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NONREGULAR ULTRAFILTERS AND LARGE CARDINALS

BY

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ABSTRACT. The relationship between the existence of nonregular ultrafilters and large cardinals in the constructible universe is studied.

1. **Introduction.** Our notation and terminology follows that of the most recent set-theoretic literature: for example $|x|$ denotes the cardinality of the set x, small Greek letters $\alpha, \beta, \gamma, \ldots$ denote ordinals, cardinals are initial ordinals, the Greek letters $\kappa, \lambda, \ldots$ are reserved for denoting cardinals, and so on. Needless to say, the word 'ultrafilter', the central object of our study, refers to a maximal filter. In particular, to simplify terminology, the phrase 'U is an ultrafilter over κ' will always refer to a *uniform* ultrafilter over κ, that is, for any set x we require $x \in U \to |x| = \kappa$.

A central notion in the study of structural theory of countably incomplete ultrafilters over uncountable cardinals is the degree of regularity of the ultrafilter.

1.1. DEFINITION. An ultrafilter D is (κ, λ)-regular if there is a set $S \subseteq D$ of cardinality λ such that

$$T \subseteq S \wedge |T| = \kappa \to \bigcap T = 0.$$

An ultrafilter is λ-regular if it is (ω, λ)-regular.

This notion is due to Keisler. He showed that ultrapowers taken using fairly regular ultrafilters have a great model-theoretic significance. Keisler also showed that for every cardinal λ there is a λ-regular ultrafilter over λ. (For details, see for example Chang and Keisler [2].) The reverse direction here, namely the existence of suitably nonregular ultrafilters was left completely open. It is obvious that every ultrafilter over a cardinal λ is (λ, λ)-regular. Beyond this, our usual set-theoretic axioms do not seem to tell too much. On the other hand, in the constructible universe, K. Prikry [13] has shown that every ultrafilter over a successor cardinal κ^+ is (κ, κ^+)-regular. R. Jensen [3] extending the work of Prikry showed that every ultrafilter over cardinals of type ω_n $(n < \omega)$ is regular.

The introduction of large cardinal axioms seems to be the most natural approach to the problem of existence of nonregular ultrafilters.

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1.2. DEFINITION. A cardinal λ is measurable if there is a λ-complete ultrafilter D over λ; i.e.

$$X \subseteq D \wedge |X| < \lambda \rightarrow \bigcap X \in D.$$

It is then easy to see that every such D is not (ρ, λ)-regular for any $\rho < \lambda$. The cardinal λ here is huge indeed; using ordinary forcing techniques we can produce a weakly inaccessible cardinal λ' below the continuum carrying an ultrafilter which is not (ρ, λ')-regular for any $\rho < \lambda'$ (for details, see Prikry [12]). However, the question of Keisler and Gillman, namely the existence of a nonregular ultrafilter over ω_1, is completely open. As was mentioned before, Prikry [13] showed that in the constructible universe, every uniform ultrafilter over ω_1 is regular. Benda and Ketonen [1] extended this further to show that Prikry's result actually follows from Kurepa's Hypothesis. Here we have the first inklings of the 'large cardinality nature' of nonregular ultrafilters over ω_1: it then follows immediately that ω_2 must be inaccessible in L whenever such ultrafilters exist. In this paper we shall show that ω_1 itself must then be a very large cardinal in L.

We shall mainly work with the problem of (κ, κ^+)-regularity of ultrafilters over κ^+. A great deal of progress has been made in this area recently, the following being the main results:

1.3. THEOREM (BENDA AND KETONEN [1]). *If D is a non-(κ, κ^+)-regular ultrafilter over κ^+, then D is a P-point, i.e. if f: $\kappa^+ \rightarrow \kappa^+$ is unbounded* (mod D), *then there is a set $X \in D$ such that for every $\alpha < \kappa^+$:*

$$|f^{-1}(\{\alpha\}) \cap X| < \kappa.$$

1.4. DEFINITION. Given two ultrafilters D, U over a cardinal λ, say that D is less than U in the Rudin-Keisler order, in symbols, $D \leqslant_{RK} U$, if there is a function f: $\lambda \rightarrow \lambda$ such that for any $X \subseteq \lambda$:

$$X \in D \leftrightarrow f^{-1}(X) \in U.$$

In this case we also denote: $D = f_*(U)$. Similarly, given two functions f, g: $\lambda \rightarrow \lambda$ say $f \leqslant_{RK} g$ (mod D) if there is a function h: $\lambda \rightarrow \lambda$ so that $f = h \circ g$ (mod D).

Hence, if $f \leqslant_{RK} g$ (mod D), then $f_*(D) \leqslant_{RK} g_*(D)$. For more on this order, see for example Kunen [10].

