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A PLANCHEREL FORMULA FOR IDYLLIC NILPOTENT LIE GROUPS(1)

BY

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ABSTRACT. A procedure is developed which can be used to compute the Plancherel measure for a certain class of nilpotent Lie groups, including the Heisenberg groups, free groups, two-and three-step groups, the nilpotent part of an Iwasawa decomposition of the R-split form of the classical simple groups A_l, C_l, G_2.

Let G be a connected, simply connected nilpotent Lie group. The Plancherel formula for G can be expressed in terms of Plancherel measure of a normal subgroup N and projective Plancherel measures of certain subgroups of G/N. To get an explicit measure for G, we need an explicit formula for (1) the disintegration of Plancherel measure of N under the action of G on $\hat{N}$, and (2) projective Plancherel measures of G_γ/N, where G_γ is the stability subgroup at γ in $\hat{N}$. When both N and G_γ/N are abelian, the measures (1) and (2) are obtained as special cases of more general problems. These measures combine into Plancherel measure for G.

0. Introduction. For a connected, simply connected, real nilpotent Lie group G, Dixmier [8], Kirillov [12], [13] and Pukańszky [19] have shown that the generic representations $\pi \in \hat{G}$ can be parametrized by a Zariski-open subset of a finite-dimensional real vector space $\mathbf{R}^k$, and that Plancherel measure for G (see [7], [18], [22]), μ_G, is then a rational function times Lebesgue measure on $\mathbf{R}^k - R(y)\,dy$. The main result of this paper is a technique for computing the rational function $R(y)$ in terms of the structure constants of the Lie algebra of G.

Kleppner and Lipsman's [14], [15] Plancherel formulation of the Mackey machine for expressing $\hat{G}$ in terms of $\hat{N}$ and irreducible projective representations of certain subgroups of G/N (the little groups), for $N < G$, is used to compute μ_G for a certain class of nilpotent Lie groups G. The procedure obtained for computing μ_G is explicit and can be carried out without too much trouble if the

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projective measures are reasonable. The method works for those connected, simply connected, nilpotent Lie groups G which have an abelian normal Lie subgroup N such that for μ_N almost all $\gamma \in \hat{N}$, G_γ/N is abelian, where G_γ is the stability subgroup at γ for the action of G on $\hat{N}$. Such a nilpotent Lie group is called idyllic.

When N is abelian, $\hat{N}$ is $\mathfrak{n}'$, the dual of the Lie algebra $\mathfrak{n}$ of N, and μ_N is Lebesgue measure on $\mathfrak{n}'$. The orbit space $\hat{N}/G$ is $\mathfrak{n}'/G$, the orbit space of the coadjoint representation of G in $\mathfrak{n}'$. We need an explicit formulation of the disintegration of Lebesgue measure on $\mathfrak{n}'$ into a measure on $\mathfrak{n}'/G$ and measures on the orbits of G in $\mathfrak{n}'$. When G_γ/N is abelian, the projective Plancherel measure can be computed. $\gamma \in \hat{N}$ extends to an ω_γ-representation of G_γ. When G_γ/N is abelian, the multiplier ω_γ on G_γ/N is the exponential of an alternating bilinear form on G_γ/N.

Let H be a finite-dimensional real vector space, $A : H \times H \longrightarrow R$ an alternating bilinear form on H, and ω_A the multiplier on H defined by $\omega_A(x, y) = e^{iA(x,y)/2}$. In §1, we compute the projective Plancherel measure on the space of irreducible ω_A-representations of H corresponding to a given Haar measure on H.

Let G be a connected, simply connected, nilpotent Lie group with Lie algebra $\mathfrak{g}$. In §2, we define a particular Haar measure m_G on G and show its invariance under certain types of changes of coordinates on G (Lemma 2.1). Theorem 2.1 gives a formula (2.4) expressing m_G in terms of a specific Haar measure on a certain type of closed subgroup $H \subset G$ and a specific G-invariant measure on the quotient space G/H.

In §3, the action on V' contragredient to a unipotent action of G on a finite-dimensional vector space V is analyzed by means of the structure matrix (3.6). Theorem 3.1 tells how to parametrize the stability subgroup G_γ for almost all $\gamma \in V'$, and describes a G-invariant measure on the orbit of γ and a Haar measure on G_γ which combine to give m_G (formula (3.8)). Theorem 3.2 describes a section for the orbits of G in a nonempty Zariski open subset of V'. Theorem 3.3 gives an explicit formula (3.13) for the disintegration of Lebesgue measure on V' under the contragredient action of G. The orbit measures in (3.13) are those in (3.8).

In §4, the results of §§1, 2, and 3 are combined via Kleppner and Lipsman's Plancherel formula for group extensions [15] to obtain a procedure for computing Plancherel measure for idyllic G (Theorem 4.1).

The following groups are known to be idyllic: free nilpotent Lie groups; Heisenberg groups; groups in Kirillov's second example; groups of dimension $\leqslant 5$; 2-step groups; the nilpotent part of an Iwasawa decomposition of the $\mathbf{R}$-split form of the classical simple groups G_2, A_l and C_l. Plancherel formulas are listed in Table I.

1. A projective Plancherel measure. Let H be a q-dimensional vector space over R. Suppose $A : H \times H \longrightarrow \mathbf{R}$ is bilinear and skew symmetric. Let $\omega : H \times H \longrightarrow T$ be the multiplier $\omega(x, y) = e^{iA(x,y)/2}$. ($T = \{z \in C: |z| = 1\}$.) Let $\{u_1, \ldots, u_q\}$ be a basis of H, and m_H the Haar measure on H defined by

$$\int_H f(x)\, dm_H(x) = \int_{\mathbf{R}^q} f\Big(\sum_{i=1}^q x^i u_i\Big)\, dm_{\mathbf{R}^q}(x^1, \ldots, x^q).$$

In this section, we compute the measure μ on the space of equivalence classes of irreducible ω-representations of H, denoted $(H, \omega)^\wedge$, such that

$$\int_H |f(x)|^2\, dm_H(x) = f *_\omega f^*(0) = \int_{(H,\omega)^\wedge} \operatorname{tr}[\sigma(f *_\omega f^*)]\, d\mu(\sigma), \qquad f \in L^1(H) \cap L^2(H).$$

Here, $f *_\omega f^*(x) = \int_H f(x - y) f^*(y)\omega(y, -x)\, dm_H(y)$, and $f^*(x) = \overline{f(-x)}$.

Suppose $\operatorname{rank}_A = 2l$, and $q = 2l + m$. Then [3, p. 81] there is a $q \times q$ nonsingular matrix $P = (P_i^j)$ such that

$$P(A(u_i, u_j))_{1 \leqslant i,j \leqslant q}\, {}^tP = \begin{array}{c} \begin{array}{cc} \overbrace{\hphantom{0 \quad I_l}}^{2l} & \overbrace{\hphantom{0}}^{m} \end{array} \\ \left[\begin{array}{c|c|c} 0 & I_l & 0 \\ \hline -I_l & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{l} \left.\vphantom{\begin{array}{c}0\\0\end{array}}\right\} 2l \\ \left.\vphantom{0}\right\} m \end{array}$$

Let $f_i = \sum_{j=1}^q P_i^j u_j$. Then $\{f_1, \ldots, f_q\}$ is a basis for H, and $A(f_i, f_j) = PA(u_i, u_j)^t P$ – that is, $A(f_i, f_{l+i}) = 1 = -A(f_{l+i}, f_i)$, for $1 \leqslant i \leqslant l$. The map $\kappa_P : (\mathbf{R}^l \times \mathbf{R}^l) \times \mathbf{R}^m \longrightarrow H$ defined by

$$\kappa_P((x, y), z) = \sum_{i=1}^l x^i f_i + \sum_{i=1}^l y^i f_{l+i} + \sum_{i=1}^m z^i f_{2l+i},$$

for $x = (x^1, \ldots, x^l)$, $y = (y^1, \ldots, y^l)$, and $z = (z^1, \ldots, z^m)$, is an isomorphism with the property that

$$\begin{aligned} &\omega(\kappa_P(x_1, y, z_1), \kappa_P(x_2, y_2, z_2)) \\ &\quad = e^{i[x_1 \cdot y_2 - x_2 \cdot y_1]/2} = \omega_1((x_1, y_1), (x_2, y_2)) \\ &\quad = (\omega_1 \times 1)(((x_1, y_1), z_1), ((x_2, y_2), z_2)), \end{aligned}$$

where $\omega_1 : (\mathbf{R}^l \times \mathbf{R}^l) \times (\mathbf{R}^l \times \mathbf{R}^l) \longrightarrow T$ is the multiplier $\omega_1((x_1, y_1), (x_2, y_2)) = e^{i[x_1 \cdot y_2 - x_2 \cdot y_1]/2}$. Here for $x = (x^1, \ldots, x^l) \in \mathbf{R}^l$, $y = (y^1, \ldots, y^l) \in \mathbf{R}^l$, $x \cdot y$ denotes the inner product, $x \cdot y = \sum_{i=1}^l x^i y^i$. Thus, the map ${}^t\kappa_P : (H, \omega)^\wedge \longrightarrow ((\mathbf{R}^l \times \mathbf{R}^l) \times \mathbf{R}^m, \omega_1 \times 1)^\wedge$ given by ${}^t\kappa_P(\sigma)((x, y), z) = \sigma(\kappa_P((x, y), z))$ for $\sigma \in (H, \omega)^\wedge$, $((x, y), z) \in (\mathbf{R}^l \times \mathbf{R}^l) \times \mathbf{R}^m$, is an isomorphism. Hence

