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INNER PRODUCT MODULES ARISING FROM COMPACT AUTOMORPHISM GROUPS OF VON NEUMANN ALGEBRAS

BY

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ABSTRACT. Let M be a von Neumann algebra of operators on a separable Hilbert space H, and G a compact, strong-operator continuous group of $*$-automorphisms of M. The action of G on M gives rise to a faithful, ultraweakly continuous conditional expectation of M on the subalgebra $N = \{A \in M: g(A) = A \forall g \in G\}$, which in turn makes M into an inner product module over N. The inner product module M may be "completed" to yield a self-dual inner product module $\overline{M}$ over N; our most general result states that the W^*-algebra $A(\overline{M})$ of bounded N-module maps of $\overline{M}$ into itself is isomorphic to a restriction of the crossed product $M \times G$ of M by G. When G is compact abelian, we give conditions for $A(\overline{M})$ and $M \times G$ to be isomorphic and show, among other things, that if G acts faithfully on M, then $M \times G$ is a factor if and only if N is a factor. As an example, we discuss certain compact abelian automorphism groups of group von Neumann algebras.

1. **Crossed products.** Let G be a locally compact group (with left Haar measure dg) acting as a strong-operator continuous automorphism group on M. That is, there is a homomorphism $g \longrightarrow g(\cdot)$ of G into the group of $*$-automorphisms of M such that for each $A \in M$, the map $g \longrightarrow g(A)$ of G into M is continuous with respect to the strong-operator topology on M. We recall the construction of the crossed product of M by G as set forth by M. Takesaki in [8].

Let $L^2(G, H)$ denote the set of functions $\Phi: G \longrightarrow H$ satisfying (i) for each $\xi \in H$, the complex-valued function $g \longrightarrow (\Phi(g), \xi)$ on G is measurable; and (ii) $\int_G \|\Phi(g)\|^2\, dg < \infty$. $L^2(G, H)$ is then a Hilbert space with inner product $(\Phi, \Psi) = \int_G (\Phi(g), \Psi(g))\, dg$. For each $A \in M$, define a bounded operator $\widetilde{A}$ on $L^2(G, H)$ by $(\widetilde{A}\Phi)(g) = g^{-1}(A)(\Phi(g))$; for each $g \in G$, let L_g be the unitary operator on $L^2(G, H)$ defined by $L_g\Phi(h) = \Phi(g^{-1}h)$. We let $M \times G$, the crossed product of M by G, denote the von Neumann algebra of operators on $L^2(G, H)$ generated by the operators $\widetilde{A}$ $(A \in M)$ and L_g $(g \in G)$.

Suppose now that G is compact (with normalized Haar measure dg). For $A \in M$, $\xi \in H$, we define the function $A \odot \xi \in L^2(G, H)$ by $(A \odot \xi)(g) = g^{-1}(A)\xi$ for $g \in G$. Notice that $\widetilde{B}(A \odot \xi) = BA \odot \xi$ and $L_g(A \odot \xi) = g(A) \odot \xi$

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$\forall B \in M, g \in G$; hence the closed linear span $M \odot H$ of $\{A \odot \xi: A \in M, \xi \in H\}$ is an invariant subspace of $L^2(G, H)$ for $M \times G$. It is also invariant for the commutant $(M \times G)'$ of $M \times G$. To see this, take $T \in (M \times G)'$. For $\xi \in H$, we have $T(I \odot \xi) = TL_g(I \odot \xi) = L_g T(I \odot \xi) \forall g \in G$, whence it follows that $T(I \odot \xi) = I \odot \eta$ for some $\eta \in H$. For $A \in M$, we then have $T(A \odot \xi) = T\widetilde{A}(I \odot \xi) = \widetilde{A}T(I \odot \xi) = \widetilde{A}(I \odot \eta) = A \odot \eta$. We record this observation below.

PROPOSITION 1.2. *If G is compact, the projection of $L^2(G, H)$ on $M \odot H$ belongs to the center of $M \times G$.*

In general, of course, $M \odot H$ may be a proper subspace of $L^2(G, H)$. For a simple example, let M be the algebra of complex 2×2 matrices acting on 2-dimensional Hilbert space H, G the group of unitary 2×2 matrices with determinant 1, and let G act as inner automorphisms on M in the obvious way (i.e. $V(A) = VAV^*$ for $A \in M$, $V \in G$). Easy calculations show that for each $\xi \in H$, the map $\Phi \in L^2(G, H)$ defined by $\Phi(V) = V^*\xi$ $(V \in G)$ is orthogonal to $M \odot H$.

2. **Inner product modules.** Let N be a von Neumann algebra of operators on H, and let X be a right N-module with vector space structure compatible with that of N. An *N-valued inner product* on X is a conjugate-bilinear map $\langle \cdot, \cdot \rangle$: $X \times X \longrightarrow N$ satisfying

(i)	$\langle x, x \rangle \geqslant 0$	$\forall x \in X$;
(ii)	$\langle x, x \rangle = 0$	only if $x = 0$;
(iii)	$\langle x, y \rangle^* = \langle y, x \rangle$	$\forall x, y \in X$;
(iv)	$\langle x \cdot A, y \rangle = \langle x, y \rangle A$	$\forall x, y \in X, A \in N$

(where $x \cdot A$ denotes the right action of $A \in N$ on $x \in X$). A right N-module equipped with an N-valued inner product will be called an *inner product module* over N. Notice that if X is an inner product module over N, the set $\{\langle x, y \rangle$: $x, y \in X\}$ spans a two-sided ideal of N; we let $\overline{\langle X, X \rangle}$ denote the ultraweak closure of this ideal. It is shown in [3] that X is normed by $\|x\|_X = \|\langle x, x \rangle\|^{1/2}$ $(x \in X)$. We write X' for the set of N-module maps from X to N which are bounded with respect to $\|\cdot\|_X$. We make X' into a right N-module by defining scalar multiplication and right action of N on X' by $(\lambda\tau)(x) = \overline{\lambda}\tau(x)$ and

$$(\tau \cdot A)(x) = A^*\tau(x) \; (\lambda \in \mathbf{C}, \tau \in X', x \in X, A \in N).$$

We shall regard X as a submodule of X' by identifying $x \in X$ with the map $\langle \cdot, x \rangle$: $X \longrightarrow N$. We call X *self-dual* if $X = X'$. The module X' may be thought of as the "completion" of X in the sense that the inner product on X can be extended to an N-valued inner product on X' in such a way as to make X' self-dual, with the extended inner product satisfying $\langle x, \tau\rangle = \tau(x)\ \forall x \in X,\ \tau \in X'$ (3.2 of [3]).

Suppose now that Y is a self-dual inner product module over N. Each bounded module map T: $Y \longrightarrow Y$ possesses an adjoint T^*: $Y \longrightarrow Y$ such that $\langle T^*x, y\rangle = \langle x, Ty\rangle\ \forall x, y \in Y$.