1.5. THEOREM (KANAMORI [5]). *If D is a non-(ω, λ)-regular ultrafilter over a regular cardinal λ, then there is an ultrafilter U below D in the Rudin-Keisler order which extends the closed unbounded filter on λ.*

Combining Theorems 1.3 and 1.5, we have:

1.6. THEOREM (KANAMORI [5]). *If D is a non-(κ, κ^+)-regular ultrafilter over κ^+, then D has a first function f: $\kappa^+ \rightarrow \kappa^+$; i.e., every function $< f$* (mod D)

is bounded by a constant $< \kappa^+$ (mod D) *and* f *itself is not bounded by a constant* (mod D).

1.7. DEFINITION. An ultrafilter D over a cardinal λ is weakly normal if every pressing down function (i.e., a function f such that for any $\alpha > 0$: $f(\alpha) < \alpha$) on λ has range of cardinality $< \lambda$ on a set of D-measure 1.

As a corollary to Theorem 1.6. we have:

1.8. THEOREM (KANAMORI [5]). *If* D *is a non-*(κ, κ^+)*-regular ultrafilter over* κ^+, *then there is a weakly normal ultrafilter below* D *in the Rudin-Keisler order.*

For weakly normal ultrafilters we have the following characterization of nonregularity:

1.9. THEOREM (KETONEN [7]). *If* D *is a weakly normal ultrafilter over a regular cardinal* λ, *then* D *is* (μ,λ)*-regular if and only if*

$$\{\alpha \mid cf(\alpha) < \mu\} \in D.$$

Combining Theorems 1.8 and 1.9, we get:

1.10. THEOREM (KANAMORI [5]). *If* κ *is singular, then every ultrafilter over* κ^+ *is* (κ, κ^+)*-regular.*

Here again we wish to point out the similarities with large cardinals: If D is a λ-complete ultrafilter over λ, then it is a well-known result (Scott [15]) that there is an ultrafilter U below D which is actually normal: every pressing down function is constant (mod U). In this connection we wish to note the following result which has an analogue (due to Scott [15]) in the measurable case.

1.11. THEOREM (BENDA AND KETONEN [1]). *If there is a non-*(κ, κ^+)*-regular ultrafilter over* κ^+, *then*

$$2^\kappa = \kappa^+ \rightarrow 2^{\kappa^+} = \kappa^{++}.$$

1.12. THEOREM (BENDA AND KETONEN [1]). (1) *Suppose* $f_\alpha: \lambda \rightarrow \mu$ $(\alpha < \lambda)$ *is a family of eventually different functions* (mod F), *where* F *is a* λ*-complete filter over* λ. *Then an ultrafilter* $D \supseteq F$ *is non-*(μ, λ)*-regular if and only if the* f_α *are cofinal in the ultrapower of* μ.

(2) *If there is a non-*(κ, κ^+)*-regular ultrafilter over* κ^+, *then* κ^{++} *is inaccessible in* L.

Our methods in this paper are based on ideas of J. Silver [16] with an infusion of Benda-style techniques. Also, the paper of Vopěnka and Hrbáček [20] is relevant.

Our main results are the following:

1.13. THEOREM. *If there is a uniform, non-*$(\kappa, \kappa+)$*-regular ultrafilter over* κ^+, *then* $0^\#$ *exists.*

1.14. THEOREM. *If there is a uniform, weakly normal ultrafilter over a regular cardinal κ which is non-(γ, κ)-regular for all $\gamma < \kappa$, then $0^{\#}$ exists.*

Here we shall not bother to give the formal definition of $0^{\#}$; we only need a statement equivalent to the existence of $0^{\#}$.

1.15. DEFINITION (KUNEN [10]). Suppose M is a transitive class model of ZFC and κ is a cardinal in M. Then D is an M-ultrafilter on κ if:

(I) D is a proper subset of $P(\kappa) \cap M$ containing no singletons.

(II) $\forall x, y: x \subseteq y \in P(\kappa) \cap M \wedge x \in D \to y \in D$.

(III) $\forall x \in P(\kappa) \cap M$: $x \in D$ or $\kappa - x \in D$.

(IV) If $\eta < \kappa$ and $\langle x_\xi | \xi < \eta \rangle \in M$ and each $x_\xi \in D$, then $\bigcap \{x_\xi | \xi < \eta\} \in M$.

(V) If $\langle x_\xi | \xi < \kappa \rangle \in M$, then $\{\xi | x_\xi \in D\} \in M$.

The following result will then be used.

1.16. THEOREM (KUNEN [9]). $0^{\#}$ *exists if and only if there is an L-ultrafilter D over a cardinal λ in L such that every countable intersection of elements of D is nonempty, if and only if there is an ultrafilter over some $P(\lambda) \cap L$ (λ a cardinal in L) such that the ultrapower of L with respect to D is well founded.*

For more on $0^{\#}$, see Solovay [18].

By Theorem 1.8, it clearly suffices to prove only 1.13. In §2 we prove that under hypotheses of Theorem 1.13, κ is weakly compact in L and $\kappa^{+(L)} < \kappa^+$. Using these two facts, we then prove the existence of $0^{\#}$ in §3.

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2. Weakly normal ultrafilters and the constructible universe. In the following, suppose that D is a uniform ultrafilter over a regular cardinal λ.