$$(H, \omega)\hat{} \simeq ((\mathbf{R}^l \times \mathbf{R}^l) \times \mathbf{R}^m, \omega_1 \times 1)\hat{} = (\mathbf{R}^l \times \mathbf{R}^l, \omega_1)\hat{} \times (\mathbf{R}^m, 1)\hat{}$$
$$= \{\sigma_1\} \times \mathbf{R}^m = \{\sigma_{1,t} = \sigma_1 \cdot \chi_t : t \in \mathbf{R}^m\}$$

where σ_1 is the unique irreducible ω_1-representation of $\mathbf{R}^{2l}$(see, for example, [17, Example 1, p. 305]), and χ_t is a character of $\mathbf{R}^m$. $\sigma_{1,t}$ can be realized on $L^2(\mathbf{R}^l)$ as follows. If $h = ((x, y), z) \in (\mathbf{R}^l \times \mathbf{R}^l) \times \mathbf{R}^m$, then

$$\begin{aligned}(\sigma_{1,t}(h)F)(v) &= \chi_t(z)(\sigma_1(x, y)F)(v)\\ &= e^{i(t\cdot z)}\, e^{i\{y\cdot v + (x\cdot y)/2\}} F(v + x).\end{aligned}$$

From [14, p. 490] the projective Plancherel measure for $(\mathbf{R}^{2l}, \omega_1)$ is $\mu_{(\mathbf{R}^{2l}, \omega_1)}(\sigma_1) = 1/(2\pi)^l$ – that is,

$$\int_{\mathbf{R}^{2l}} |\phi(x, y)|^2 \, dm_{\mathbf{R}^{2l}}(x, y) = \frac{1}{(2\pi)^l} \operatorname{tr}(\sigma_1(\phi *_{\omega_1} \phi^*)),$$

$\phi \in L^1(\mathbf{R}^{2l}) \cap L^2(\mathbf{R}^{2l})$. (Here $m_{\mathbf{R}^{2l}}$ is Lebesgue measure $m_{\mathbf{R}^{2l}}$ such that $m_{\mathbf{R}^{2l}}([0, 1]^{2l}) = 1$.)

Plancherel measure for $\mathbf{R}^m$ is $\mu_{\mathbf{R}^m} = (2\pi)^{-m} m_{\mathbf{R}^m}$ – i.e.,

$$\int_{\mathbf{R}^m} |f(z)|^2 \, dm_{\mathbf{R}^m}(z) = \frac{1}{(2\pi)^m} \int_{\mathbf{R}^m} |\chi_t(f)|^2 \, dm_{\mathbf{R}^m}(t),$$

where

$$\chi_t(f) = \hat{f}(t) = \int_{\mathbf{R}^m} f(s) e^{i(s\cdot t)} \, dm_{\mathbf{R}^m}(t), \qquad f \in L^1(\mathbf{R}^m) \cap L^2(\mathbf{R}^m).$$

($m_{\mathbf{R}^m}$ is Lebesgue measure on $\mathbf{R}^m$ such that $m_{\mathbf{R}^m}([0, 1]^m) = 1$.)

Let ν_H be the image of Lebesgue measure on $(\mathbf{R}^l \times \mathbf{R}^l) \times \mathbf{R}^m$ under the map κ_P. Then

$$\begin{aligned}\int_H f(h)\, d\nu_H(h) &= \int_{(\mathbf{R}^l \times \mathbf{R})^l \times \mathbf{R}^m} f(\kappa_P((x, y), z))\, dm_{(\mathbf{R}^l \times \mathbf{R})^l \times \mathbf{R}^m}((x, y), z)\\ &= \int_{\mathbf{R}^{2l+m}} f\left(\sum_{i=1}^{2l+m} h^i f_i\right) dm_{\mathbf{R}^{2l+m}}(h^1, \ldots, h^{2l+m})\\ &= \int_{\mathbf{R}^q} f\left(\sum_{j=1}^{q} \left(\sum_{i=1}^{q} h^i P_i^j\right) u_j\right) dm_{\mathbf{R}^q}(h^1, \ldots, h^q)\\ &= |\det P|^{-1} \int_{\mathbf{R}^q} f\left(\sum_{j=1}^{q} h^j u_j\right) dm_{\mathbf{R}^q}(h^1, \ldots, h^q)\\ &= |\det P|^{-1} \int_H f(h)\, dm_H(h),\end{aligned}$$

so that $m_H = |\det P| \nu_H = |\det P| (\kappa_P(m_{(\mathbf{R}^l \times \mathbf{R}^l) \times \mathbf{R}^m}))$. It follows that

$$\begin{aligned}\mu_{(H,\omega)} &= |\det P|^{-1} ({}^t\kappa_P)^{-1} (\mu_{(\mathbf{R}^l \times \mathbf{R}^l) \times \mathbf{R}^m, \omega_1 \times 1})\\ &= |\det P|^{-1} ({}^t\kappa_P)^{-1} (\mu_{(\mathbf{R}^{2l}, \omega_1)} \times \mu_{\mathbf{R}^m})\\ &= |\det P|^{-1} ({}^t\kappa_P)^{-1} ((2\pi)^{-l} \times (2\pi)^{-m} m_{\mathbf{R}^m}),\end{aligned}$$

i.e., that $\mu_{(H,\omega)}$ is the image of the measure $|\det P|^{-1}(2\pi)^{-(l+m)}m_{\mathbf{R}^m}$ on $\mathbf{R}^m$ under the map

$$\psi_P : t \longrightarrow ({}^t\kappa_P)^{-1}(\sigma_{1,t}) : \mathbf{R}^m \longrightarrow (H, \omega)\hat{}.$$

Kleppner and Baggett [1, Corollary, p. 310] prove that this map is a homeomorphism. To see that

$$\int_H |f(h)|^2\, dm_H(h) = |\det P|^{-1}\frac{1}{(2\pi)^{l+m}}\int_{\mathbf{R}^m} \operatorname{tr}[({}^t\kappa_P)^{-1}(\sigma_{1,t})(f *_\omega f^*)]\, dm_{\mathbf{R}^m}(t),$$

we calculate that

$$\begin{aligned}({}^t\kappa_P)^{-1}(\sigma_{1,t})(f *_\omega f^*) &= |\det P|\,\sigma_{1,t}((f *_\omega f^*) \circ \kappa_P)\\ &= |\det P|^2\,\sigma_{1,t}((f \circ \kappa_P) *_{\omega_1 \times 1} (f \circ \kappa_P)^*).\end{aligned}$$

Hence,

$$\begin{aligned}&\frac{|\det P|^{-1}}{(2\pi)^{l+m}}\int_{\mathbf{R}^m} \operatorname{tr}[({}^t\kappa_P)^{-1}(\sigma_{1,t})(f *_\omega f^*)]\, dm_{\mathbf{R}^m}(t)\\ &\quad= \frac{|\det P|}{(2\pi)^{l+m}}\int_{\mathbf{R}^m} \operatorname{tr}[\sigma_{1,t}((f \circ \kappa_P) *_{\omega_1 \times 1} (f \circ \kappa_P)^*)]\, dm_{\mathbf{R}^m}(t)\\ &\quad= |\det P|\int_{\mathbf{R}^{2l}\times\mathbf{R}^m} |f \circ \kappa_P((x, y), z)|^2\, dm_{\mathbf{R}^{2l}\times\mathbf{R}^m}((x, y), z)\end{aligned}$$

(since $(2\pi)^{-(l+m)}m_{\mathbf{R}^m}$ is the projective Plancherel measure for $(\mathbf{R}^{2l} \times \mathbf{R}^m, \omega_1 \times 1)$)

$$= |\det P|\int_H |f(h)|^2\, dv_H(h) = \int_H |f(h)|^2\, dm_H(h).$$

The projective Plancherel measure $\mu_{(H,\omega)} = (2\pi)^{-(l+m)}|\det P|^{-1}\psi_P(m_{\mathbf{R}^m})$ on (H, ω) corresponding to Haar measure m_H on H depends on the choice of the matrix P. If A is nondegenerate, then $|\det P|^{-1} = \text{Pfaffian}\,(A(u_i, u_j))_{1 < i,j < q}$ [3, pp. 82–84] is uniquely determined by A. However, if A is degenerate, then P is quite arbitrary on the null space of $(A(u_i, u_j))_{1<i,j<q}$, and $|\det P|$ is not unique.