The *-algebra $A(Y)$ consisting of all such T is a W^*-algebra; a norm-bounded net $\{T_\alpha\}$ in $A(Y)$ converges ultraweakly to $T \in A(Y)$ if and only if $\langle T_\alpha x, y\rangle \longrightarrow \langle Tx, y\rangle$ ultraweakly in $N\ \forall x, y \in Y$ (3.10 of [3]). In [4], M. A. Rieffel constructs a faithful normal *-representation of $A(Y)$ on Hilbert space as follows. Let $Y \otimes H$ be the algebraic tensor product of Y with H. Define $[\cdot, \cdot]: Y \otimes H \times Y \otimes H \longrightarrow \mathbf{C}$ on elementary tensors by $[x \otimes \xi, y \otimes \eta] = (\langle x, y\rangle\xi, \eta)$ $(x, y \in Y,\ \xi, \eta \in H)$. Arguing as in 1.7 of [4] (or using 6.1 of [3] to see that for $x_1, x_2, \ldots, x_n \in Y$, the $n \times n$ operator matrix $[\langle x_i, x_j\rangle]$ is nonnegative) one shows that $[\cdot, \cdot]$ extends to a well-defined positive semidefinite conjugate-bilinear form on $Y \otimes H$. Let $Z = \{w \in Y \otimes H\colon [w, w] = 0\}$, so Z is a subspace of $Y \otimes H$ and $K_0 = (Y \otimes H)/Z$ is a pre-Hilbert space with inner product $(w_1 + Z, w_2 + Z) = [w_1, w_2]$.

REMARK 2.1. For $x \in Y$, $B \in N$, $\xi \in H$, direct computation shows that $(x \cdot B) \otimes \xi + Z = x \otimes B\xi + Z$.

Let K be the Hilbert space completion of K_0. For $T \in A(Y)$, define a linear map $\theta_0(T)$: $Y \otimes H \longrightarrow Y \otimes H$ by

$$\theta_0(T)(x \otimes \xi) = Tx \otimes \xi \qquad (x \in Y,\ \xi \in H).$$

It is shown in 5.3 of [4] that $\theta_0(T)$ induces a bounded linear map $\theta(T)$: $K \longrightarrow K$ satisfying $\theta(T)(x \otimes \xi + Z) = Tx \otimes \xi + Z$. One checks without difficulty that θ is a faithful normal *-representation of $A(Y)$ on K.

For $x, y \in Y$, define $x \otimes y$: $Y \longrightarrow Y$ by $x \otimes y(w) = x \cdot \langle w, y\rangle$ $(w \in Y)$. One checks that $x \otimes y \in A(Y)$ with $(x \otimes y)^* = y \otimes x$ and $T(x \otimes y) = Tx \otimes y\ \forall T \in A(Y)$. It follows that $\{x \otimes y\colon x, y \in Y\}$ spans a two-sided ideal of $A(Y)$.

PROPOSITION 2.2. *If $\overline{\langle Y, Y\rangle} = N$, then $\{x \otimes y\colon x, y \in Y\}$ spans an ultraweakly dense two-sided ideal of $A(Y)$.*

PROOF. If the span of $\{x \otimes y\colon x, y \in Y\}$ is not ultraweakly dense in $A(Y)$, there is a nonzero projection $P \in A(Y)$ such that $P(x \otimes y) = 0\ \forall x, y \in Y$. We then have $\langle Px, x\rangle\langle y', y\rangle = \langle P(x \otimes y)y', x\rangle = 0\ \forall x, y, y' \in Y$. Since $\overline{\langle Y, Y\rangle} = N$, this forces $\langle Px, x\rangle = 0\ \forall x \in X$ and hence $P = 0$, a contradiction.

Now let X be an inner product module, not necessarily self-dual, over N with $\overline{\langle X, X\rangle} = N$. In the sequel, we will need to know that the set $\{x \otimes y\colon x, y \in X\}$ spans an ultraweakly dense *-subalgebra of $A(X')$. (Here, as elsewhere, we are regarding X as a submodule of X', so $x \otimes y(\tau) = x \cdot \langle \tau, y\rangle = x \cdot \tau(y)^*$ for

$\tau \in X'$.) It is clear that the span of this set is a *-subalgebra of $A(X')$; unfortunately, the fact that it is ultraweakly dense seems not to follow directly from 2.2, but rather depends on certain details of the construction of the extended inner product in [3]. We will require the following lemma.

LEMMA 2.3. *For each $\tau \in X'$, there is a net $\{x_\alpha\}$ in X such that $\langle \tau - x_\alpha, \tau - x_\alpha \rangle \longrightarrow 0$ ultraweakly in N.*

PROOF. Let $s = \{f_1, f_2, \ldots, f_n\}$ be a finite set of normal positive linear functionals on N, and set $f = f_1 + f_2 + \cdots + f_n$. If we let $Z_f = \{x \in X: f(\langle x, x\rangle) = 0\}$, one checks that Z_f is a subspace of X and that X/Z_f is a pre-Hilbert space with inner product $(x + Z_f, y + Z_f) = f(\langle x, y\rangle)$. Since

$$\tau(x)^*\tau(x) \leqslant \|\tau\|^2\langle x, x\rangle \quad \forall x \in X$$

by 2.8 of [3], the map $x + Z_f \longrightarrow f(\tau(x))$ is a well-defined bounded linear functional on X/Z_f, so there is a vector τ_f in the Hilbert space completion of X/Z_f such that

$$(x + Z_f, \tau_f) = f(\tau(x)) \quad \forall x \in X.$$

Let $x_s \in X$ be such that $(x_s + Z_f - \tau_f, x_s + Z_f - \tau_f) \leqslant 1/n$. We have $f(\langle \tau, \tau\rangle) = (\tau_f, \tau_f)$ by the construction of the extended inner product on X' in 3.2 of [3], so $f(\langle \tau - x_s, \tau - x_s\rangle) \leqslant 1/n$ and hence $f_i(\langle \tau - x_s, \tau - x_s\rangle) \leqslant 1/n$ for $1 \leqslant i \leqslant n$. It is now clear that the net $\{x_s\}$ (indexed by the inclusion-directed family of finite sets of normal positive linear functionals on N) has the required property.

Let $\theta: A(X') \longrightarrow B(K)$ (the algebra of bounded operators on K) be the *-representation of $A(X')$ on the Hilbert space K constructed at the beginning of this section.

LEMMA 2.4. *Let $\tau \in X'$ and let $\{x_\alpha\}$ be as in 2.3. Then for each $\xi \in H$, the net $\{x_\alpha \otimes \xi + Z\}$ converges to $\tau \otimes \xi + Z$ in the norm of K. Moreover, for $\psi, \psi' \in X'$, the net $\{\psi' \otimes \psi(x_\alpha)^*\xi + Z\}$ converges to $\psi' \otimes \langle \psi, \tau\rangle\xi + Z$ weakly in K.*

PROOF. The first statement follows from the identity

$$[(\tau - x_\alpha) \otimes \xi, (\tau - x_\alpha) \otimes \xi] = (\langle \tau - x_\alpha, \tau - x_\alpha\rangle\xi, \xi).$$

If $\psi \in X'$ and f is a normal positive linear functional on N, we have

$$|f(\langle \psi, x_\alpha\rangle) - f(\langle \psi, \tau\rangle)| \leqslant f(\langle \psi, \psi\rangle)^{1/2} f(\langle \tau - x_\alpha, \tau - x_\alpha\rangle)^{1/2},$$

so $\langle \psi, x_\alpha\rangle = \psi(x_\alpha)^* \longrightarrow \langle \psi, \tau\rangle$ ultraweakly in N and hence $\psi(x_\alpha)^*\xi \longrightarrow \langle \psi, \tau\rangle\xi$ weakly in H. For $\psi' \in X'$, the map $\eta \longrightarrow \psi' \otimes \eta + Z$ is a bounded linear transformation from H into K, so $\psi' \otimes \psi(x_\alpha)^*\xi + Z \longrightarrow \psi' \otimes \langle \psi, \tau\rangle\xi + Z$ weakly in K.