2.1. DEFINITION. (1) If $f, g: \lambda \to V$, then

$$f \sim g \leftrightarrow \{\alpha | f(\alpha) = g(\alpha)\} \in D.$$

(2) If $f: \lambda \to V$, then

$$[f]_D = \{g | g \sim f \text{ and } \forall h(h \sim f \to \text{rank}(h) \geqslant \text{rank}(g))\}.$$

Here the rank of a set x is the usual set-theoretic rank. We can now define the ultrapowers we need. Let C be a transitive classmodel of set theory.

2.2. DEFINITION. (1) The ultrapower of the class C with respect to D is the class $\Pi_D C = \{[f]_D | f: \lambda \to C\}$.

(2) The restricted ultrapower of the class C with respect to D is the class $\Pi^*_D C = \{[f]_D | f: \lambda \to C \text{ and } |\text{range}(f)| < \lambda\}$.

(3) In either case, we define the ϵ-relation on the ultrapower to be $[f]_D E [g]_D \leftrightarrow \{\alpha | f(\alpha) \in g(\alpha)\} \in D$.

The idea of using a restricted ultrapower in set-theory appears in the paper of Vopěnka-Hrbáček [20]. It is actually a special case of Keisler's notion of a limit-ultrapower (see Keisler [6]). A word of caution: All the models which we will construct will most often be not well founded.

2.3. PROPOSITION (KEISLER [6]). *Define an embedding* $i: C \to \Pi^*_D C$ *by setting* $i(x) = [x]_D$, *the equivalence class of the constant function* x, *and let* $j: \Pi^*_D C \to \Pi_D C$ *be the map induced by the inclusion map. Then both* i, j *are elementary embeddings, and they induce a commutative diagram.*

Now assume that D is weakly normal. Let $\lambda^* = [\lambda]_D$ and $\rho^* = [\mathrm{id}]_D$. The following result is esentially due to Vopěnka and Hrbáček.

2.4. PROPOSITION. *The map* j *is an onto map when restricted to* λ^* *in* $\Pi^*_D C$:

$$j: \{[f]_D \mid |\mathrm{rng} f| < \lambda \text{ and } [f]_D E \lambda^*\} \xrightarrow{\text{onto}} \{[f]_D \; [f]_D E \rho^*\}.$$

This is clear, since every function $<$ [id] (mod D) has range of cardinality $< \lambda$(mod D).

Thus we have an order-isomorphism via the map j between the predecessors of λ^* in $\Pi^*_D C$ and predecessors of ρ^* in $\Pi_D C$. The 'ordinal' λ^* gets mapped into a bigger ordinal than ρ^* by j; thus in a sense λ^* is the first ordinal moved. Here is the basic idea of Silver [16]: Well foundedness will be replaced by isomorphisms between structures. The next result is essentially due to Kunen [9] and in its present context to Silver [16]. It follows directly from Proposition 2.4.

2.5. PROPOSITION. *Define a collection* U *of 'subsets' of* λ^* *as follows*:

$$U = \{[f]_D \mid \Pi^*_D C \models [f]_D \subseteq \lambda^* \text{ and } \Pi_D C \models \rho^* E j[f]_D\}.$$

Then the following statements hold:

I. *If* $x \in U$ *then*

$$\Pi^*_D C \models |x| = \lambda^*.$$

II. *For* $x, y \in \Pi^*_D C$,

$$\Pi^*_D C \models x \cup y = \lambda^* \to x \in U \text{ or } y \in U,$$

$$\Pi^*_D C \models x \cap y = 0 \to x \notin U \text{ or } y \notin U.$$

III. *If* $F \in \Pi^*_D C$ *and* $\Pi^*_D C \models F: \lambda^* \to \lambda^*$ *is pressing down, then there is a* $y \in \Pi^*_D C$ *such that there is a* $z \in U$ *with*

$$\Pi^*_D C \models y < \lambda^* \wedge z = f^{-1}(\{y\}).$$

IV. *If* $F \in \Pi^*_D C$ *and* $\Pi^*_D C \models F: y \to V \wedge y < \lambda^*$ *and for any* a *such that*

$\Pi^*_D C \models a < y$ *we have* $F(a) \in U$, *then, if*

$$\Pi^*_D C \models z = \bigcap\{f(u) \mid u < y\},$$

$z \in U$.

Thus, U is a '$\Pi^*_D C$-ultrafilter' in the sense of Definition 1.14 with the possible exception of the following property.

V. If $F \in \Pi^*_D C$ and $\Pi^*_D C \models F\colon \lambda^* \to V$, then there is a $Z \in \Pi^*_D C$ such that

$$F(a) \in U \longleftrightarrow \Pi^*_D C \models aEZ.$$

To accomplish this, we need to extend the isomorphism of Proposition 2.4 to subsets of λ^* in $\Pi^*_D C$. Some kind of 'smoothness' of the model C is required. From now on we shall assume that C is the constructible universe.