$\psi_P : \mathbf{R}^m \longrightarrow (H, \omega)\hat{}$ is the following map. Let $Q = (Q_i^j)_{1<i,j<2l+m} = P^{-1}$. If A is nondegenerate, then $M = 0$; and $(H, \omega)\hat{}$ consists of one point, $\psi_P = {}^t\kappa_P^{-1}(\sigma_1)$. If $x = \Sigma_{i=1}^{2l} x^i u_i \in H$, then

$$\psi_P(x) = \sigma_1(\kappa_P^{-1}(x)) = \sigma_1(xQ^{(l)}, xQ^{(2l)}),$$

where

$$xQ^{(l)} = \left(\sum_{i=1}^{2l} x^i Q_i^1, \ldots, \sum_{i=1}^{2l} x^i Q_i^l\right),$$

$$xQ^{(2l)} = \left(\sum_{i=1}^{2l} x^i Q_i^{l+1}, \ldots, \sum_{i=1}^{2l} x^i Q_i^{2l}\right).$$

If $m > 0$, then $\psi_P : \mathbf{R}^m \xrightarrow{\sim} (H, \omega)^\wedge$ is given by

$$\psi_P(t)(x) = \sigma_{1,t}(\kappa_P^{-1}(x)) = \sigma_1(xQ^{(l)}, xQ^{(2l)})e^{i(xQ^{(m)}t)},$$

where $x = \Sigma_{i=1}^{2l+m} x^i u_i \in H$, $t = (t_1, \ldots, t_m) \in \mathbf{R}^m$, and

$$xQ^{(l)} = \left(\sum_{i=1}^{2l+m} x^i Q_i^1, \ldots, \sum_{i=1}^{2l+m} x^i Q_i^l \right),$$

$$xQ^{(2l)} = \left(\sum_{i=1}^{2l+m} x^i Q_i^{l+1}, \ldots, \sum_{i=1}^{2l+m} x^i Q_i^{2l} \right),$$

$$xQ^{(m)}t = \sum_{a=1}^{m} \sum_{i=1}^{2l+m} x^i Q_i^{2l+a} t_a.$$

If $l = 0$, then $A = 0$; $\omega = 1$; $(H, \omega)^\wedge = \hat{H}$, the character group of H; $m = q$; and P may be taken as the identity. In this case, $\psi_P : \mathbf{R}^m \xrightarrow{\sim} \hat{H}$ is given by $\psi_P(t)(x) = e^{i\Sigma_{a=1}^m x^a t_a} = \chi_t(x)$, for $x = \Sigma_{a=1}^m x^a u_a \in H$, $t = (t_1, \ldots, t_m) \in \mathbf{R}^m$.

2. **Some formulas for Haar measure on G.** Let G be a connected, simply connected nilpotent Lie group over $\mathbf{R}$ with Lie algebra $\mathfrak{g}$. This section is devoted to establishing formulas for Haar measure on G in terms of certain coordinate systems for G. Suppose $\dim \mathfrak{g} = s$. The exponential map, denoted exp, is a diffeomorphism of $\mathfrak{g}$ onto G. Hence the choice of a basis $\{e_1, \ldots, e_s\}$ in $\mathfrak{g}$ determines a coordinate system for G by the map $\xi : \mathbf{R}^s \longrightarrow G$ given by $\xi(x^1, \ldots, x^s) = \exp(\Sigma_{i=1}^s x^i e_i)$. The image of Lebesgue measure on $\mathbf{R}^s$ under this map is a Haar measure on G, called the measure on G defined in terms of the basis $\{e_1, \ldots, e_s\}$ of $\mathfrak{g}$.

Let B be a basis of $\mathfrak{g}$. A linear order "$<$" on B is called a Jordan-Hölder order if, for each ν in B, $[\mathfrak{g}, \nu] = 0$ if ν is maximal; otherwise, $[\mathfrak{g}, \nu] \subset \operatorname{span}\{\omega \in B : \nu < \omega\}$. Suppose $e_1 < \cdots < e_s$ is a basis of $\mathfrak{g}$ in Jordan-Hölder order, i.e., $[\mathfrak{g}, e_s] = 0$, $[\mathfrak{g}, e_i] \subset \operatorname{span}(e_{i+1}, \ldots, e_s)$ for $1 \leq i \leq s-1$. Let $m_{\mathbf{R}^s}$ be Lebesgue measure on $\mathbf{R}^s$ such that $m_{\mathbf{R}^s}([0, 1]^s) = 1$. m_G will denote the Haar measure on G defined in terms of $\{e_1, \ldots, e_s\}$; so that

$$\int_G f(A)\, dm_G(A) = \int_{\mathbf{R}^s} f\left(\exp\left(\sum_{i=1}^{s} x^i e_i\right)\right) dm_{\mathbf{R}^s}(x^1, \ldots, x^s).$$

Invariance of m_G under left and right translation follows from the Campbell-Baker-Hausdorff formula, $\exp x \exp y = \exp(x + y + \tfrac{1}{2}[x, y] + \cdots)$, and the fact that $e_1 < \cdots < e_s$ is a Jordan-Hölder basis of $\mathfrak{g}$. Then the fact that the measure on G defined in terms of any basis of $\mathfrak{g}$ is a Haar measure follows. Indeed, if m is the measure on G defined in terms of the basis $\{\omega_1, \ldots, \omega_s\}$ of $\mathfrak{g}$, and if $\omega_i = \Sigma_{j=1}^s a_i^j e_j$ for $1 \leq i \leq s$, then $m = |\det A|^{-1} m_G$, where $A = (a_i^j)_{1 \leq i,j \leq s}$.

Because $e_1 < \cdots < e_s$ is a Jordan-Hölder basis of $\mathfrak{g}$, the measure on G given in terms of the coordinate system $\xi(x^1, \ldots, x^s) = \exp(\Sigma_{i=1}^s x^i e_i)$ is the same as the measure on G obtained by taking the image of Lebesgue measure on $\mathbf{R}^s$ under the map $\eta(x^1, \ldots, x^s) = \exp x^1 e_1 \cdots \exp x^s e_s$. In fact, any sum and any permutation is allowed in the sense of the following lemma.

LEMMA 2.1. *Let $\{e_1, \ldots, e_s\}$ be a Jordan-Hölder basis of $\mathfrak{g}$ such that $[\mathfrak{g}, e_s] = 0$, $[\mathfrak{g}, e_i] \subset \operatorname{span}\{e_{i+1}, \ldots, e_s\}$ for $1 \leqslant i \leqslant s-1$. Let σ be a permutation of $\{1, \ldots, s\}$. If $f \in C_0(G)$ (= continuous functions with compact support), then, for $1 \leqslant m \leqslant s$,*

$$\int_{\mathbf{R}^s} f\left(\exp\left(\sum_{i=1}^s x^i e_i\right)\right) dm_{\mathbf{R}^s}(x^1, \ldots, x^s)$$

$$= \int_{\mathbf{R}^s} f\left[\exp\left(\sum_{i=1}^m x^{\sigma(i)} e_{\sigma(i)}\right) \prod_{i=m+1}^s \exp x^{\sigma(i)} e_{\sigma(i)}\right] dm_{\mathbf{R}^s}(x^1, \ldots, x^s). \tag{2.1}$$

PROOF. For $x = (x^1, \ldots, x^s) \in \mathbf{R}^s$, put

$$T(x) = \exp\left(\sum_{i=1}^m x^{\sigma(i)} e_{\sigma(i)}\right) \prod_{i=m+1}^s \exp x^{\sigma(i)} e_{\sigma(i)}.$$

The Campbell-Baker-Hausdorff formula,

$$\begin{aligned} &\exp v \exp w \\ &\quad = \exp\left(v + w + \frac{1}{2}[v, w] + \frac{1}{12}([v, [v, w]] - [w, [v, w]]) + \cdots\right), \end{aligned} \tag{2.2}$$

where $v, w \in \mathfrak{g}$, shows that $T(x) = \exp(\Sigma_{k=1}^S x^k e_k + B(x))$, where $B(x) \in \mathfrak{g}$ is a sum of terms of the form

$$[\cdots [x^j e_j, [\cdots [x^l e_l, x^t e_t] \cdots]] \cdots]. \tag{$*$}$$

Let $\phi^k(x)$ denote the kth component with respect to the basis $\{e_i\}_{i=1}^s$ of $\mathfrak{g}$ of $B(x)$. Since $e_1 < \cdots < e_s$ is a Jordan-Hölder basis of $\mathfrak{g}$, ϕ^k is independent of $(x^k, \ldots, x^s)$. Indeed, if $j \geqslant k$, then

$$[\cdots [x^j e_j, [\cdots [\ ,\ ] \cdots]] \cdots] \in \operatorname{span}\{e_{j+1}, \ldots, e_s\} \subset \operatorname{span}\{e_{k+1}, \ldots, e_s\}.$$

Thus the only terms $(*)$ in $B(x)$ which can have a nonzero component in the direction of e_k are those brackets involving only $x^1 e_1, \ldots, x^{k-1} e_{k-1}$. Hence ϕ^k is a function of $(x^1, \ldots, x^{k-1})$. Therefore,

$$T(x) = \exp\left(x^1 e_1 + x^2 e_2 + \sum_{k=3}^S (x^k + \phi^k(x^1, \ldots, x^{k-1})) e_k\right). \tag{2.3}$$

(2.1) follows from (2.3) by Fubini's Theorem. Considering the right-hand side of (2.1) as an iterated integral and using (2.3), we make $s-2$ successive substitutions $x^{s-i} \longrightarrow x^{s-i} - \phi^{s-i}(x^1, \ldots, x^{s-i-1})$ holding $x^1, \ldots, x^{s-i-1}$ fixed, for $i = 0, 1, \ldots, s-3$. The result is the left-hand side of (2.1).

The following lemma and theorem establish a formula for m_G in terms of coordinates on a certain type of Lie subgroup H of G and on the quotient manifold G/H.