PROPOSITION 2.5. *Let X be an inner product module over N such that $\overline{\langle X, X\rangle} = N$. Then $\{x \otimes y: x, y \in X\}$ spans an ultraweakly dense *-subalgebra of $A(X')$.*

PROOF. By 2.2, it will suffice to show that if $W \in B(K)$ commutes with all of the operators $\theta(x \otimes y)$ $(x, y \in X)$, then W commutes with all of the operators $\theta(\tau \otimes \psi)$ $(\tau, \psi \in X')$. We first show that W commutes with $\theta(\tau \otimes y)$ $\forall \tau \in X', y \in X$. Let $\{x_\alpha\}$ be a net in X approximating τ as in 2.3. For $\psi_1, \psi_2 \in X'$, $\xi_1, \xi_2 \in H$, we have

$$\begin{aligned}
&(W\theta(\tau \otimes y)(\psi_1 \otimes \xi_1 + Z), \psi_2 \otimes \xi_2 + Z)\\
&= (W((\tau \cdot \psi_1(y)^*) \otimes \xi_1 + Z), \psi_2 \otimes \xi_2 + Z)\\
&= (W(\tau \otimes \psi_1(y)^*\xi_1 + Z), \psi_2 \otimes \xi_2 + Z) \quad \text{(by 2.1)}\\
&= \lim_\alpha (W(x_\alpha \otimes \psi_1(y)^*\xi_1 + Z), \psi_2 \otimes \xi_2 + Z) \quad \text{(by 2.4)}\\
&= \lim_\alpha (W\theta(x_\alpha \otimes y)(\psi_1 \otimes \xi_1 + Z), \psi_2 \otimes \xi_2 + Z)\\
&= \lim_\alpha (\theta(x_\alpha \otimes y)W(\psi_1 \otimes \xi_1 + Z), \psi_2 \otimes \xi_2 + Z)\\
&= \lim_\alpha (W(\psi_1 \otimes \xi_1 + Z), \theta(y \otimes x_\alpha)(\psi_2 \otimes \xi_2 + Z))\\
&= \lim_\alpha (W(\psi_1 \otimes \xi_1 + Z), y \otimes \psi_2(x_\alpha)^*\xi_2 + Z) \quad \text{(by 2.1)}\\
&= (W(\psi_1 + \otimes \xi_1 + Z), y \otimes \langle \psi_2, \tau\rangle\xi_2 + Z) \quad \text{(by 2.4)}\\
&= (\theta(\tau \otimes y)W(\psi_1 \otimes \xi_1 + Z), \psi_2 \otimes \xi_2 + Z).
\end{aligned}$$

We conclude that $\theta(\tau \otimes y)W = W\theta(\tau \otimes y)$ $\forall \tau \in X', y \in Y$. In exactly the same way, we can now show that W commutes with all of the operators $\theta(\tau \otimes \psi)$ $(\tau, \psi \in X')$, which concludes the proof.

In [5], Rieffel investigated the notion of "Morita equivalence" among W^*-algebras. Roughly speaking, two W^*-algebras are *Morita equivalent* if their categories of nondegenerate normal *-representations on Hilbert space are equivalent via a linear functor which is "normal" in an appropriate sense. One of the main results of [5] states that two W^*-algebras N_1 and N_2 are Morita equivalent if and only if there is a self-dual inner product module Y over N_1 with $\overline{\langle Y, Y\rangle} = N_1$ such that $A(Y)$ is isomorphic to N_2. We conclude this section with a discussion of the relationship between N and $A(Y)$, where Y is a self-dual inner product module over N, which may be thought of as giving a characterization of Morita equivalence in terms of tensor products with type I factors.

Let J be an index set, and let N_j be a copy of N (regarded as a self-dual inner product module over N) for each $j \in J$. Form $X = \mathrm{UDS}\{N_j: j \in J\}$ as in §3 of [3], so X is a self-dual inner product module over N.

PROPOSITION 2.6. *$A(X)$ and $N \otimes B(l^2(J))$ are isomorphic.*

PROOF. This follows routinely from 1.22.14 of [6], where for each $j \in J$ we let V_j be the partial isometry in $A(X)$ which takes a J-tuple $\{B_k\}_{k\in J} \in X$ to the J-tuple with B_j in the j_0-slot and 0's elsewhere ($j_0 \in J$ fixed).

PROPOSITION 2.7. *Let Y be a self-dual inner product module over N such that $\overline{\langle Y, Y\rangle} = N$. There is a Hilbert space H_1 and a projection $E \in N \otimes B(H_1)$ with central cover $I \otimes I_{H_1}$ such that $A(Y)$ and $E(N \otimes B(H_1))E$ are isomorphic.*

PROOF. By 3.12 of [3], there is an index set J and a collection $\{P_j: j \in J\}$ of (not necessarily distinct) nonzero projections in N such that Y and UDS $\{P_jN: j \in J\}$ are isomorphic as inner product modules over N. Hence, we may regard Y as a submodule of the inner product module X defined above. Define a projection E in $A(X)$ by $E\{B_j\}_{j\in J} = \{P_jB_j\}_{j\in J}$ for $\{B_j\}_{j\in J} \in X$; E is the projection of X onto Y. It is clear that $A(Y)$ and $EA(X)E$ are isomorphic. If Q is a central projection in $A(X)$ majorizing E, then $Qy = y \ \forall y \in Y$ and hence for y_1, $y_2 \in Y$ and $x \in X$, we have $Qx \cdot \langle y_1, y_2\rangle = Q(x \otimes y_2)(y_1) = (x \otimes y_2)(Qy_1) = (x \otimes y_2)(y_1) = x \cdot \langle y_1, y_2\rangle$. Since $\overline{\langle Y, Y\rangle} = N$, this implies that $Qx = x \ \forall x \in X$, showing that the central cover of E in $A(X)$ is the identity map. To conclude the proof, take $H_1 = l^2(J)$ and apply 2.6.