2.6. PROPOSITION. *If D is a weakly normal ultrafilter over λ, then there exists an isomorphism G such that for any relation $R \subseteq \lambda \times \lambda$,*

$$G\colon \prod_{\substack{D\\ \alpha<\lambda}} \langle L_\alpha, \epsilon, R \cap (\alpha \times \alpha)\rangle \cong \Pi^*_D \langle L_\lambda, \epsilon, R\rangle.$$

2.7. PROPOSITION. *If D is a weakly normal ultrafilter over λ, then there exists an isomorphism H between the structure $\Pi^*_D \langle L_{\lambda^+}, \epsilon\rangle$, where λ^+ is the (real) successor of λ, and an initial E-segment of the structure $\prod_{D,\, \alpha<\lambda} \langle L_{\alpha^+}, \epsilon\rangle$.*

These two propositions are directly modeled after those of Silver [16] and their proofs are similar. The proof of Proposition 2.6 is a straightforward application of weak normality. For the sake of completeness, we shall include the proof of Proposition 2.7.

PROOF OF PROPOSITION 2.7. Given an ordinal $\alpha < \lambda^+$, let $R^\alpha \subseteq \lambda \times \lambda$ be a (possibly nonconstructible) relation coding $\langle L_\alpha, \epsilon\rangle$; i.e., the structures $\langle \lambda, R^\alpha\rangle$ and $\langle L_\alpha, \epsilon\rangle$ are isomorphic. Then the set

$$C^\alpha = \{\gamma \mid \langle \gamma, R^\alpha \cap (\gamma \times \gamma)\rangle \text{ is an elementary substructure of } \langle \lambda, R^\alpha\rangle\}$$

is a closed unbounded set and therefore belongs to D. Let g^α be a function so that for $\gamma \in C^\alpha$, $g^\alpha(\gamma) < \gamma^+$ and

$$\langle L_{g^\alpha(\gamma)}, \epsilon\rangle \cong \langle \gamma, R^\alpha \cap (\gamma \times \gamma)\rangle.$$

We have by Proposition 2.6,

$$\begin{aligned}\Pi^*_D \langle L_\alpha, \epsilon\rangle &\cong \Pi^*_D \langle \lambda, R^\alpha\rangle \\ &\cong \prod_{\substack{D\\ \gamma<\lambda}} \langle \gamma, R^\alpha \cap (\gamma \times \gamma)\rangle \cong \prod_{\substack{D\\ \gamma<\lambda}} \langle L_{g^\alpha(\gamma)}, \epsilon\rangle.\end{aligned}$$

Thus we have a canonical isomorphism for $\alpha < \lambda^+$

$$H^\alpha\colon \Pi^*_D \langle L_\alpha, \epsilon\rangle \stackrel{!}{\cong} \prod_{\substack{D\\ \gamma<\lambda}} \langle L_{g^\alpha(\gamma)}, \epsilon\rangle.$$

It remains to show that for $\alpha < \beta$,

$$H^\alpha = H^\beta \quad \text{on } \Pi_D^*\langle L_\alpha, \epsilon\rangle.$$

This is an immediate consequence of the following fact: For any $\tau < \lambda^+$, let P_τ be the isomorphism

$$P_\tau\colon \langle\lambda, R^\tau\rangle \cong \langle L_\tau, \epsilon\rangle.$$

Then for any $\alpha < \beta$ the set

$$\{\gamma | L_\alpha \cap P''_\beta(\{\delta | \delta < \gamma\}) = L_\alpha \cap P''_\alpha(\{\delta | \delta < \gamma\})\}$$

contains a closed unbounded subset of λ. □

The functions $g^\alpha(\gamma)$ constructed in the above proof have an important property:

2.8. Proposition. *If $\alpha < \beta < \lambda^+$, then there is a closed unbounded set $C \subseteq \lambda$ such that $\gamma \in C \to g^\alpha(\gamma) < g^\beta(\gamma)$.*

Proof. If $\alpha < \beta < \lambda^+$, then there is an isomorphism i from $\langle\lambda, R^\alpha\rangle$ onto a R^β-proper initial segment of the structure $\langle\lambda, R^\beta\rangle$. Hence there is an ordinal $\delta < \lambda$ s.t. $i(\gamma)R^\beta\delta$ for $\gamma < \lambda$. Let $C = \{\gamma|\gamma > \delta \text{ and } i\colon \gamma \to \gamma \text{ and } \gamma \in C^\alpha \cap C^\beta\}$. This set satisfies our requirements. □

2.9. Theorem. *If D is a weakly normal ultrafilter over λ such that D is not (γ, λ)-regular for any $\gamma < \lambda$, then there is an isomorphism*

$$H\colon \Pi_D^*\langle L_{\lambda^+}, \epsilon\rangle \cong \prod_{\alpha<\lambda}{}_D\langle L_{\alpha^+}, \epsilon\rangle$$

extending the isomorphism G^{-1} of Proposition 2.6.