LEMMA 2.2. *Suppose* $\mathfrak{h}$ *is a subalgebra of* $\mathfrak{g}$, *and* $H = \exp \mathfrak{h}$ *is the corresponding Lie subgroup of* G. *Suppose* $\dim(\mathfrak{g}/\mathfrak{h}) = r$, *and* $\mathfrak{h} = \mathfrak{h}_{r+1} \subset \mathfrak{h}_r \subset \cdots \subset \mathfrak{h}_1 = \mathfrak{g}$ *is an ascending sequence of subalgebras of* $\mathfrak{g}$ *such that*

$$\dim(\mathfrak{h}_i/\mathfrak{h}_{i+1}) = 1 \quad \text{for } 1 \leqslant i \leqslant r.$$

Suppose ω_k *is in* $\mathfrak{h}_i$, *not in* $\mathfrak{h}_{i+1}$, *for* $1 \leqslant i \leqslant r$. *Then the map* $(t^1, \ldots, t^r) \longrightarrow H \exp t^r\omega_r \cdots \exp t^1\omega_1$ *is a homeomorphism of* $\mathbf{R}^r$ *onto* G/H. *The image of Lebesgue measure on* $\mathbf{R}^r$ *under this map is a G-invariant measure on* G/H.

PROOF. Pukańszky gives a proof in [19, pp. 85, 97].

This measure will be called the measure on G/H defined in terms of the basis $\{\omega_1, \ldots, \omega_r\}$ of $\mathfrak{g}/\mathfrak{h}$.

If m_H is any Haar measure on H, and ν is any G-invariant measure on G/H, then ν and m_H combine to give a Haar measure on G, i.e.,

$$\int_G f(x)\,dx = \int_{G/H}\int_H f(hx)\,dm_H(h)\,d\nu(\bar{x})$$

defines a Haar measure on G. For the subgroups of G which occur in the sequel, the measures ν and m_H can be chosen so that the resulting Haar measure on G is exactly m_G. The following theorem gives the conditions that will arise and the proof for this type of subgroup $H \subset G$.

THEOREM 2.1. *Suppose* $\mathfrak{h}$ *is a subalgebra of* $\mathfrak{g}$ *having a basis* $\{u_1, \ldots, u_q\}$ *with the following property. There is a partition* $\{1, \ldots, s\} = \{m_1 < \cdots < m_q\} \cup \{i_1 < \cdots < i_r\}$ *such that*

$$u_b = e_{m_b} - \sum_{\{t:\, m_b < i_t\}} \lambda^{i_t}_{m_b} e_{i_t} \quad \text{for } 1 \leqslant b \leqslant q.$$

Let $H = \exp \mathfrak{h}$, *and let* m_H *be the Haar measure on H defined in terms of* $\{u_1, \ldots, u_q\}$.

Then the map $(t^1, \ldots, t^r) \longrightarrow H \exp t^r e_{i_r} \cdots \exp t^1 e_{i_1}$ *is a homeomorphism of* $\mathbf{R}^r$ *with* G/H. *The image of Lebesgue measure on* $\mathbf{R}^r$ *under this map is a G-invariant measure,* ν, *on* G/H. m_G, ν, *and* m_H *satisfy*

$$\int_G f(A)\,dm_G(A) = \int_{G/H}\int_H f(hA)\,dm_H(h)\,d\nu(\bar{A}),$$

i.e.,

$$\int_{\mathbf{R}^s} f\left(\exp\left(\sum_{i=1}^{s} x^i e_i\right)\right) dm_{\mathbf{R}^s}(x^1, \ldots, x^s)$$

$$(2.4) \qquad = \int_{\mathbf{R}^r}\left[\int_{\mathbf{R}^q} f\left(\exp\left(\sum_{i=1}^{q} z^i u_i\right) \exp t^r e_{i_r} \cdots \exp t^1 e_{i_1}\right) dm_{\mathbf{R}^q}(z^1, \ldots, z^q)\right] dm_{\mathbf{R}^r}(t^1, \ldots, t^r).$$

PROOF. Let $\mathfrak{h}_{r+1} = \mathfrak{h}$, and $\mathfrak{h}_k = \mathfrak{h}_{k+1} \oplus (e_{i_k})$ for $r \geq k \geq 1$. Then $\mathfrak{h} = \mathfrak{h}_{r+1} \subset \mathfrak{h}_r \subset \cdots \subset \mathfrak{h}_1 = \mathfrak{g}$ is an increasing sequence of subspaces of $\mathfrak{g}$ such that $\dim(\mathfrak{h}_k/\mathfrak{h}_{k+1}) = 1$; and e_{i_k} is in $\mathfrak{h}_k$, not in $\mathfrak{h}_{k+1}$, for $1 \leq k \leq r$. Thus, the fact that the map $\psi : (t^1, \ldots, t^r) \longrightarrow H \exp t^r e_{i_r} \cdots \exp t^1 e_{i_1}$ is a homeomorphism of $\mathbf{R}^r$ onto G/H and that $\nu = \psi(m_{\mathbf{R}^r})$ is a G-invariant measure on G/H is just Lemma 2.2, once it is shown that each $\mathfrak{h}_k$, $r \geq k \geq 1$, is a subalgebra of $\mathfrak{g}$.

To prove that each $\mathfrak{h}_k$ is a subalgebra of $\mathfrak{g}$, we first prove, by calculating brackets, that $[\mathfrak{g}, e_{i_k}] \subset \mathfrak{h}_{k+1}$ for $r \geq k \geq 1$. Let $x \in \mathfrak{g}$ and $1 \leq k \leq r$. Then

$$\begin{aligned}
[x, e_{i_k}] &= \sum_{n=(i_k)+1}^{s} a_{i_k}^{n}(x) e_n \\
&= \sum_{\{b : m_b > i_k\}} a_{i_k}^{m_b}(x) e_{m_b} + \sum_{\{s : i_s > i_k\}} a_{i_k}^{i_s}(x) e_{i_s} \\
&= \sum_{\{b : m_b > i_k\}} a_{i_k}^{m_b}(x) \left(u_b + \sum_{\{t : i_t > m_b\}} \lambda_{m_b}^{i_t} e_{i_t}\right) + \sum_{\{s : i_s > i_k\}} a_{i_k}^{i_s}(x) e_{i_s}
\end{aligned}$$

(by the hypothesis on $\{u_1, \ldots, u_q\}$).

Thus $[x, e_{i_k}]$ is in $\operatorname{span}(\{u_b : m_b > i_k\} \cup \{e_{i_s} : i_s > i_k\})$, which is contained in $\mathfrak{h} \oplus (e_{i_r}) \oplus \cdots \oplus (e_{i_{(k+1)}}) = \mathfrak{h}_{k+1}$.

That each $\mathfrak{h}_k$ is a subalgebra of $\mathfrak{g}$ follows by induction. $\mathfrak{h}_{r+1} = \mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ by hypothesis. Assume $\mathfrak{h}_{k+1}$ is a subalgebra of $\mathfrak{g}$. Then, for $\mathfrak{h}_k = \mathfrak{h}_{k+1} + (e_{i_k})$, we have $[\mathfrak{h}_k, \mathfrak{h}_k] = [\mathfrak{h}_{k+1}, \mathfrak{h}_{k+1}] + [\mathfrak{h}_{k+1}, e_{i_k}]$ contained in $\mathfrak{h}_{k+1}$, since $[\mathfrak{h}_{k+1}, \mathfrak{h}_{k+1}] \subset \mathfrak{h}_{k+1}$ by inductive hypothesis, and $[\mathfrak{h}_{k+1}, e_{i_k}] \subset [\mathfrak{g}, e_{i_k}] \subset \mathfrak{h}_{k+1}$ by the preceding calculation. Since $\mathfrak{h}_{k+1}$ is contained in $\mathfrak{h}_k$, this shows that $\mathfrak{h}_k$ is a subalgebra of $\mathfrak{g}$.

The rest of the proof is an application of Lemma 2.1 to show that m_G, ν, and m_H satisfy (2.4). The set $\{e_{i_1}, \ldots, e_{i_r}\} \cup \{u_1, \ldots, u_q\}$ is a basis of $\mathfrak{g}$. For $1 \leq k \leq s$, let

$$\begin{aligned}
f_k &= e_{i_t} \quad \text{if } k = i_t \\
&= u_b \quad \text{if } k = m_b.
\end{aligned}$$

Then $f_1, \ldots, f_s$ is a Jordan-Hölder basis of $\mathfrak{g}$ such that $[\mathfrak{g}, f_s] = 0$, $[\mathfrak{g}, f_k] \subset \operatorname{span}(f_{k+1}, \ldots, f_s)$, for $1 \leqslant k \leqslant s-1$. Indeed, if $k = i_t$, then

$$[\mathfrak{g}, f_k] = [\mathfrak{g}, e_{i_t}] \subset \operatorname{span}(\{u_b : m_b > i_t\} \cup \{e_{i_s} : i_s > i_t\})$$

by the preceding calculation. Since $\{u_b : m_b > i_t\} = \{f_l : l = m_b > k\}$, and $\{e_{i_s} : i_s > i_t\} = \{f_l : l = i_s > k\}$, we have $[\mathfrak{g}, f_k] \subset \operatorname{span}(f_{k+1}, \ldots, f_s)$. If $k = m_b$, then $[\mathfrak{g}, f_k] = [\mathfrak{g}, u_b] = [\mathfrak{g}, e_{m_b} - \Sigma_{\{s:i_s > m_b\}} \lambda^{i_s}_{m_b} e_{i_s}]$, which is contained in $\operatorname{span}\{e_l : l > m_b\}$. Now

$$\operatorname{span}\{e_l : l > m_b\} = \operatorname{span}(\{u_a : m_a > m_b\} \cup \{e_{i_s} : i_s > m_b\})$$

(since $e_{m_a} = u_a + \Sigma_{\{s:i_s > m_a\}} \lambda^{i_s}_{m_a} e_{i_s}$). Since $\{u_a : m_a > m_b\} = \{f_l : l = m_a > m_b = k\}$, and $\{e_{i_s} : i_s > m_b\} = \{f_l : l = i_s > m_b = k\}$, we have $[\mathfrak{g}, f_k] \subset \operatorname{span}(f_{k+1}, \ldots, f_s)$.