3. **Compact automorphism groups.** In this section, M will be a von Neumann algebra of operators on a separable Hilbert space H, and G a compact group acting as a strong-operator continuous group of *-automorphisms of M. Let $N = \{A \in M: g(A) = A \ \forall g \in G\}$ be the fixed-point subalgebra for G. Define Γ: $M \longrightarrow M$ by $\Gamma(A) = \int_G g(A)\,dg$ for $A \in M$. (All integrals of this sort are taken in the sense of §1 of [1]. For $A \in M$, $\Gamma(A)$ is the unique operator in M such that $f(\Gamma(A)) = \int_G f(g(A))\,dg$ for all ultraweakly continuous linear functionals f on M.) It is clear that Γ is an ultraweakly continuous faithful conditional expectation of M on N. If we define $\langle\cdot, \cdot\rangle$: $M \times M \longrightarrow N$ by $\langle A, B\rangle = \Gamma(B^*A)$, it follows that $\langle\cdot, \cdot\rangle$ is an N-valued inner product on M (viewed as a right N-module). In general, M is not complete with respect to the norm obtained from this inner product, much less self-dual, but we can form the self-dual inner product module $\overline{M}$ of bounded N-module maps of M into N. Here, "bounded" means "bounded with respect to the norm obtained from the inner product". We write $\overline{M}$ instead of M' to avoid any possible confusion with the commutant of M. By 2.8 of [3], $\overline{M}$ consists precisely of all (complex -) linear $\tau: M \longrightarrow N$ for which there is a real $r \geqslant 0$ such that $\tau(A)^*\tau(A) \leqslant r\langle A, A\rangle \ \forall A \in M$. (For such maps, it turns out that N-linearity is automatic.)

Let K be the Hilbert space constructed from the tensor product $\overline{M} \otimes H$ as in §2, and θ the associated faithful normal *-representation of $A(\overline{M})$ on K.

By 2.4, the subspace of K spanned by $\{A \otimes \xi + Z\colon A \in M,\ \xi \in H\}$ is dense in K. For $A_1, A_2, \ldots, A_n \in M$ and $\eta_1, \eta_2, \ldots, \eta_n \in H$, we have

$$\begin{aligned}\left\|\sum_{i=1}^{n} A_i \otimes \eta_i + Z\right\|^2 &= \sum_{i,j}(\langle A_i, A_j\rangle \eta_i, \eta_j)\\ &= \sum_{i,j} \int_G (g(A_j^* A_i)\eta_i, \eta_j)\, dg\\ &= \sum_{i,j} \int_G (g^{-1}(A_i)\eta_i, g^{-1}(A_j)\eta_j)\, dg\\ &= \left\|\sum_{i=1}^{n} A_i \odot \eta_i\right\|^2\end{aligned}$$

We conclude that there is an isometry V of K onto the subspace $M \odot H$ of $L^2(G, H)$ defined in §1 such that $V(A \otimes \eta + Z) = A \odot \eta\ \forall A \in M,\ \eta \in H$. If we define $\rho\colon A(\overline{M}) \longrightarrow B(M \odot H)$ by $\rho(T) = V\theta(T)V^*$ for $T \in A(\overline{M})$, ρ will be a faithful normal $*$-representation of $A(\overline{M})$ on $M \odot H$.

THEOREM 3.1. $\rho(A(\overline{M})) = M \times G|_{M \odot H}$.

PROOF. Notice that if $T \in A(\overline{M})$ is such that $TM \subseteq M$, then $\rho(T)(C \odot \eta) = T(C) \odot \eta\ \forall C \in M,\ \eta \in H$. For $A, B \in M$ we have $g(B^*A^*AB) \leqslant \|A\|^2 g(B^*B)$ $\forall g \in G$, so $\langle AB, AB\rangle \leqslant \|A\|^2\langle B, B\rangle$. It follows that for each $A \in M$, the map $B \longrightarrow AB$ of M into M is a bounded N-module map, and hence extends uniquely to a map $m_A \in A(\overline{M})$. We have $\rho(m_A)(C \odot \eta) = AC \odot \eta = \widetilde{A}(C \odot \eta)\ \forall C \in M$, $\eta \in H$, so $\rho(m_A) = \widetilde{A}|_{M \odot H}$. Likewise each $g \in G$ gives rise to a map $k_g \in A(\overline{M})$ such that $k_g(B) = g(B)\ \forall B \in M$. We have $\rho(k_g)(C \odot \eta) = g(C) \odot \eta = L_g(C \odot \eta)\ \forall C \in M,\ \eta \in H$, so $\rho(k_g) = L_g|_{M \odot H}$. We conclude that $M \times G|_{M \odot H} \subseteq \rho(A(\overline{M}))$.

For the reverse inclusion, consider the projection $I \otimes I \in A(\overline{M})$. We have $\rho(I \otimes I)(C \odot \eta) = I\langle C, I\rangle \odot \eta = \Gamma(C) \odot \eta = I \odot \Gamma(C)\eta$ (since $g(\Gamma(C)) = \Gamma(C)$ $\forall g \in G$) $\forall C \in M,\ \eta \in H$, whence it follows that $\rho(I \otimes I)$ is the projection of $M \odot H$ onto the closed subspace $I \odot H = \{I \odot \xi\colon \xi \in H\}$. If S is an operator on $M \odot H$ commuting with $M \times G|_{M \odot H}$, then $S(I \odot \xi) = SL_g(I \odot \xi) = L_gS(I \odot \xi)\ \forall \xi \in H,\ g \in G$; we conclude that $S(I \odot H) \subseteq I \odot H$, which shows that $\rho(I \otimes I) \in M \times G|_{M \odot H}$. For $A, B, C \in M,\ \eta \in H$, we compute that

$$\begin{aligned}\rho(A \otimes B)(C \odot \eta) &= A\langle C, B\rangle \odot \eta = A\Gamma(B^*C) \odot \eta\\ &= \widetilde{A}(\Gamma(B^*C) \odot \eta) = \widetilde{A}\rho(I \otimes I)\widetilde{B}^*(C \odot \eta).\end{aligned}$$

Hence, $\rho(A \otimes B) \in M \times G|_{M \odot H}\ \forall A, B \in M$. By 2.5, then, $\rho(A(\overline{M})) \subseteq M \times G|_{M \odot H}$, which concludes the proof.

COROLLARY 3.2. *Let G be a compact, strong-operator continuous group* *of automorphisms of M, with fixed-point algebra N. Then $M \times G$ is a factor if and only if N is a factor and $M \odot H = L^2(G, H)$.*

PROOF. This follows from 3.1, 1.2, and the fact that Morita equivalent W^*-algebras have isomorphic centers (8.1 of [5]).

REMARK 3.3. If $M \odot H = L^2(G, H)$, then G must act faithfully on M, i.e. the homomorphism $g \longrightarrow g(\cdot)$ of G into the group of *-automorphisms of M must be injective. (If $g \in G$ is such that $g(A) = A\ \forall A \in M$, then $L_g\Phi = \Phi\ \forall\Phi \in M \odot H$, which forces $g = e$, the identity of G, if $M \odot H = L^2(G, H)$.) The example at the end of §1 shows that it is possible for G to act faithfully on M with N a factor without having $M \odot H = L^2(G, H)$.

4. Compact abelian automorphism groups. In this section we impose the additional requirement that the compact group G of automorphisms of M be abelian. One of our main purposes will be to shed some light in the abelian case on the condition $M \odot H = L^2(G, H)$ which insures the isomorphism of $A(\overline{M})$ with $M \times G$.