Proof. By Proposition 2.7 it suffices to show that if the functions g^α $(\alpha < \lambda^+)$ are not cofinal (mod D) in the ultraproduct

$$\prod_{\alpha<\lambda}{}_D\langle\alpha^+, \epsilon\rangle,$$

then D is (γ, λ)-regular for some $\gamma < \lambda$. If the g^α are not cofinal, there is a function $h\colon \lambda \to \lambda$ s.t. $h(\gamma) < \gamma^+$ and for all $\gamma < \lambda$ and for all $\alpha < \lambda^+$ we have: $g^\alpha < h$(mod D). Now, let k_γ be a one-to-one function from $\lambda \to \lambda$ which maps $h(\gamma) \to \gamma$. Define for $\alpha < \lambda^+, \gamma < \lambda$: $h^\alpha(\gamma) = k_\gamma(g^\alpha(\gamma))$. Each h^α is a pressing down function (mod D). Therefore we may, without loss of generality, assume that there is a $\xi < \lambda$ such that $h^\alpha \leqslant \xi$ (mod D) for all $\alpha < \lambda^+$. But, by Proposition 2.8, the functions g^α are mutually eventually different modulo the closed unbounded filter. Therefore, by Benda's Theorem 1.2, D is (ξ, λ)-regular. □

The following result is then a straightforward analog of Silver's Theorem 1.5 in [16].

2.10. THEOREM. *If D is not-(γ, λ)-regular for any $\gamma < \lambda$ and D is weakly normal, then the ultrafilter U satisfies condition V.*

PROOF. Suppose that $F \in \Pi_D^* C$ and $\Pi_D^* C \models F\colon \lambda^* \to V$. Let $y \in \Pi_D C$ so that

$$\Pi_D C \models y = \{s < \rho^* \mid \rho^* E(jF)(s)\}.$$

Then $H^{-1}(y)$ will satisfy the requirements of condition V. □

The way is finally clear for large cardinality results:

2.11. THEOREM. *If D is weakly normal over λ and D is non-(γ, λ)-regular for any $\gamma < \lambda$, then λ is Π_n^1-indescribable for every $n < \omega$ in L. That is, for any constructible relation R on λ, Π_n^1 sentence ϕ: If $(\langle \lambda, <, R\rangle \models \phi)^L$, then there is an $\alpha < \lambda$ such that $(\langle \alpha, <, R|\alpha\rangle \models \phi)^L$. As a matter of fact,*

$$\{\alpha \mid (\langle \alpha, <, R|\alpha\rangle \models \phi)^L\} \in D.$$

PROOF. This is clear from Theorem 2.9, since ϕ becomes a first order statement in L_{λ^+}. □

Thus, we now know that λ is weakly compact under the hypotheses of Theorem 2.11. We shall prove that $\lambda^{+(L)} < \lambda^+$ in this situation by contradiction.

2.12. PROPOSITION. *If X is a function $\lambda \to P(\lambda) \cap L$ with range of cardinality $< \lambda$, then*

$$[X] \in U \leftrightarrow \{\alpha \mid \alpha \in X(\alpha)\} \in D.$$

This is immediate from the definition of U.

2.13. PROPOSITION. *For any sequence $\langle m(\alpha) | \alpha < \lambda\rangle$, where $m(\alpha)$ is a constructible subset of α, there exists a function $A\colon \lambda \to P(\lambda) \cap L$ with range of cardinality $< \lambda$ such that*

$$\{\alpha \mid m(\alpha) = A(\alpha) \cap \alpha\} \in D.$$

PROOF. The 'set' $[m]_D \in \Pi_D L$. Let $A\colon \lambda \to P(\lambda) \cap L$ so that $H([A]_D) = [m]_D$. Then A satisfies our requirements. Note that by 2.9, if $x \subseteq \lambda$ and $x \cap \alpha \in L$ for all $\alpha < \lambda$, then $x \in L$. □

There is a useful modification of our basic construction: Instead of looking at all functions, we can restrict our attention to constructible functions: Define

$$\Pi_D^L L = \{[f]_D \mid f \in L \wedge f\colon \lambda \to L\},$$

$$\Pi_D^{**} L = \{[f]_D \mid f \in L \wedge f\colon \lambda \to L \wedge |\mathrm{rng}(f)| < \lambda\}.$$

2.14. PROPOSITION. *If $\lambda^{+L} = \lambda^+$ and D is weakly normal and not (γ, λ)-regular for any $\gamma < \lambda$, then*

$$\Pi_D^{**}\langle L_{\lambda^+}, \epsilon\rangle \cong \Pi_{\substack{D\\ \alpha<\lambda}}^L \langle L_{\alpha^+}, \epsilon\rangle.$$

PROOF. For $\alpha < \lambda^+$ we can then require the relations R^α of Proposition 2.6 to be constructible. Theorem 2.9 then implies that

$$\Pi_D^*\langle L_{\lambda^+}, \epsilon\rangle \cong \Pi_{D \atop \alpha<\lambda} \langle L_{\alpha^+}, \epsilon\rangle$$

from which our claim follows since the $g^\alpha(\alpha < \lambda^+)$ are then cofinal in the ultrapower. □