To apply Lemma 2.1, let $\sigma \in S_s$ be a permutation such that $i_t = \sigma(s-t+1)$ for $1 \leqslant t \leqslant r$, and $m_b = \sigma(b)$ for $1 \leqslant b \leqslant q$. Now, taking $f \in C_0(b)$ and using Fubini's theorem, the right-hand side of (2.4) may be written as

$$\int_{\mathbf{R}^r} \left[\int_{\mathbf{R}^q} f\left(\exp\left(\sum_{b=1}^{q} x^{m_b} u_b \right) \exp x^{i_r} e_{i_r} \cdots \exp x^{i_1} e_{i_1} \right) dm_{\mathbf{R}^q}(x^{m_1}, \ldots, x^{m_q}) \right] dm_{\mathbf{R}^r}(x^{i_1}, \ldots, x^{i_r})$$

$$= \int_{\mathbf{R}^s} f\left(\exp\left(\sum_{b=1}^{q} x^{m_b} u_b \right) \exp x^{i_r} e_{i_r} \cdots \exp x^{i_1} e_{i_1} \right) dm_{\mathbf{R}^s}(x^1, \ldots, x^s)$$

(by Fubini)

$$= \int_{\mathbf{R}^s} f\left(\exp\left(\sum_{b=1}^{q} x^{m_b} f_{m_b} \right) \exp x^{i_r} f_{i_r} \cdots \exp x^{i_1} f_{i_1} \right) dm_{\mathbf{R}^s}(x^1, \ldots, x^s)$$

(by definition of $\{f_k : 1 \leqslant k \leqslant s\}$)

$$= \int_{\mathbf{R}^s} f\left(\exp\left(\sum_{b=1}^{q} x^{\sigma(b)} f_{\sigma(b)} \right) \exp x^{\sigma(q+1)} f_{\sigma(q+1)} \cdots \exp x^{\sigma(s)} f_{\sigma(s)} \right) dm_{\mathbf{R}^s}(x^1, \ldots, x^s)$$

(by definition of σ)

$$= \int_{\mathbf{R}^s} f\left(\exp\left(\sum_{k=1}^{s} x^k f_k \right) \right) dm_{\mathbf{R}^s}(x^1, \ldots, x^s)$$

(by Lemma 2.1).

If $f_k = \Sigma_{j=1}^{S} a^j_k e_j$, $1 \leqslant k \leqslant s$, then $|\det(a^j_k)_{1 \leqslant j, k \leqslant s}| = 1$, since $f_{m_b} = u_b \equiv e_{m_b}$ $(e_{(m_b)+1}, \ldots, e_s)$, $1 \leqslant b \leqslant q$, and $f_{i_t} = e_{i_t}$, $1 \leqslant t \leqslant r$.

Thus the final integral above is equal to

$$\int_{\mathbf{R}^s} f\left(\exp\left(\sum_{k=1}^{s} x^k e_k\right)\right)\, dm_{\mathbf{R}^s}(x^1, \ldots, x^s),$$

which is the left-hand side of (2.4).

3. **A disintegration theorem.** Suppose G is a connected, simply connected nilpotent Lie group over $\mathbf{R}$ with Lie algebra $\mathfrak{g}$; V a finite-dimensional vector space over $\mathbf{R}$; and $G \times V \longrightarrow V : (A, v) \longrightarrow Av$ a unipotent action of G on V. This section is devoted to analyzing the contragredient action of G on the dual space V' of V: $V' \times G \longrightarrow V' : (\gamma, A) \longrightarrow (v \longrightarrow \langle \gamma, Av \rangle)$. After establishing terminology, notation, and preliminary facts about orbits, stability subgroups, and the relation between the action of G and that of $\mathfrak{g}$, we develop a technique for (1) computing almost all the stability subgroups for the action of G on V', (2) coordinatizing almost all the orbits of G in V', and (3) coordinatizing almost all the orbit space V'/G. We establish a formula (3.8) giving Haar measure on G in terms of Haar measure on the stability subgroup G_γ and a G-invariant measure on the orbit G/G_γ. Lebesgue measure on V', denoted $m_{V'}$, is decomposed by G into a measure on the orbit space V'/G and measures on the corresponding orbits. We prove an explicit formula (3.13) for this disintegration of $m_{V'}$ by G, in which the orbit measures are those appearing in (3.8). This coincidence of the orbit measures is necessary for the proof of the Plancherel formula in §4.

Let G be a connected, simply connected nilpotent Lie group over $\mathbf{R}$ with Lie algebra $\mathfrak{g}$. Suppose V is a K-dimensional vector space over $\mathbf{R}$ on which G acts smoothly as a group of unipotent automorphisms, i.e., the mapping $G \times V \longrightarrow V : (A, v) \longrightarrow Av$ is differentiable. Then for each v in V the map $F_v : G \longrightarrow V$ given by $F_v(A) = Av$, $A \in G$, is differentiable. Its derivative defines an action of $\mathfrak{g}$ as a nilpotent Lie algebra of endomorphisms of V by $av = (d/dt)(\exp ta)(v)|_{t=0}$, $a \in \mathfrak{g}$, $v \in V$. If $a \in \mathfrak{g}$, $v \in V$, then $(\exp a)(v) = (1 + a + a^2/2! + \cdots + a^k/k!)(v)$.

Let V' denote the dual space of V. The contragredient action of G (resp. $\mathfrak{g}$) on V' is given by $V' \times G \longrightarrow V'$ (resp. $V' \times \mathfrak{g} \longrightarrow V'$): $(\gamma, A) \longrightarrow \gamma A$, where $\langle \gamma A, v \rangle = \langle \gamma, Av \rangle$ for $A \in G$ (resp. $\mathfrak{g}$), $\gamma \in V'$, $v \in V$. For γ in V', let $F_\gamma : G \longrightarrow V'$ be the map $F_\gamma(A) = \gamma \cdot A$. Let $O_\gamma = F_\gamma(G)$ denote the orbit of γ in V'; $G_\gamma = \{A \in G : \gamma \cdot A = \gamma\}$, the stabilizer of γ in G. F_γ is differentiable. Its derivative at A in G, denoted $dF_\gamma(A)$, maps the tangent space to G at A, $T_A(G) = dL_A(e)(\mathfrak{g})$ (where $L_A B = AB$ for $A, B \in G$), into the tangent space to V' at $F_\gamma(A) = \gamma \cdot A$, $T_{\gamma \cdot A}(V')$. If $x \in \mathfrak{g} = T_e(G)$, then

$$dF_\gamma(e)x = \frac{d}{dt} F_\gamma(\exp tx)\,|_{t=0} \tag{3.1}$$
$$= \frac{d}{dt}(\gamma \cdot \exp tx)\,|_{t=0} = \gamma \cdot x.$$

Let $\mathfrak{g}_\gamma = \operatorname{Ker} dF_\gamma(e) = \{x \in g : \gamma \cdot x = 0\}$.

PROPOSITION 3.1.

(i) *O_γ is closed in V'.*

(ii) *G_γ is a Lie subgroup of G, and $T_e(G_\gamma) = \operatorname{Ker} dF_\gamma(e) = \mathfrak{g}_\gamma$.*

(iii) *O_γ is a submanifold (C^∞) of V'; $h_\gamma : G/G_\gamma \longrightarrow O_\gamma$: $G_\gamma x \longrightarrow \gamma \cdot x$ is a diffeomorphism of the quotient manifold (analytic) G/G_γ onto the manifold O_γ; and the tangent space at γ to O_γ, $T_\gamma(O_\gamma) = \operatorname{im} dF_\gamma(e)$.*

PROOF. (i) is in [2, p. 7]. (ii) and (iii) are in [4, Chapitre 3, Proposition 14, p. 108]. ((i) is necessary for (iii) since one needs O_γ to be a Baire space and G to be separable to show that $h_\gamma : G/G_\gamma \longrightarrow O_\gamma$ is open.)

Proposition 3.1(ii) implies that $G_\gamma = \exp \mathfrak{g}_\gamma$, since $\exp : \mathfrak{g} \longrightarrow G$ is a diffeomorphism, and $\exp x \in G_\gamma$ implies $x \in \mathfrak{g}_\gamma$ in this case.