We let $\hat{G}$ denote the dual group of G. For each character $\lambda \in \hat{G}$, define a subspace M_λ of M and a linear map $\Gamma_\lambda: M \longrightarrow M$ by

$$M_\lambda = \{A \in M: g(A) = \overline{\lambda(g)}A \quad \forall g \in G\},$$

$$\Gamma_\lambda(A) = \int_G \lambda(g)g(A)\, dg \qquad (A \in M).$$

(It is not hard to see that, in the terminology of [1], M_λ is the *spectral subspace* of M corresponding to $\{\lambda\} \subseteq \hat{G}$ and that the *spectrum* of the action of G on M is precisely $\{\lambda \in \hat{G}: M_\lambda \neq 0\}$.)

REMARK 4.1. The following properties of the subspaces M_λ and the maps Γ_λ are all easily checked:

(i) each M_λ is a two-sided N-module;
(ii) $\langle A, B\rangle = B^*A$ for $A, B \in M_\lambda$;
(iii) $M_\lambda^* = M_{\bar{\lambda}}$;
(iv) $\langle M_{\lambda_1}, M_{\lambda_2}\rangle = 0$ for $\lambda_1 \neq \lambda_2$;
(v) $M_{\lambda_1}M_{\lambda_2} \subseteq M_{\lambda_1\lambda_2}$;
(vi) Γ_λ is a projection of M onto M_λ and a bounded N-module map;
(vii) $\Gamma_{\lambda_1}\Gamma_{\lambda_2} = 0$ for $\lambda_1 \neq \lambda_2$.

Notice in particular that the subspace of M spanned by the M_λ's is a *-subalgebra of M.

PROPOSITION 4.2. *The subspace spanned by $\{M_\lambda: \lambda \in \hat{G}\}$ is ultraweakly dense in M.*

PROOF. Let f be an ultraweakly continuous linear functional on M such that $f(M_\lambda) = 0\ \forall\lambda \in \hat{G}$. Take $A \in M$ and define a complex-valued continuous function ϕ on G by $\phi(g) = f(g^{-1}(A))$. For each $\lambda \in \hat{G}$, we have

$$\begin{aligned}\hat{\phi}(\lambda) &= \int_G \overline{\lambda(g)} f(g^{-1}(A))\, dg \\ &= \int_G \lambda(g) f(g(A))\, dg = f(\Gamma_\lambda(A)) = 0,\end{aligned}$$

so $\phi = 0$ and in particular $f(A) = 0$.

As noted above, each M_λ is an inner product module over N with N-valued inner product $\langle A, B\rangle = B^*A$ $(A, B \in M_\lambda)$.

PROPOSITION 4.3. *Each M_λ is self-dual.*

PROOF. Let $\tau: M_\lambda \longrightarrow N$ be a bounded N-module map and for simplicity assume that $\|\tau\| \leq 1$. We must produce an operator $B \in M_\lambda$ such that $\tau(A) = B^*A$ $\forall A \in M_\lambda$. By 2.8 of [3] we have $\tau(A)^*\tau(A) \leq A^*A$ $\forall A \in M_\lambda$, and it follows from 6.1 of [3] that for $A_1, A_2, \ldots, A_n \in M_\lambda$, the matrices $[\tau(A_i)^*\tau(A_j)]$ and $[A_i^*A_j]$ in $N_{(n)}$, the W^*-algebra of $n \times n$ matrices with entries in N, satisfy

$$[\tau(A_i)^*\tau(A_j)] \leq [A_i^*A_j].$$

Hence for $\xi_1, \xi_2, \ldots, \xi_n \in H$, we have

$$\begin{aligned}\left\|\sum_{i=1}^{n} \tau(A_i)\xi_i\right\|^2 &= \sum_{i,j}(\tau(A_i)^*\tau(A_j)\xi_j, \xi_i) \\ &\leq \sum_{i,j}(A_i^*A_j\xi_j, \xi_i) = \left\|\sum_{i=1}^{n} A_i\xi_i\right\|^2,\end{aligned}$$

so there is a bounded linear map B_0: $[M_\lambda H] \longrightarrow H$ (where $[M_\lambda H]$ is the closed linear span of $\{A\xi: A \in M_\lambda, \xi \in H\}$) satisfying $B_0A\xi = \tau(A)\xi$ $\forall A \in M_\lambda$, $\xi \in H$. Extend B_0 to a bounded operator on H by setting $B_0([M_\lambda H]^\perp) = 0$. Take $T \in M'$. Clearly, $[M_\lambda H]$ and $[M_\lambda H]^\perp$ are invariant subspaces for T, so $TB_0\eta = 0 = B_0T\eta$ for $\eta \in [M_\lambda H]^\perp$, and

$$TB_0A\xi = T\tau(A)\xi = \tau(A)T\xi = B_0AT\xi = B_0TA\xi \quad \forall A \in M_\lambda, \xi \in H.$$

We conclude that $B_0 \in M$. Set $B = \Gamma_\lambda(B_0^*)$, so $B \in M_\lambda$. For $A \in M_\lambda$, we have

$$B^*A = \int_G \overline{\lambda(g)} g(B_0)A\, dg = \int_G g(B_0A)\, dg = \tau(A),$$

since $B_0A = \tau(A)$ and $g(\tau(A)) = \tau(A)$ $\forall g \in G$.

Form the self-dual inner product module $\mathrm{UDS}\{M_\lambda: \lambda \in \hat{G}\}$ over N as in §3 of [3].

PROPOSITION 4.4. *$\overline{M}$ and* $\mathrm{UDS}\{M_\lambda: \lambda \in \hat{G}\}$ *are isomorphic as inner product modules over N.*

PROOF. For each $\tau \in \overline{M}$ and $\lambda \in \hat{G}$, 4.3 gives a unique operator $B(\tau)_\lambda \in M_\lambda$ such that $\tau(A) = B(\tau)_\lambda^* A$ $\forall A \in M_\lambda$. For $\lambda_1, \lambda_2, \ldots, \lambda_n$ distinct characters in $\hat{G}$, we have

$$\left\|\sum_{i=1}^n B(\tau)_{\lambda_i}^* B(\tau)_{\lambda_i}\right\|^2 = \left\|\tau\left(\sum_{i=1}^n B(\tau)_{\lambda_i}\right)\right\|^2$$

$$\leqslant \|\tau\|^2 \left\|\left\langle \sum_{i=1}^n B(\tau)_{\lambda_i}, \sum_{i=1}^n B(\tau)_{\lambda_i}\right\rangle\right\|$$

$$= \|\tau\|^2 \left\|\sum_{i=1}^n B(\tau)_{\lambda_i}^* B(\tau)_{\lambda_i}\right\|,$$

the last equality holding by virtue of (ii) and (iv) of 4.1. Hence, $\|\Sigma_{i=1}^n B(\tau)_{\lambda_i}^* B(\tau)_{\lambda_i}\| \leqslant \|\tau\|^2$. We conclude that the map $\tau \longrightarrow \{B(\tau)_\lambda\}_{\lambda\in\hat{G}}$ is a bounded N-module map of $\overline{M}$ into UDS$\{M_\lambda: \lambda \in \hat{G}\}$. If τ has the form $\langle \cdot, A\rangle$ for some $A \in M$, it is clear that $B(\tau)_\lambda = \Gamma_\lambda(A)$ $\forall \lambda \in \hat{G}$, so in consequence the $\hat{G}$-tuple $\{\Gamma_\lambda(A)\}_{\lambda\in\hat{G}} \in$ UDS$\{M_\lambda: \lambda \in \hat{G}\}$ $\forall A \in M_\lambda$. Let f be a normal positive linear functional on N, and take $A \in M$, $\tau \in \overline{M}$. Since $\tau(B)^*\tau(B) \leqslant \|\tau\|^2 B^*B$ $\forall B \in M$ (and hence $\|\tau(B)\xi\| \leqslant \|\tau\|\,\|B\xi\|$ $\forall B \in M$, $\xi \in H$), it follows that τ is continuous with respect to the strong-operator topology of M. Hence, the function ϕ on G defined by $\phi(g) = f(\tau(g(A)))$ is continuous. We have

$$\hat{\phi}(\lambda) = \int_G \overline{\lambda(g)} f(\tau(g(A)))\,dg = f(\tau(\Gamma_{\bar{\lambda}}(A)))$$

$$= f(B(\tau)_{\bar{\lambda}}^*\ \Gamma_{\bar{\lambda}}(A))\ \forall \lambda \in \hat{G}.$$

