.

(2.21) A subset $B$ of a filter $F \in \mathcal{F}(\mathcal{P})$ is called a *basis* for $F$ if $B$ is totally ordered, and for any $x \in F$ there exists $b \in B$ with $b \leqslant x$.

THEOREM. *Let $\mathcal{P}$ be a CW poset. Then $m_0$: $\operatorname{Spec}_0 \widetilde{\mathcal{P}} \longrightarrow \operatorname{Spec}\mathcal{P}$ is a homeomorphism onto the dense set of all $F \in \operatorname{Spec}\mathcal{P}$ which have a basis. If $\mathcal{P}$ is countable, then $m_0$ is a homeomorphism onto* $\operatorname{Spec}\mathcal{P}$.

PROOF. Using the definition of $m$, together with Lemma (2.20)(ii), it is easy to see that $m_0$ is injective. Since $m = m_0 \circ q_{(\operatorname{Spec}\widetilde{\mathcal{P}},\tau)}$ where $q_{(\operatorname{Spec}\widetilde{\mathcal{P}},\tau)}$ is the surjection of (2.19), it is clear that $\operatorname{im} m = \operatorname{im} m_0$. Now suppose $\widetilde{F} \in \operatorname{Spec}\widetilde{\mathcal{P}}$. Then $\{p(T) \mid T \in \widetilde{F}\}$ forms a basis for $m(\widetilde{F})$. Hence $\operatorname{im} m \subseteq \{F \in \operatorname{Spec}\mathcal{P} \mid F \text{ has a basis}\}$. For the opposite containment, I will use a lemma.

LEMMA. *Let $\mathcal{P}$ be a CW poset, $F \in \operatorname{Spec}\mathcal{P}$, and $G$ a maximal totally ordered subset of $F$. For $x \in G$, let $T_x = (\uparrow x) \cap G$. Then $T_x \in \widetilde{\mathcal{P}}$ and $\widetilde{F} = \{T_x \mid x \in G\} \in \operatorname{Spec}\widetilde{\mathcal{P}}$.*

PROOF. Let $x \in G$. Since $G$ is maximal in $F$ and $\uparrow x \subseteq F$, $\uparrow x \cap G$ is maximal in $\uparrow x$; thus $T_x \in \widetilde{\mathcal{P}}$. To see that $\widetilde{F} \in \mathcal{F}(\widetilde{\mathcal{P}})$ is easy, so it remains to see that $\widetilde{F}$ is recursive. By Lemma (2.17), it suffices to show that, for $x \in G$, $\underline{x} = \varnothing$ or $\underline{x} \cap G \neq \varnothing$. But this is obviously the case, since $F$ is recursive and $G$ is maximal. □

Suppose now that $F$ has a basis. By Zorn's lemma, $F$ has a maximal basis

$G$. Defining $\widetilde{F}$ as in the lemma, one sees that $m(\widetilde{F}) = F$, and so $F \in \operatorname{im} m$.

Next, let $a \in \mathcal{P}$, $F \in \mathcal{S}(a) \neq \varnothing$. Let $C$ be a chain maximal in $F$ and containing $a$. The lemma shows that $\widetilde{F} = \{\uparrow x \cap C \mid x \in C\} \in \operatorname{Spec} \widetilde{\mathcal{P}}$, and clearly $m(\widetilde{F}) \in \mathcal{S}(a)$. Thus the image is dense. To show that $m_0$ is a homeomorphism, it will suffice to show that for $a \in \mathcal{P}$, $m[\mathcal{W}(a)] = \mathcal{S}(a) \cap \operatorname{im} m_0$. In fact, from the definitions of $\mathcal{W}(a)$ and $m$, it is clear that for all $a \in \mathcal{P}$, and for all $\widetilde{F} \in \operatorname{Spec} \widetilde{\mathcal{P}}$, $F \in \mathcal{W}(a)$ if and only if $m(\widetilde{F}) \in \mathcal{S}(a)$.