If $A \in G$, let $\pi(A) : V' \longrightarrow V'$ be $\pi(A)(\gamma) = \gamma \cdot A$. Then for $\gamma \in V'$, $F_{\gamma \cdot A} = F_\gamma \circ L_A = \pi(A) \circ F_\gamma \circ C_A$, where $C_A : G \longrightarrow G : x \longrightarrow AxA^{-1}$. By the chain rule,

$$\begin{aligned} dF_{\gamma \cdot A}(e) &= dF_\gamma(A)\, dL_A(e) \\ &= d\pi(A)(\gamma)\, dF_\gamma(e)\, dC_A(e) = d\pi(A)(\gamma)\, dF_\gamma(e) \operatorname{Ad}(A). \end{aligned} \tag{3.2}$$

$dL_A(e)$, $d\pi(A)(\gamma)$, and $dC_A(e) = \operatorname{Ad}(A)$ are isomorphisms. Therefore,

$$\operatorname{rank}_{\mathbf{R}}(dF_{\gamma \cdot A}(e)) = \operatorname{rank}_{\mathbf{R}}(dF_\gamma(A)) = \operatorname{rank}_{\mathbf{R}}(dF_\gamma(e)). \tag{3.3}$$

Thus, from Proposition 3.1(iii),

$$\dim(T_{\gamma \cdot A}(O_\gamma)) = \dim(\operatorname{im} dF_{\gamma \cdot A}(e)) = \dim(\operatorname{im} dF_\gamma(e)) = \dim(T_\gamma(O_\gamma)). \tag{3.4}$$

Also, by (3.2) $x \in \mathfrak{g}$ is in $\operatorname{Ker} dF_{\gamma \cdot A}(e)$ if and only if $dL_A(e)x$ is in $\operatorname{Ker} dF_\gamma(A)$ if and only if $\operatorname{Ad}(A)x$ is in $\operatorname{Ker} dF_\gamma(e)$ if and only if x is in $\operatorname{Ad}(A)^{-1}(\operatorname{Ker} dF_\gamma(e))$. Hence

$$\mathfrak{g}_{\gamma \cdot A} = \operatorname{Ker} dF_{\gamma \cdot A}(e) = \operatorname{Ad}(A^{-1})(\operatorname{Ker} dF_\gamma(e)) = \operatorname{Ad}(A^{-1})(\mathfrak{g}_\gamma). \tag{3.5}$$

To develop computational machinery, we take bases in V and $\mathfrak{g}$. Let $v_1 < \cdots < v_K$ be a basis for V in Jordan-Hölder order relative to $\mathfrak{g}$, i.e., $\mathfrak{g} v_K = 0$, $\mathfrak{g} v_i \subset \operatorname{span}\{v_{i+1}, \ldots, v_K\}$ for $1 \leqslant i \leqslant K-1$. Let $\{v^1, \ldots, v^K\}$ be the dual basis of V', and let $m_{V'}$ denote the measure on V' defined in terms of this basis, i.e.,

$$\int_{V'} f(\gamma)\, dm_{V'}(\gamma) = \int_{\mathbf{R}^K} f\left(\sum_{i=1}^K \gamma_i v^i\right) dm_{\mathbf{R}^K}(\gamma_1, \ldots, \gamma_K).$$

For A in G, put $\langle A(m_{V'}), f\rangle = \int_{V'} f(\gamma \cdot A)\, dm_{V'}(\gamma)$. Then $A(m_{V'}) = m_{V'}$, since the determinant of $(\gamma \longrightarrow \gamma \cdot A)$ is one for all A in G. Let m_G denote the Haar measure on G defined in terms of the Jordan-Hölder basis $e_1 < \cdots < e_s$ of $\mathfrak{g}$ as in §2.

Consider the matrix

$$M = (e_i v_j)_{1 \leqslant i \leqslant s, 1 \leqslant j \leqslant K}. \tag{3.6}$$

The entries $e_i v_j$ are vectors in V, so are elements in the field of fractions of the symmetric algebra of V, denoted F_V. If R is in F_V, then $R = P/Q$, for P, Q in the symmetric algebra, S_V, of V. S_V is isomorphic to the ring of polynomial functions on V' by the map $P \longrightarrow (\gamma \longrightarrow P(\gamma))$, where

$$P(\gamma) = P(\gamma_1, \ldots, \gamma_K) = \sum a_{i_1 \cdots i_K} \gamma_1^{i_1} \cdots \gamma_K^{i_K}, \quad \text{for } \gamma = \sum_{i=1}^{K} \gamma_i v^i V',$$

$P = \Sigma a_{i_1 \cdots i_K} v_1^{i_1} \cdots v_K^{i_K} \in S_V$. If $R = P/Q \in F_V$, and $\gamma \in V'$, then define $R(\gamma) = P(\gamma)/Q(\gamma)$ whenever $Q(\gamma) \neq 0$. The map $R \longrightarrow (\gamma \longrightarrow R(\gamma))$ is an isomorphism of F_V with the field of rational functions on V'. (As an element in F_V, a vector $v \in V$ corresponds to the function $\gamma \longrightarrow v(\gamma) = \langle \gamma, v \rangle$ on V'.)

M is called the *structure matrix* for the action of $\mathfrak{g}$ on V. Since the elements in M are rational functions on V', properties of M–its rank, its independent rows and columns, its minors–are useful in analyzing the contragredient action of $\mathfrak{g}$, hence of G, on V'. In fact, all the major formulas in this paper come via M. M works because $\mathfrak{g}$ is nilpotent, and $\{e_1 < \cdots < e_s\}$, $\{v_1 < \cdots < v_K\}$ are Jordan-Hölder bases.

For $\gamma \in V'$, let $M(\gamma)$ denote the matrix $(\langle \gamma, e_i v_j \rangle)_{1 \leqslant i \leqslant s, 1 \leqslant j \leqslant K}$. Since $\langle \gamma, e_i v_j \rangle = \langle \gamma e_i, v_j \rangle = \langle dF_\gamma(e) e_i, v_j \rangle$ by (3.1), $M(\gamma)$ is the matrix for $dF_\gamma(e) : \mathfrak{g} \longrightarrow V'$ in terms of the basis $\{e_1, \ldots, e_s\}$ of $\mathfrak{g}$, and $\{v^1, \ldots, v^K\}$ of V'. Thus by (3.4)

$$\begin{aligned} \operatorname{rank}_{\mathbf{R}}(M(\gamma)) &= \operatorname{rank}_{\mathbf{R}}(dF_\gamma(e)) = \dim T_\gamma(O_\gamma) \\ &= (\text{the dimension of the orbit of } \gamma \text{ under } G). \end{aligned} \tag{3.7}$$

Suppose $\operatorname{rank}_{F_V} M = r > 0$. Let $d = K - r$, $q = s - r$. For $1 \leqslant i \leqslant s$, $1 \leqslant j \leqslant K$, let $R_i = (e_i v_1, \ldots, e_i v_K)$ denote the ith row of M, and

$$C_j = \begin{pmatrix} e_1 v_j \\ \vdots \\ e_s v_j \end{pmatrix}$$

denote the jth column of M. Choose indices $1 \leqslant i_1 < \cdots < i_r \leqslant s$ (resp. $1 \leqslant l_1 < \cdots < l_r \leqslant K$) as follows: i_r (resp. l_r) in the largest integer ($1 \leqslant i_r \leqslant s$) such that $R_{i_r} \neq 0$ (resp. ($l_r \neq 0$)). Having chosen i_k (resp. l_k), i_{k-1} (resp. l_{k-1}) is the largest integer ($1 \leqslant i_{k-1} < i_k$) such that $R_{i_{k-1}}$ (resp. $C_{l_{k-1}}$) is linearly independent in $(F_V)^K$ (resp. $(F_V)^s$) from $R_{i_k}, \ldots, R_{i_r}$ (resp. $C_{l_k}, \ldots, C_{l_r}$). Next, choose $1 \leqslant m_1 < \cdots < m_q \leqslant s$ (resp. $1 \leqslant j_1 < \cdots < j_d = K$) such that $\{i_1, \ldots, i_r\}$, $\{m_1, \ldots, m_q\}$ (resp. $\{l_1, \ldots, l_r\}$, $\{j_1, \ldots, j_d\}$) is a partition of $\{1, \ldots, s\}$ (resp. $\{1, \ldots, K\}$).

In a sense (to be made precise), the dependent columns $\{C_{j_1}, \ldots, C_{j_d}\}$ of M provide a coordinate system for almost all of V'/G; and the independent rows $\{R_{i_1}, \ldots, R_{i_r}\}$ of M provide coordinates for almost all the orbits of V' under G; while the dependent rows $\{R_{m_1}, \ldots, R_{m_q}\}$ parametrize almost all the stability subalgebras $\mathfrak{g}_\gamma \subset \mathfrak{g}$.