Thus

$$\sum_{\lambda\in G} |\hat{\phi}(\lambda)| \leqslant \sum_{\lambda\in\hat{G}} f(B(\tau)_\lambda^* B(\tau)_\lambda)^{1/2} f(\Gamma_\lambda(A)^*\Gamma_\lambda(A))^{1/2}$$

$$\leqslant f\left(\sum_{\lambda\in G} B(\tau)_\lambda^* B(\tau)_\lambda\right)^{1/2} f\left(\sum_{\lambda\in G} \Gamma_\lambda(A)^*\Gamma_\lambda(A)\right)^{1/2}$$

$$< \infty \text{ because } \{B(\tau)_\lambda\}, \{\Gamma_\lambda(A)\} \in \mathrm{UDS}\{M_\lambda: \lambda \in \hat{G}\}.$$

We conclude that

$$f(\tau(A)) = \sum_{\lambda\in\hat{G}} \hat{\phi}(\lambda)$$

$$= \sum_{\lambda\in\hat{G}} f(B(\tau)_\lambda^*\Gamma_\lambda(A))$$

$$= f(\langle\{\Gamma_\lambda(A)\}, \{B(\tau)_\lambda\}\rangle).$$

This shows that $\tau(A) = \langle\{\Gamma_\lambda(A)\}, \{B(\tau)_\lambda\}\rangle$ $\forall \tau \in \overline{M}$, $A \in M$ and it follows from the construction of the extended inner product of $\overline{M}$ that $\langle \tau, \psi\rangle = \langle\{B(\tau)_\lambda\},$

$\{B(\psi)\}\rangle\ \forall\tau, \psi \in \overline{M}$. Hence the map $\tau \longrightarrow \{B(\tau)_\lambda\}$ is an inner product module isomorphism of $\overline{M}$ into $\text{UDS}\{M_\lambda: \lambda \in \hat{G}\}$. But it is clear that if $\{B_\lambda\} \in \text{UDS}\{M_\lambda: \lambda \in \hat{G}\}$, then $\tau(A) = \langle\{\Gamma_\lambda(A)\}, \{B_\lambda\}\rangle$ $(A \in M)$ defines a $\tau \in \overline{M}$ such that $B(\tau)_\lambda = B_\lambda\ \forall\lambda \in \hat{G}$. This completes the proof.

For $\lambda \in \hat{G}$, let $L(M_\lambda) = \{A \in M: AM_\lambda = 0\}$ and let Q_λ be the left-annihilating projection of M_λ in M, i.e. the projection in M such that $L(M_\lambda) = MQ_\lambda$.

PROPOSITION 4.5. *Q_λ is the projection of H on $[M_\lambda H]^\perp$ and belongs to the center of N.*

PROOF. Let P_λ be the projection of H on $[M_\lambda H]$. Since this latter subspace is invariant under M', we have $P_\lambda \in M$, and $I - P_\lambda \in L(M_\lambda)$, so $Q_\lambda \geqslant I - P_\lambda$. But also $I - Q_\lambda \geqslant P_\lambda$, since $Q_\lambda P_\lambda = 0$, so $Q_\lambda = I - P_\lambda$, which proves the first statement. Next, we show that $Q_\lambda \in N$. For $A \in M_\lambda$ and $g \in G$, we have $g(Q_\lambda)A = \lambda(g)g(Q_\lambda A) = 0$, so $g(Q_\lambda) \leqslant Q_\lambda\ \forall g \in G$ and we conclude that $g(Q_\lambda) = Q_\lambda\ \forall g \in G$, i.e. $Q_\lambda \in N$. Finally, note that $[M_\lambda H]$ is an invariant subspace for N, so Q_λ lies in the center of N.

We can now give a characterization of the condition $M \odot H = L^2(G, H)$ when G is compact abelian.

PROPOSITION 4.6. *The following are equivalent*:
(i) $M \odot H = L^2(G, H)$;
(ii) $[M_\lambda H] = H\ \forall\lambda \in \hat{G}$;
(iii) $L(M_\lambda) \cap \text{center}\ (N) = 0\ \forall\lambda \in \hat{G}$.

PROOF. The equivalence of (ii) and (iii) follows from 3.5; we show that (i) and (ii) are equivalent. Suppose first that (i) holds, and take $\lambda \in \hat{G}$. If $\eta \in [M_\lambda H]^\perp$, define $\Phi \in L^2(G, H)$ by $\Phi(g) = \lambda(g)\eta$ $(g \in G)$. For $A \in M$, $\xi \in H$, we have $(A \odot \xi, \Phi) = \int_G (g^{-1}(A)\xi, \lambda(g)\eta)\,dg = \int_G (\lambda(g)g(A)\xi, \eta)\,dg = (\Gamma_\lambda(A)\xi, \eta) = 0$, so $\eta = 0$ by (i). Now suppose (ii) holds, and take $\Phi \in (M \odot H)^\perp$. Given $\xi \in H$, define $\phi \in L^2(G)$ by $\phi(g) = (\Phi(g), \xi)$. We claim that $\hat{\phi}(\lambda) = 0\ \forall\lambda \in \hat{G}$, which will show that $\Phi = 0$. Indeed, for $\lambda \in \hat{G}$ and $\epsilon > 0$, (i) gives $A_1, A_2, \ldots, A_n \in M_\lambda$ and $\xi_1, \xi_2, \ldots, \xi_n \in H$ such that $\|\xi - \sum_{i=1}^n A_i\xi_i\| \leqslant \epsilon$. Hence

$$|\hat{\phi}(\lambda)| = \left|\int_G \overline{\lambda(g)}(\Phi(g), \xi)\,dg\right|$$

$$\leqslant \left|\int_G \overline{\lambda(g)}\left(\Phi(g), \sum_{i=1}^n A_i\xi_i\right)dg\right| + \epsilon\|\Phi\|.$$

For $i = 1, 2, \ldots, n$, we have

$$\int_G \overline{\lambda(g)}(\Phi(g), A_i\xi_i)\,dg = \int_G (\Phi(g), \lambda(g)A_i\xi_i)\,dg$$
$$= \int_G (\Phi(g), g^{-1}(A_i)\xi_i)\,dg$$
$$= (\Phi, A_i \odot \xi_i) = 0,$$

so $|\hat{\phi}(\lambda)| \leq \epsilon\|\Phi\|$. As $\epsilon > 0$ was arbitrary, we have $\hat{\phi}(\lambda) = 0 \ \forall \lambda \in \hat{G}$, which completes the proof.