To complete the proof of the theorem, let $F = \{x_i \mid i \in \mathcal{N}\}$ be a countable filter. Define a basis $\{y_i\}$ for $F$ as follows: $y_0 = x_0$. Suppose $y_i$ defined. Then since $F$ is a filter, $(y_i \wedge x_{i+1}) \cap F \neq \varnothing$. Let $y_{i+1}$ be an arbitrary element of this set. □

(2.22) LEMMA a. *Let $\mathcal{P}$ be a CW poset, $a \in \mathcal{P}$, and choose $T_a \in \widetilde{\mathcal{P}}$ with $p(T_a) = a$. If $q$: Spec $\widetilde{\mathcal{P}} \longrightarrow$ Spec$_0$ $\widetilde{\mathcal{P}}$ is the map of* (2.19), *then* $q[\mathcal{W}(a)] = q[\mathcal{S}(Ta)]$.

PROOF. One knows that $\mathcal{W}(a) \supseteq \mathcal{S}(T_a)$. Thus let $F \in \mathcal{W}(a)$, and choose $T_0 \in F$ such that $p(T_0) \leqslant a$. Then the set $[\bigcup_{S \in \downarrow T_0 \cap F} p(S)] \cup T_a$ is a chain contained in $m(F)$. Choose a chain $C$ containing it and maximal in $m(F)$. By Lemma (2.21), $G = \{\uparrow x \cap C \mid x \in C\} \in \operatorname{Spec} \widetilde{\mathcal{P}}$. Clearly $G \in \mathcal{S}(T_a)$, and $m(G) = \bigcup_{x \in C} \uparrow x = m(F)$. By Theorem (2.21), $q(G) = q(F)$. This proves the lemma. □

LEMMA b. *Let $\mathcal{P}$ be a laterally countable well-graded poset and suppose $a \in \mathcal{P}$ is finitely based. Then $\mathcal{S}(a)$ is quasi-compact.*

PROOF. Let $T_a$ be an element of $\widetilde{\mathcal{P}}$ such that $p(T_a) = a$. Combining information from (2.21) and (2.22), one has $m(\mathcal{S}(T_a)) = \mathcal{S}(a)$. By Lemmas (2.14) and (2.17), $\mathcal{S}(T_a)$ is compact. Thus $\mathcal{S}(a)$, being a continuous image of a compact set, is quasi-compact. □

(2.23) A topological space is called *spectral* if

(i) $X$ is $T_0$.

(ii) Every irreducible set in $X$ is a singleton closure.

(iii) $X$ has a basis of quasi-compact open sets.

(2.8) and (2.22) now prove:

THEOREM. *Let $\mathcal{P}$ be a well-graded countable poset which is spectrally finitely based. Then* Spec $\mathcal{P}$ *is a spectral space. If $\mathcal{P}$ is laterally finite,* Spec $\mathcal{P}$ *is a quasi-compact spectral space.*

**3. The structure spaces of *AF*-algebras.** Throughout this section, $A$ will denote an $AF$ $C^*$-algebra with identity, $\mathcal{D}(A) = (\mathcal{D}(A), \geqslant)$ its Bratteli poset.

(3.1) LEMMA. *Let* $\Lambda \subseteq \mathcal{D}(A)$. *Consider the following conditions on* $\Lambda$.

(i) *For all* $x \in \Lambda$, $\downarrow x \subseteq \Lambda$.

(ii) *For all* $x \in \mathcal{D}(A)$, *if* $\downarrow x \backslash \{x\} \subseteq \Lambda$ *then* $x \in \Lambda$.

(iii) *If* $x, y \in \Lambda^c$ *then there exists* $z \in \Lambda^c$ *such that* $z \leqslant x$ *and* $z \leqslant y$.

*If* $\Lambda$ *satisfies* (i) *and* (ii), *let* $\Lambda^n = \Lambda \cap [\mathcal{D}(A)]^n$, *put* $I = \bigcup_{n=1}^{\infty} \bigoplus_{x \in \Lambda^n} M_{(x)}$, *and let* $I_\Lambda = \bar{I}$. *Then* $I_\Lambda$ *is a norm-closed ideal of* $A$. $\Lambda \longrightarrow I_\Lambda$ *is a bijection of the set of all subsets of* $\mathcal{D}(A)$ *satisfying* (i) *and* (ii) *above onto the set of norm-closed ideals of* $A$. *Let* $\Lambda \subseteq \mathcal{D}(A)$ *satisfying* (i) *and* (ii). *Then* $I_\Lambda$ *is a primitive ideal of* $A$ *if and only if* $\Lambda$ *satisfies* (iii).