Let $M^{(r)}$ denote the $r \times r$ matrix $(e_{i_a} v_{l_b})_{1 \leqslant a,b \leqslant r}$. Since $\operatorname{rank}_{F_V} M = r$, and $R_{i_1}, \ldots, R_{i_r}$ (resp. $C_{l_1}, \ldots, C_{l_r}$) are linearly independent rows (resp. columns) of M,

$$\operatorname{rank}_{F_V} M^{(r)} = \operatorname{rank}_{F_V} [(e_{i_a} v_{l_b})_{1 \leqslant a, b \leqslant r}] = r.$$

Therefore $\det M^{(r)} = \Sigma_{\sigma \in S_r} (\operatorname{sign} \sigma)(e_{i_1} v_{l_{\sigma(1)}}) \cdots (e_{i_r} v_{l_{\sigma(r)}})$ is a nonzero element in S_V, so there is a $\gamma \in V'$ such that the polynomial

$$(\det M^{(r)})(\gamma) = \sum_{\sigma \in S_r} (\operatorname{sign} \sigma) \langle \gamma, e_{i_1} v_{l_{\sigma(1)}} \rangle \cdots \langle \gamma, e_{i_r} v_{l_{\sigma(r)}} \rangle$$
$$= \det(M^{(r)}(\gamma)) \neq 0.$$

Let $E = \{\gamma \in V' : \det M^{(r)}(\gamma) \neq 0\}$. E is a nonempty Zariski open set in V'.

LEMMA 3.1. *E is a G-invariant set containing only maximal dimension orbits.*

PROOF. $\operatorname{rank}_{F_V} M = r$ implies that every $(r + 1) \times (r + 1)$ minor of M is zero. Hence, if $\gamma \in V'$, then every $(r + 1) \times (r + 1)$ minor of $M(\gamma)$ is zero. Thus, $\operatorname{rank}_{\mathbf{R}}(M(\gamma)) \leqslant r$. If $\gamma \in E$, then $\operatorname{rank}_{\mathbf{R}} M(\gamma) = r$. By (3.7), $\operatorname{rank}_{\mathbf{R}}(M(\gamma))$ is the dimension of the orbit of γ under G. Thus, if $\gamma \in E$, then O_γ has maximum possible dimension.

For $1 \leqslant j \leqslant K$, let $M_j = (e_i v_k)_{1 \leqslant i \leqslant s, j \leqslant k \leqslant K}$; $r_j = \operatorname{rank}_{F_V} M_j$ (then $0 = r_K \leqslant r_{K-1} \leqslant \cdots \leqslant r_1 = r$); $U_j = \{\gamma \in V' : \operatorname{rank}_{\mathbf{R}} M_j(\gamma) = r_j\}$; and $U = \bigcap_{j=1}^K U_j$. Each U_j is a nonempty Zariski open set in V'. (The set B_j of all $r_j \times r_j$ minors of M_j is a family of polynomial functions on V', and $U_j = \{\gamma \in V' : P(\gamma) \neq 0$ for some $P \in B_j\}$.)

To show that U_j is G-invariant, we must show that $\operatorname{rank}_{\mathbf{R}} M_j(\gamma \cdot A) = \operatorname{rank}_{\mathbf{R}} M(\gamma)$ for all $A \in G$. Note that $\{v^1, \ldots, v^K\}$ (the basis of V' dual to the basis $\{v_1, \ldots, v_K\}$ of V) is a Jordan-Hölder basis for V' relative to $\mathfrak{g}$ such that $v^1 \cdot \mathfrak{g} = 0$, and $v^i \cdot \mathfrak{g} \subset \operatorname{span}\{v^1, \ldots, v^{i-1}\}$ for $2 \leqslant i \leqslant K$. (For $x \in \mathfrak{g}$, the (v^a)th component of $v^i \cdot x$ is $(v^i \cdot x)(v_a) = v^i(xv_a)$. Since $xv_a \in \operatorname{span}\{v_{a+1}, \ldots, v_K\}$, $v^i(xv_a)$ is zero if $a > i - 1$.) Let $V_1 = (0)$, $V_j = \operatorname{span}\{v^1, \ldots, v^{j-1}\}$ for $2 \leqslant j \leqslant K + 1$. Each V_j is invariant under G, so G acts on $V'/V_j \simeq \operatorname{span}\{v^j, \ldots, v^K\}$ by $P_j(\gamma) \cdot A = P_j(\gamma \cdot A)$, where $\gamma \in V'$, $A \in G$, and $P_j : V' \longrightarrow V'/V_j$ is the projection. Let $F_{P_j(\gamma)} : G \longrightarrow V'/V_j$ be the map $F_{P_j(\gamma)}(A) = P_j(\gamma) \cdot A$. Then for $j \leqslant k \leqslant K$, $1 \leqslant i \leqslant s$,

$$(dF_{P_j(\gamma)}(e)(e_i))(v_k) = (P_j(\gamma)\cdot e_i)(v_k) = P_j(\gamma)(e_iv_k) = \sum_{t=j}^{K} \gamma_t v^t(e_iv_k)$$
$$= \sum_{t=1}^{K} \gamma_t v^t(e_iv_k) = \gamma(e_iv_k).$$

($\sum_{t=1}^{j-1} \gamma_t v^t(e_iv_k) = 0$ because $(e_iv_k) \in \operatorname{span}\{v_{k+1}, \ldots, v_K\}$ and $j-1 < j \leqslant k$.) Thus the matrix for $dF_{P_j(\gamma)}(e) : \mathfrak{g} \longrightarrow V'/V_j$ in terms of the basis $\{e_1, \ldots, e_s\}$ of $\mathfrak{g}$ and $\{v^j, \ldots, v^K\}$ of V'/V_j is $M_j(\gamma)$. Hence, if $A \in G$, we have, by (3.3),

$$\operatorname{rank}_{\mathbf{R}}(M_j(\gamma)) = \operatorname{rank}_{\mathbf{R}}(dF_{P_j(\gamma)}(e)) = \operatorname{rank}_{\mathbf{R}}(dF_{P_j(\gamma)\cdot A}(e))$$
$$= \operatorname{rank}_{\mathbf{R}}(dF_{P_j(\gamma\cdot A)}(e)) = \operatorname{rank}_{\mathbf{R}}(M_j(\gamma\cdot A)).$$

Since each U_j is G-invariant, $U = \bigcap_{j=1}^{K} U_j$ is G-invariant.

For $1 \leqslant i \leqslant s$, let $N_i = (e_tv_j)_{i\leqslant t\leqslant s, 1\leqslant j\leqslant K}$; $d_i = \operatorname{rank}_{F_V} N_i$ (then $0 \leqslant d_s \leqslant d_{s-1} \leqslant \cdots \leqslant d_1 = r$); $D_i = \{\gamma \in V : \operatorname{rank}_{\mathbf{R}} N_i(\gamma) = d_i\}$; and $D = \bigcap_{i=1}^{s} D_i$. Each D_i is a nonempty Zariski open set in V'.

To show D_i is G-invariant we must show that $\operatorname{rank}_{\mathbf{R}} N_i(\gamma\cdot A) = \operatorname{rank}_{\mathbf{R}} N_i(\gamma)$ for all $A \in G$. Recall that $\{e_1, \ldots, e_s\}$ is a Jordan-Hölder basis of $\mathfrak{g}$ such that $[e_s, \mathfrak{g}] = 0$, and $[e_i, \mathfrak{g}] \subset \operatorname{span}\{e_{i+1}, \ldots, e_s\}$ for $1 \leqslant i \leqslant s-1$. Therefore $\mathfrak{h}_i = \operatorname{span}\{e_i, \ldots, e_s\}$ is an ideal in $\mathfrak{g}$, and $H_i = \exp \mathfrak{h}_i$ is a normal Lie subgroup of G. The restriction of the action of G (resp. $\mathfrak{g}$) to H_i (resp. $\mathfrak{h}_i$) defines a smooth action of H_i (resp. $\mathfrak{h}_i$) on V'. Let $F_\gamma^i = F_\gamma|_{H_i} : H_i \longrightarrow V'$. Then $dF_\gamma^i(e)$: $\mathfrak{h}_i \longrightarrow V'$, and by (3.1), for $i \leqslant t \leqslant s$, $1 \leqslant j \leqslant K$, $(dF_\gamma^i(e)(e_t))(v_j) = (\gamma\cdot e_t)(v_j) = \gamma(e_tv_j)$ so that the matrix for $dF_\gamma^i(e)$ in terms of the basis $\{e_i, \ldots, e_s\}$ of $\mathfrak{h}_i$ and $\{v^1, \ldots, v^K\}$ of V' is $N_i(\gamma)$. Since H_i is normal in G, if $A \in G$, then $F_{\gamma\cdot A}^i = \pi(A)F_\gamma^i C_A$; so that (as in (3.3))

$$\operatorname{rank}(N_i(\gamma\cdot A)) = \operatorname{rank}(dF_{\gamma\cdot A}^i(e)) = \operatorname{rank}(dF_\gamma^i(e)) = \operatorname{rank}(N_i(\gamma)).$$

Since each D_i is G-invariant, $D = \bigcap_{i=1}^{s} D_i$ is G-invariant. Hence $U \cap D$ is G-invariant.