2.15. PROPOSITION. *Under the hypotheses of Proposition* 2.15: *If* $\langle X_\xi | \xi < \lambda\rangle$ *is a constructible sequence of subsets of* λ, *then there is a constructible function* A: $\lambda \to P(\lambda) \cap L$ *with range* $< \lambda$ *such that for any constructible* f: $\lambda \to \lambda$ *with range* $< \lambda$ *we have*

$$\{\alpha | \alpha \in X_{f(\alpha)}\} \in D \leftrightarrow \{\alpha | f(\alpha) \in A(\alpha)\} \in D.$$

PROOF. This follows immediately from the constructible analogues of Theorem 2.10 and Proposition 2.12. □

Let n be a positive integer. For a set x, $[x]^n$ denotes the set of all unordered n-tuples from the set x.

2.16. PROPOSITION. *Under the hypotheses of Proposition* 2.15: *If* F: $[\lambda]^n \to \{0, 1\}$ *is constructible, then there is a function* X: $\lambda \to P(\lambda)$ *with range of cardinality* $< \lambda$ *such that* $X \in L$ *and*

(a) $\forall\alpha < \lambda$, $X(\alpha)$ *is homogeneous for* F; *i.e.*, $F''[X(\alpha)]^2$ *is a singleton.*

(b) $\{\alpha|\alpha \in X(\alpha)\} \in D$.

PROOF. Using the constructible analogue of Theorem 2.10 and standard techniques of, say, Kunen [10], we can show that there is a $[X]_D \in U \cap \Pi_D^{**}L$ such that $\Pi_D^{**}L \models [X]_D$ is homogeneous for $i(F)$. □

We shall present the proof of the entirely analogous 2.20 in more detail.

2.17. DEFINITION. F: $[\lambda]^n \to \lambda$ is pressing down if for all $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_n$,

$$F(\{\alpha_1 \ldots \alpha_n\}) < \alpha_1.$$

2.18. THEOREM. *Under the hypotheses of Proposition* 2.14: *If* F: $[\lambda]^n \to \lambda$ *is a constructible pressing down function, then there is a set* $X \in D \cap L$ *and a* $\xi < \lambda$ *such that* $F''[X]^n \subseteq \xi$.

PROOF. Take $n = 2$. For each $\alpha < \lambda$ define

$$g_\alpha(\beta) = F(\{\alpha, \beta\}) \qquad (\beta < \alpha).$$

Thus $g_\alpha \subseteq \alpha \times \alpha$. By the constructible analog of 2.14, there is a constructible map T: $\lambda \to {}^\lambda\lambda$ with range of cardinality $\xi < \lambda$ so that for any $\alpha < \lambda$, $T(\alpha)$ is a pressing down function and

$$U = \{\alpha | g_\alpha = T(\alpha)|\alpha\} \in D.$$

By a variant of Theorem 1.8 the set $Y = \{\alpha | cf^L(\alpha) > (\xi^+)^L\} \in D$. This can be seen for example as follows: Suppose that $Z = \{\alpha | cf^L(\alpha) \leqslant (\xi^+)^L\} \in D$. Then we can find a constructible sequence $\{A_\alpha | \alpha < \lambda\}$ of sets such that $L \models \forall \alpha < \lambda$: $A_\alpha \subseteq \alpha$ has order type $\leqslant (\xi^+)^L$, and if $\alpha \in Z$, then A_α is cofinal in α. But then, by Proposition 2.13, we can find a constructible $A \subseteq \lambda$ such that the set $\{\alpha \in Z | A_\alpha = A \cap \alpha\}$ has cardinality λ, contradiction by Theorem 2.11. Hence, if

$$K(y) = \sup\{T(\alpha)(y) | \alpha < \lambda\},$$

K is pressing down on Y. By weak normality, there is a $\xi < \lambda$ so that

$$Z = \{y \in Y | K(y) \leqslant \xi\} \in D \cap L.$$

It then follows that $F \leqslant \xi$ on $[Z \cap U]^2$. □

2.19. THEOREM. *Under the hypotheses of Proposition* 2.15: *Suppose that* $\{f_\alpha | \alpha < \lambda\}$ *is a constructible family of bounded functions* $\lambda \to \lambda$. *Then there is a constructible function* g: $\lambda \to \xi$ $(\xi < \lambda)$ *such that for every* $\alpha < \lambda$

$$f_\alpha \leqslant_{\mathrm{RK}} g \quad (\mathrm{mod}\, D \cap L).$$

PROOF. Define a pressing down function F by:

$$F(\{\alpha, \beta\}) = \begin{cases} 0 & \text{if } \forall \gamma < \alpha < \beta\colon f_\gamma(\alpha) = f_\gamma(\beta), \\ \text{least } \gamma < \alpha \text{ so that } f_\gamma(\alpha) \neq f_\gamma(\beta) & \text{otherwise.} \end{cases}$$