As a corollary, we obtain the following abelian version of 3.2.

COROLLARY 4.7. *Let G be a compact abelian strong-operator continuous group of automorphisms of M, with fixed-point algebra N. Then $M \times G$ is a factor if and only if N is a factor and G acts faithfully on M.*

PROOF. In light of 3.2, 3.3 and 4.6, we need only show that if G acts faithfully on M, and N is a factor, then $[M_\lambda H] = H \ \forall \lambda \in \hat{G}$, i.e. $Q_\lambda = 0 \ \forall \lambda \in \hat{G}$. Notice that since each Q_λ lies in the center of N, we have $Q_\lambda = 0$ or $I \ \forall \lambda \in \hat{G}$. Let $S = \{\lambda \in \hat{G}: Q_\lambda = 0\} = \{\lambda \in \hat{G}: Q_\lambda \neq I\} = \{\lambda \in \hat{G}: M_\lambda \neq 0\}$. We claim that S is a subgroup of $\hat{G}$. If $\lambda \in S$, then $\overline{\lambda} \in S$ by (iii) of 4.1.

If $\lambda_1, \lambda_2 \in S$, then $[M_{\lambda_1}H] = H$ and $[M_{\lambda_2}H] = H$, so $M_{\lambda_1}M_{\lambda_2} \neq 0$, so $M_{\lambda_1\lambda_2} \neq 0$ by (v) of 4.1 and hence $\lambda_1\lambda_2 \in S$. To see that $S = \hat{G}$, suppose that $g \in G$ is such that $\lambda(g) = 1 \ \forall \lambda \in S$. For $\lambda \in S$ and $A \in M_\lambda$ we have $g(A) = \overline{\lambda(g)}A = A$, and for $\lambda \notin S$, we have $M_\lambda = 0$. Hence, g fixes each M_λ, so by 4.2, $g(A) = A \ \forall A \in M$. Since G acts faithfully on M, g must be the identity element of G. We conclude that $S = \hat{G}$.

Takesaki's duality theorem for crossed products (4.5 of [8]) can be used in connection with 4.7 above in determining what sorts of von Neumann algebras can appear as fixed-point algebras for actions of compact abelian groups on M. As a sample application, we will prove the following proposition after establishing a preliminary lemma.

PROPOSITION 4.8. *Suppose that there is a faithful strong-operator continuous action of the circle group $T = \{z \in \mathbf{C}: |z| = 1\}$ as automorphisms on M for which the fixed-point algebra N is a type* I *factor. Then M must be of type* I.

The proof of 4.8 requires the following lemma.

LEMMA 4.9. *Let K be a separable Hilbert space and let the discrete group Z of integers act as automorphisms on $B(K)$. Then $B(K) \times Z$ is of type* I.

PROOF OF LEMMA. Let τ_n be the automorphism of $B(K)$ corresponding to $n \in Z$. Since all $*$-automorphisms of $B(K)$ are inner, there is a unitary $U \in B(K)$ such that $\tau_n(A) = U^nAU^{-n} \ \forall A \in B(K), n \in Z$. For $\phi \in l^2(Z)$, $\xi \in K$, define a function $\phi \otimes \xi \in L^2(Z, K)$ by $\phi \otimes \xi(n) = \phi(n)U^{-n}\xi$. It is easily checked that the $\phi \otimes \xi$'s span a dense subspace of $L^2(Z, K)$ and that $(\phi_1 \otimes \xi_1, \phi_2 \otimes \xi_2) = (\phi_1, \phi_2)(\xi_1, \xi_2)\ \forall \phi_1, \phi_2 \in l^2(Z), \xi_1, \xi_2 \in K$, so we may identify $L^2(Z, K)$ in this way with the Hilbert space tensor product $l^2(Z) \otimes K$. For $A \in B(K)$, $\phi \in l^2(Z)$, $\xi \in K$, we have

$$\widetilde{A}(\phi \otimes \xi)(n) = \tau_n^{-1}(A)(\phi \otimes \xi)(n) = \phi(n)U^{-n}AU^nU^{-n}\xi$$
$$= (\phi \otimes A\xi)(n) \qquad (n \in Z),$$

so $\widetilde{A}(\phi \otimes \xi) = \phi \otimes A\xi$. If we let V_m (for $m \in Z$) denote the unitary operator on $l^2(Z)$ defined by $(V_m\phi)(n) = \phi(n-m)$, we have

$$\begin{aligned} L_m\widetilde{U}^{-m}(\phi \otimes \xi)(n) &= L_m(\phi \otimes U^{-m}\xi)(n) \\ &= \phi(n-m)U^{m-n}U^{-m}\xi \\ &= \phi(n-m)U^{-n}\xi = (V_m\phi \otimes \xi)(n), \end{aligned}$$

so $L_m\widetilde{U}^{-m}(\phi \otimes \xi) = V_m\phi \otimes \xi$. Since $B(K) \times Z$ is generated by the operators $\widetilde{A}(A \in B(K))$ and $L_m\widetilde{U}^{-m}$ $(m \in Z)$, it follows that $B(K) \times Z$ is isomorphic with the tensor product of $B(K)$ by the (abelian) von Neumann algebra of operators on $l^2(Z)$ generated by the V_m's. This proves the lemma.

PROOF OF 4.8. We have $M \times T$ isomorphic with $A(\overline{M})$ by 4.6 and 4.7. $M \times T$, being Morita equivalent to the type I factor N, must be a type I factor by 2.7, and hence isomorphic with $B(K)$ for some Hilbert space K, which must be separable because $M \times T$ acts on a separable Hilbert space. By the duality result of Takesaki alluded to above, there is an action of Z as automorphisms of $B(K)$ such that $B(K) \times Z$ is isomorphic with $M \otimes B(L^2(T))$. By the lemma, then, this tensor product must be of type I and hence M is of type I.

We remark that it is not clear whether this line of argument can be made to yield an analogous result for arbitrary compact abelian groups acting faithfully on M. The problem lies in generalizing Lemma 4.9, where one encounters the difficulty that an arbitrary abelian group of automorphisms of $B(K)$ need not be implemented by a *group* of unitary operators (although each individual automorphism will of course be inner).

5. **A class of examples.** In this section we construct a class of examples in which several of the features of the analysis developed above stand out in high relief. As a by-product, we obtain what appears to be a new method for constructing outer automorphisms (and indeed compact abelian groups of outer automorphisms) on many II_1-factors.