PROOF. This follows from Theorems 3.3 and 3.8 of [3]. □

(3.2) LEMMA. *Let* $\Lambda \subseteq \mathcal{D}(A)$. *Then* $\Lambda$ *satisfies* (3.1)(i), (ii) *and* (iii) *if and only if* $\Lambda^c$ *is a recursive filter of* $\mathcal{D}(A)$.

Combining (3.1) with this lemma, we see that $\beta$: $F \longmapsto I_{F^c}$: Spec $\mathcal{D} \longrightarrow$ Prim $A$ is a bijection.

(3.3) LEMMA. *A basis for the open sets of the Jacobson topology on* Prim $A$ *is given by* $O_x = \{\varphi \in \text{Prim } A \mid \varphi \not\supseteq M_{(x)}\}$, $x \in \mathcal{D}(A)$.

PROOF. It suffices to show that any closed set in Prim $A$ is an intersection of sets of the form $O_x^c = \{\varphi \in \text{Prim } A \mid \varphi \supseteq M_{(x)}\}$. Recall that the closed sets of Prim $A$ are $C_I = \{\varphi \in \text{Prim } A \mid \varphi \supseteq I\}$ where $I$ runs through all closed ideals of $A$. Since, for $\varphi \in$ Prim $A$, $\varphi \supseteq M_{(x)}$ if and only if $\varphi \supseteq$ the ideal generated by $M_{(x)}$, each of the $O_x^c$ is clearly closed. Let $\bar{I}$ be a closed ideal of $A$, where $I = \bigcup_{n=1}^{\infty} \bigoplus_{x \in \Lambda_I^n} M_{(x)}$. The closed set $C_{\bar{I}}$ is easily seen to equal $\bigcap_{n=1}^{\infty} \bigcap_{x \in \Lambda_I^n} M_{(x)}$. □

THEOREM. *Let* $A$ *be an AF C*-algebra with identity,* $\mathcal{D}(A)$ *its Bratteli poset. Then the map* $\beta$: Spec $\mathcal{D}(A) \longrightarrow$ Prim $A$ *is a homeomorphism.*

PROOF. One has $\beta[S(x)] = O_x$. The theorem follows. □

COROLLARY a. *Let* $A$ *be an abelian AF C*-algebra with identity. Then* Prim $A$ *is a compact zero-dimensional Polish space. Every compact zero-dimensional Polish space arises as* Prim $A$ *for some AF C*-algebra* $A$.

(2.16), combined with Gelfand-Naimark duality, now shows that any abelian *AF C**-algebra with identity is a *C**-quotient algebra of the AF *C**-algebra $C(K)$, and that any separable abelian *C**-algebra with identity is a subalgebra of $C(K)$.

COROLLARY b. *Let* $A$ *be an AF C*-algebra with identity. Then* Prim $A$ *is a quasi-compact spectral space.*

Corollary a is proved as Proposition 3.1 of [4]. Corollary b is an improvement on the results of [4], since it contains the assertion that Prim $A$ has a basis of quasi-compact open sets. That irreducible sets in Prim $A$ are one point closures is proved as Lemma 4.2 of [4], but the proof is considerably less elementary than the proof of Theorem (2.8). Corollary b has since been proved in [5], by functional analytic methods. [5] also contains a partial converse to Corollary b; if $X$ is a spectral space with the additional property that the intersection of two quasi-compact open sets is quasi-compact, then $X$ arises as Prim $A$ for some $AF$ $C^*$-algebra $A$.

In conclusion, I remark that, although I have assumed throughout that $A$ has identity, all the results of this paper may easily be generalized to the case where $A$ is without identity. One may associate with $A$ (again via a diagram), a well-graded countable poset, $(\mathcal{D}(A), \geqslant)$ which does not necessarily have finite layers, but which does have the property that it is spectrally finitely based. One again has Prim $A$ homeomorphic to Spec $\mathcal{D}(A)$, and consequently, Corollary b generalizes to this case.

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