By Theorem 2.20 we can find an $\eta < \lambda$ and a set $X \in D \cap L$ such that $F \leqslant \eta$ on $[X]^2$, i.e., for $\alpha, \beta \in X$ either $\forall \gamma < \alpha < \beta$: $f_\gamma(\alpha) = f_\gamma(\beta)$ or there is a $\mu < \eta$ such that $f_\mu(\alpha) \neq f_\mu(\beta)$. By Theorem 2.19 there is a constructible partitioning $\{Y_\xi | \xi < \theta\}$ $(\theta < \lambda)$ such that $Y = U\{Y_\xi | \xi < \theta\} \in D$, $Y \subseteq X$ and for all $\xi < \theta$ either $\alpha, \beta \in Y_\xi \to \forall \gamma < \alpha < \beta$: $f_\gamma(\alpha) = f_\gamma(\beta)$ or $\alpha, \beta \in Y_\xi \to \exists \mu < \eta$: $f_\mu(\alpha) \neq f_\mu(\beta)$. We can without a loss of generality assume that, for any $\xi < \theta$, $|Y_\xi| = \lambda$. This rules out the second possibility listed above since λ is inaccessible in L. Hence, if we set $g = \xi$ on Y_ξ, then for $\gamma < \lambda$:

$$f_\gamma \leqslant_{\mathrm{RK}} g \quad \text{on } Y - \gamma. \ \square$$

2.20. COROLLARY. *Under the hypotheses of Proposition* 2.14: *For every* $\gamma < \lambda^+$ *there is a* $\xi_\gamma < \lambda$ *and a constructible function* g_γ: $\lambda \to \xi_\gamma$ *such that for any bounded function* h: $\lambda \to \lambda$ *with* $h \in L_\gamma$ *we have*

$$h \leqslant_{\mathrm{RK}} g_\gamma \quad (\mathrm{mod}\, D \cap L).$$

The following is a trivial modification of the fundamental result of Silver [16].

2.21. THEOREM (SILVER [16]). *If D is a weakly normal ultrafilter over λ which is not (γ, λ)-regular for any $\gamma < \lambda$ such that*

$$\Pi_D^{**}\langle L_{\lambda^{+(L)}}, \epsilon\rangle \cong \Pi_{\substack{D\\ \alpha<\lambda}}^{L}\langle L_{\alpha^{+(L)}}, \epsilon\rangle$$

and

$$|\Pi_D^L \omega| < \lambda,$$

then $\mathrm{cof}(\lambda^{+(L)}) < \lambda$.

We can finally prove:

2.22. THEOREM. *If there is a weakly normal ultrafilter over λ which is not (γ, λ)-regular for any $\gamma < \lambda$, then $\lambda^{+(L)} < \lambda^+$.*

PROOF. For suppose that $\lambda^{+(L)} = \lambda^+$. By Theorem 2.22: $|\Pi_D^L\omega| \geq \lambda$. By Corollary 2.21, there is a fixed $\xi < \lambda$ and constructible g_γ: $\lambda \to \xi$ $(\gamma < \lambda^+)$ such that for any $f \in L_\gamma$

$$f \leq_{\mathrm{RK}} g_\gamma \pmod{D \cap L}.$$

Therefore

$$|\{[f]_D | f \in L_\gamma \text{ and } f\colon \lambda \to \omega\}| \leq (\xi^+)^L.$$

Since $\lambda^{+(L)} = \lambda^+$ by assumption and λ is inaccessible in L,

$$|\Pi_D^L\omega| < \lambda\,;$$

a contradiction. □

3. **The main results.** We shall now prove our main results. As was remarked before, it suffices to prove Theorem 1.14.

From now on, assume that D is a weakly normal ultrafilter over a regular cardinal λ such that D is not (γ, λ)-regular for any $\gamma < \lambda$. By the results of §2, λ is inaccessible in L and $\lambda^{+(L)} < \lambda^+$.

As before, for any $\tau < \lambda^+$ pick $R^\tau \subseteq \lambda \times \lambda$ coding $\langle L_\tau, \epsilon\rangle$ and let P_τ: $\langle\lambda, R^\tau\rangle \to \langle L_\tau, \epsilon\rangle$ be the isomorphism. Let $P = \{\tau | L_\tau < L_{\lambda^+}$ and $\tau > (\lambda^+)^L\}$. As usual, the symbol '$A < B$' means that A is an elementary substructure of B. For $\tau \in P$, let

$$B^\tau = \{\alpha | \alpha < \lambda, \langle\alpha, R^\tau \cap \alpha\rangle < \langle\lambda, R^\tau\rangle \text{ and } P_\tau''(\alpha) \cap \lambda = \alpha\}.$$