Let D be a countable discrete group with identity e, and let H be the Hilbert space $l^2(D)$; elements of H will be written as complex-valued functions on D. Let $M = VN(D)$, the von Neumann algebra generated by the left regular representation of D on H. Notice that for $\xi, \eta \in l^2(D)$, the convolution formula

$$(\xi * \eta)(s) = \sum_{t \in D} \xi(t)\eta(t^{-1}s) \ (s \in D)$$

defines a complex-valued function $\xi * \eta$ on D (which may or may not belong to $l^2(D)$). We shall call a function $x \in l^2(D)$ a *left convolver* if $x * \xi \in l^2(D) \ \forall \xi \in l^2(D)$ and the operator $\xi \longrightarrow x * \xi$ is bounded on $l^2(D)$. In II.5 of [7] it is shown that every operator in M has the form $\xi \longrightarrow x * \xi$ for some left convolver x; it is straightforward to verify that convolution on the left by any left convolver gives an operator in M. We shall henceforth identify M with the algebra of left convolvers in $l^2(D)$ (with convolution for multiplication and involution defined by $x^*(s) = \overline{x(s^{-1})}$ $(s \in D)$). For $s \in D$, we write δ_s for the function (and left convolver) in $l^2(D)$ with value 1 at s and 0 elsewhere.

Let λ be a character of D, i.e. a homomorphism of D into the circle group $T = \{z \in \mathbf{C}: |z| = 1\}$. For $x \in M$, define $a_\lambda(x) \in l^2(D)$ by $a_\lambda(x)(s) = \lambda(s)x(s)$ $(s \in D)$. For $\xi \in l^2(D), s \in D$, we have

$$\begin{aligned}(a_\lambda(x) * \xi)(s) &= \sum_{t \in D} \lambda(t)x(t)\xi(t^{-1}s) \\ &= \lambda(s) \sum_{t \in D} x(t)\overline{\lambda(t^{-1}s)}\xi(t^{-1}s) \\ &= \lambda(s)(x * (\bar{\lambda}\xi))(s),\end{aligned}$$

so

$$a_\lambda(x) * \xi = \lambda(x * (\bar{\lambda}\xi)),$$

which shows that $a_\lambda(x) \in M$. Routine computations now show that a_λ is a $*$-automorphism of M.

PROPOSITION 5.1. *If D is an infinite-conjugacy-class group (so that M is a II_1-factor) and λ is a nontrivial character of D, then a_λ is an outer automorphism of M.*

PROOF. Suppose by way of contradiction that there is a unitary $u \in M$ such that $a_\lambda(x) = u * x * u^* \ \forall x \in M$. For $s \in D$, define $\xi_s \in l^2(D)$ by $\xi_s(t) = u(s^{-1}ts)$ $(t \in D)$. We have

$$\begin{aligned}(u, \xi_s) &= \sum_{t \in D} u(t)\overline{u(s^{-1}ts)} \\ &= \sum_{t \in D} u(t)u^*(s^{-1}t^{-1}s) \\ &= \sum_{t \in D} u(t)(\delta_s * u^*)(t^{-1}s) \\ &= (u * \delta_s * u^*)(s) = a_\lambda(\delta_s)(s) \\ &= \lambda(s).\end{aligned}$$

Now u is a unit vector in $l^2(D)$ (because $u * u^* = \delta_e$) and hence so is ξ_s. Since $|(u, \xi_s)| = 1$, we conclude that ξ_s must be a scalar multiple of u, so $|u(t)| = |u(s^{-1}ts)|\ \forall s, t \in D$. By assumption, a_λ is not the identity automorphism of D, so there is a $t_0 \in D\backslash\{e\}$ such that $u(t_0) \neq 0$. Since the set $\{s^{-1}t_0 s: s \in D\}$ is infinite, we have a contradiction.

REMARK 5.2. If u is as in the proof above, we have $a_\lambda(u) = u$ and hence $\operatorname{supp}(u) \subseteq \ker(\lambda)$. We can therefore replace the assumption in 5.1 that D be an infinite-conjugacy-class group with the weaker condition that the conjugacy class in D of every $t \in \ker(\lambda)\backslash\{e\}$ be infinite.

In [2], R. Kallman shows that every outer automorphism of an infinite conjugacy class group induces an outer automorphism of the corresponding II_1-factor, and points out that there exist infinite-conjugacy-class groups all of whose automorphisms are inner. The example he mentions is the semidirect product $D = Q_* \times Q$ (where Q is the additive group of rationals, Q_* is the multiplicative group of nonzero rationals, and multiplication in D is defined by $(a_1, b_1)(a_2, b_2) = (a_1a_2, a_1b_2 + b_1)$ for $a_1, a_2 \in Q_*, b_1, b_2 \in Q$). Although it has no outer automorphisms, D has an abundance of nontrivial characters, since the abelian group Q_* is a homomorphic image of D; hence $VN(D)$ has many outer automorphisms. It would be interesting to know whether every infinite-conjugacy-class group with no outer automorphisms must necessarily possess a nontrivial character.

Returning to the general situation, let G be a compact abelian group and suppose that there exists a homomorphism $s \longrightarrow \lambda_s$ of D *onto* $\hat{G}$. We proceed to construct a faithful, strong-operator continuous action of G on $M = VN(D)$. For $g \in G$, $x \in M$, define $g(x) \in M$ by $g(x)(s) = \overline{\lambda_s(g)}x(s)$ $(s \in D)$; it is apparent that $g(\cdot) = a_\lambda$, where λ is the character of D defined by $\lambda(s) = \overline{\lambda_s(g)}$, so $g(\cdot)$ is a $*$-automorphism of M. It is equally clear that $g \longrightarrow g(\cdot)$ is an isomorphism (injective because $s \longrightarrow \lambda_s$ is onto) of G into the group of $*$-automorphisms of M. To show that the action of G on M is strong-operator continuous, take $s \in D$ and let $\{g_n\}$ be a sequence in G with limit $g \in G$. (Note that $\hat{G}$ must be countable, so the topology on G is metrizable.) For $x \in M$, we have

$$\|g_n(x) * \delta_s - g(x) * \delta_s\|^2 = \sum_{t \in D} |g_n(x)(ts^{-1}) - g(x)(ts^{-1})|$$
$$= \sum_{t \in D} |\lambda_{ts^{-1}}(g_n) - \lambda_{ts^{-1}}(g)|^2 |x(ts^{-1})|^2.$$

This sum goes to 0 as $n \longrightarrow \infty$ because $\lambda_{ts^{-1}}(g_n) \longrightarrow \lambda_{ts^{-1}}(g)\ \forall t \in D$. Since $\{\delta_s: s \in D\}$ spans a dense subset of $l^2(D)$, strong-operator continuity follows. The spectral subspaces M_λ for this action of G on M are immediately identifiable; for each $\lambda \in \hat{G}$, we have $M_\lambda = \{x \in M: \operatorname{supp}(x) \subseteq \{s \in D: \lambda_s = \lambda\}\}$. In particular,

$N = \{x \in M\colon \operatorname{supp}(x) \subseteq D_1\}$, where D_1 is the kernel of the homomorphism $s \longrightarrow \lambda_s$; thus N is isomorphic with $VN(D_1)$. Notice that for $\lambda \in \hat{G}$ and $s \in D$ such that $\lambda_s = \lambda$, we have $M_\lambda = \delta_s * N$, so M_λ and N are isomorphic as inner product modules over N, and $[M_\lambda H] = H$. By 4.6 and 3.1, $M \times G$ is isomorphic with $A(\overline{M})$, which in turn is isomorphic with $N \otimes B(l^2(\hat{G}))$ by 2.6.

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