Taking the transitive collapse of $\langle\alpha, R^\tau \cap \alpha\rangle$, for $\alpha \in B^\tau$ we get an elementary embedding Π_α^τ: $L_{g^\tau(\alpha)} \to L_{\lambda^+}$ such that $\Pi_\alpha^\tau(\alpha) = \lambda$ and α is the first ordinal moved. Here the functions g^τ are constructed as in the proof of Proposition 2.7. We can then define ultrafilters U_α^τ over $P(\alpha) \cap L_{g^\tau(\alpha)}$ as follows:

$$X \in U_\alpha^\tau \leftrightarrow \alpha \in \Pi_\alpha^\tau(X).$$

3.1. LEMMA. (a) *Let* $\tau_0 = (\lambda^+)^L$. *Then*

$$X_0 = \{\alpha | g^{\tau_0}(\alpha) = (\alpha^+)^L\} \in D.$$

(b) *For any* $\tau < \eta < \lambda^+$, $\tau, \eta \in P$: $g^\tau < g^\eta$ *on a closed unbounded set.*

(c) *For any* $f\colon \lambda \to \lambda$ *such that* $f(\alpha) < |\alpha|^+$ *there is a* $\tau \in P$ *such that* $f \leqslant g^\tau$ (mod D).

(d) *For any* $\tau < \eta < \lambda^+$, $\tau, \eta \in P$: *There is a closed unbounded set C such that*

$$C \cap X_0 \subseteq \{\alpha | U^\tau_\alpha = U^\tau_\alpha = U^\eta_\alpha\}.$$

PROOF. To prove (a), use the fact that $(\lambda^+)^L < \lambda^+$ and Theorem 2.9. (b) is simply a restatement of Proposition 2.8. (c) follows immediately from the proof of Theorem 2.9.

To prove (d), argue as follows: Given τ, η, the set

$$A = \{\alpha | L_\tau \cap P''_\eta(\{\delta | \delta < \gamma\}) = L_\eta \cap P''_\eta(\{\delta | \delta < \gamma\})\}$$

is closed unbounded. By (a), (b) there is a closed unbounded set $C \subseteq A$ so that on $C \cap X_0$

$$(\alpha^+)^L < g^\tau(\alpha) < g^\eta(\alpha).$$

Now, $L_{g^\eta(\alpha)}$ is the transitive collapse of $P''(\{\delta | \delta < \alpha\})$. Since all the subsets of α in $L_{g^\eta(\alpha)}$ appear already in $L_{g^\tau(\alpha)}$ for all $\alpha \in C \cap X_0$, $U^\tau_\alpha = U^\eta_\alpha$. $\square$

Thus, we can find ultrafilters U_α over $P(\alpha) \cap L$ for $\alpha \in X_0$ such that for every $\tau \in P$ there is a closed unbounded set C^τ such that for $\alpha \in X_0 \cap C^\tau$:

$$X \in U_\alpha \leftrightarrow \alpha \in \Pi^\tau_\alpha(X).$$

To finish off the proof, by Theorem 1.16 it suffices to prove:

3.2. LEMMA. *There is an* $\alpha \in X_0$ *such that the ultrapower*

$$\mathrm{Ult}(L, U_\alpha) = \Pi^L_{U_\alpha} L$$

is well founded.

PROOF. If this was not the case, for every $\alpha \in X_0$ we can find a sequence $f^\alpha_i\colon \alpha \to \mathrm{ORD}$ such that for all $i < \omega$:

$$\{\gamma | \gamma < \alpha, f^\alpha_i(\gamma) > f^\alpha_{i+1}(\gamma)\} \in U_\alpha$$

and each $f_i \in L_\theta$ for some ordinal θ depending only on α. Form the elementary substructure M_α of L_θ generated by the set $\alpha \cup \{f^\alpha_i | i < \omega\}$. By collapsing M_α, we get an ordinal $\beta < |\alpha|^+$ such that

$$\langle M_\alpha, \epsilon\rangle \cong \langle L_\beta, \epsilon\rangle.$$

From this follows that we can without loss of generality assume that each $f^\alpha_i\colon \alpha \to |\alpha|^+$. Let f be a function $\lambda \to \lambda$ such that $f(\alpha) < |\alpha^+|$ and for all α, i:

$$f_i^\alpha \in L_{f(\alpha)}, \qquad f(\alpha) > (\alpha^+)^L.$$

By Lemma 3.1 there is a closed unbounded set C and a $\tau \in P$ such that

$$f(\alpha) < g^\tau(\alpha) \qquad (\alpha \in C \cap X_0)$$

and

$$X \in U_\alpha \leftrightarrow \alpha \in \Pi_\alpha^\tau(X)$$

where Π_α^τ is an elementary embedding $L_{g^\tau(\alpha)} \to L_{\lambda^+}$ with α the first ordinal moved.

But, given $\alpha \in C \cap X_0$ we then have

$$(\Pi_\alpha^\tau f_1^\alpha)(\alpha) > (\Pi_\alpha^\tau f_2^\alpha)(\alpha) > (\Pi_\alpha^\tau f_3^\alpha)(\alpha) > \ldots,$$

a contradiction. □

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