To show that $U \cap D = E$, let $\gamma \in V'$. $\gamma \in E$ if and only if $\det M^{(r)}(\gamma) \neq 0$ if and only if $R_{i_1}(\gamma), \ldots, R_{i_r}(\gamma)$ are independent rows of $M(\gamma)$, and $C_{l_1}(\gamma), \ldots, C_{l_r}(\gamma)$ are independent columns of $M(\gamma)$ if and only if $\gamma \in U \cap D$. Indeed, $\gamma \in D = \bigcap_{i=1}^{s} D_i$ if and only if $\operatorname{rank}_{\mathbf{R}}(D_i(\gamma)) = d_i$, the maximal possible rank for each $i = s, s-1, \ldots, 1$. From the definition of the indices $\{i_1, \ldots, i_r\}$, i_r is the largest integer such that $d_{i_r} = 1$, $i_{(k-1)}$ is the largest integer such that $d_{i_{(k-1)}} = (d_{i_k}) + 1$ for $2 \leqslant k \leqslant r$. Thus $\gamma \in D$ if and only if $R_{i_r}(\gamma), \ldots, R_{i_1}(\gamma)$ are linearly independent rows of $M(\gamma)$. Similarly, l_r is the largest integer such that $r_{l_r} = 1$, $l_{(k-1)}$ is the largest integer such that $r_{l_{(k-1)}} = (r_{l_k}) + 1$ for $2 \leqslant k \leqslant r$. $\gamma \in U = \bigcap_{j=1}^{K} U_j$ if and only if $\operatorname{rank}_{\mathbf{R}}(M_j(\gamma)) = r_j$, the maximum possible rank for each j. Hence $\gamma \in U \Longleftrightarrow C_{l_r}(\gamma), \ldots, C_{l_r}(\gamma)$ are independent columns of $M(\gamma)$.

In general, the set $\{\gamma \in V' : \dim O_\gamma \text{ is maximum}\} = \{\gamma \in V' : \operatorname{rank}_{\mathbf{R}}(M(\gamma)) = r\} = U_1 = D_1$ properly contains $U \cap D = E$.

The following theorem coordinatizes O_γ for all γ in E, and gives a G-invariant measure on O_γ in terms of these coordinates. The proof shows how to use M to compute all the stability subalgebras $\mathfrak{g}_\gamma$ for $\gamma \in E$.

THEOREM 3.1. (a) *If $\gamma \in E$, then the mapping $t = (t^1, \ldots, t^r) \longrightarrow G_\gamma \cdot \exp t^r e_{i_r} \cdots \exp t^1 e_{i_1}$ is a homeomorphism of $\mathbf{R}^r$ onto G/G_γ. Let ν_γ be the measure on G/G_γ defined by*

$$\begin{aligned}\langle \nu_\gamma, f\rangle &= \int_{G/G_\gamma} f(G_\gamma x)\, d\nu_\gamma(G_\gamma x)\\ &= \int_{\mathbf{R}^r} f(G_\gamma \exp t^1 e_{i_r} \cdots \exp t^1 e_{i_r})\, dm_{\mathbf{R}^r}(t^1, \ldots, t^r).\end{aligned}$$

There is a basis $\{u_1(\gamma), \ldots, u_q(\gamma)\}$ of $\mathfrak{g}_\gamma$ such that if Haar measure m_{G_γ} on G_γ is taken as

$$\langle m_{G_\gamma}, f\rangle = \int_{\mathbf{R}^q} f\left(\exp \sum_{b=1}^{q} z^b u_b(\gamma)\right) dm_{\mathbf{R}^q}(z^1, \ldots, z^q),$$

then, for $f \in C_0(G)$,

$$\int_G f(x)\, dm_G(x) = \int_{G/G_\gamma} \int_{G_\gamma} f(zx)\, dm_{G_\gamma}(z)\, d\nu_\gamma(G_\gamma x). \tag{3.8}$$

(b) *If $\gamma \in E$, then the mapping $t = (t^1, \ldots, t^r) \longrightarrow \gamma \cdot \exp t^r e_{i_r} \cdots \exp t^1 e_{i_1}$ is a homeomorphism of $\mathbf{R}^r$ onto O_γ. The measure on O_γ given by $\langle \nu_\gamma, f\rangle = \int_{\mathbf{R}^r} f(\gamma \cdot \exp t^r e_{i_r} \cdots \exp t^1 e_{i_1})\, dm_{\mathbf{R}^r}(t)$ is G-invariant.*

PROOF. (b) follows from (a) by Proposition 3.1. The map $h_\gamma : G/G_\gamma \longrightarrow O_\gamma : G_\gamma x \longrightarrow \gamma \cdot x$ carries coordinates and measures on G/G_γ to O_γ.

The proof of (a) consists in showing that if $\gamma \in E$, then $\mathfrak{g}_\gamma$ has a basis $\{u_1(\gamma), \ldots, u_q(\gamma)\}$ satisfying the requirement of Theorem 2.1 with respect to the indices $i_1 < \cdots < i_r$ of the independent rows of M and $m_1 < \cdots < m_q$ of the dependent rows of M. In other words, there are scalars $\lambda^{i_s}_{m_b}(\gamma)$, $1 \leqslant b \leqslant q$, $1 \leqslant s \leqslant r$, with $\lambda^{i_s}_{m_b}(\gamma) = 0$ if $i_s < m_b$, such that the vectors $u_b(\gamma) = e_{m_b} - \Sigma_{s=1}^{r} \lambda^{i_s}_{m_b}(\gamma) e_{i_s}$, $1 \leqslant b \leqslant q$, form a basis of $\mathfrak{g}_\gamma$.

By definition, $1 \leqslant m_1 < \cdots < m_q \leqslant s$ are indices such that $\{1, \ldots, s\} = \{m_1, \ldots, m_q\} \cup \{i_1, \ldots, i_r\}$. By definition of $\{i_1, \ldots, i_r\}$, for $1 \leqslant b \leqslant q$, $R_{m_b} = \Sigma_{\{s: i_s > m_b\}} \lambda^{i_s}_{m_b} R_{i_s}$, with

$$(3.9)\qquad \lambda^{i_s}_{m_b} = \frac{\begin{vmatrix} e_{i_1}v_{l_1} & \cdots & e_{i_1}v_{l_r} \\ \vdots & & \\ e_{i_{(s-1)}}v_{l_1} & \cdots & e_{i_{(s-1)}}v_{l_r} \\ e_{m_b}v_{l_1} & \cdots & e_{m_b}v_{l_r} \\ e_{i_{(s+1)}}v_{l_1} & \cdots & e_{i_{(s+1)}}v_{l_r} \\ \vdots & & \\ e_{i_r}v_{l_1} & \cdots & e_{i_r}v_{l_r} \end{vmatrix}}{\det M^{(r)}}, \qquad 1 \leqslant s \leqslant r.$$

By definition of $\{i_1, \ldots, i_r\}$, $\lambda^{i_s}_{m_b} = 0$ if $i_s < m_b$. Hence $e_{m_b}v_a = \Sigma^r_{s=1}\lambda^{i_s}_{m_b}e_{i_s}v_a$, $1 \leqslant a \leqslant K$. If $\gamma \in E$, let

$$(3.10)\qquad u_b = u_b(\gamma) = e_{m_b} - \sum_{s=1}^{r} \lambda^{i_s}_{m_b}(\gamma)e_{i_s}, \qquad 1 \leqslant b \leqslant q.$$

Then, for $1 \leqslant a \leqslant K$, $\gamma u_b v_a = \gamma e_{m_b}v_a - \Sigma^r_{s=1}\lambda^{i_s}_{m_b}(\gamma)\gamma e_{i_s}v_a = 0$. Hence $u_b \in \mathfrak{g}_\gamma$ for $1 \leqslant b \leqslant q$. Since $\dim \mathfrak{g}_\gamma = \dim \mathfrak{g} - \dim O_\gamma = s - r = q$, and $u_1, \ldots, u_q$ are linearly independent, $\{u_1, \ldots, u_q\}$ is a basis of $\mathfrak{g}_\gamma$.

Since E is a nonempty Zariski open set in V', E is $m_{V'}$-conull. Thus, to obtain a disintegration formula for $m_{V'}$, we may restrict consideration to the G-invariant space E and the orbit space E/G. V' has dimension K, and $m_{V'}$ is essentially $m_{\mathbf{R}^K}$, Lebesgue measure on $\mathbf{R}^K$. The orbits in E are r-dimensional manifolds, and each carries a G-invariant measure ν_γ (Theorem 3.1) which is essentially $m_{\mathbf{R}^r}$, Lebesgue measure on $\mathbf{R}^r$. One would expect the measure on the orbit space V'/G in the disintegration of $m_{V'}$ by G to be essentially $m_{\mathbf{R}^d}$, where $d = K - r$ is the codimension of a maximal dimension orbit. To get the precise form of the measure on the orbit space, we need coordinates on V'/G. The advantage of E is that we can use M to compute coordinates on E/G and the measure in terms of these coordinates. The following theorem gives a coordinate system for the orbit space E/G.

THEOREM 3.2. *Let $p: V' \longrightarrow V'/G$ be the projection. Let $s: \mathbf{R}^d \longrightarrow V'$ be the map $s(y) = s(y_1, \ldots, y_d) = \Sigma^d_{k=1} y_k v^{j_k}$, where $\{j_1, \ldots, j_d\}$ are the indices previously defined for the dependent columns of M. Let $W = \{y \in \mathbf{R}^d : s(y) \in E\}$.*

Then W is a nonempty Zariski open set in $\mathbf{R}^d$, and the map $(y_1, \ldots, y_d) \longrightarrow p(\Sigma^d_{k=1} y_k v^{j_k}) : W \longrightarrow E/G$ is a homeomorphism.

PROOF. By definition of E, $W = \{y \in \mathbf{R}^d : \det M^{(r)}(s(y)) \neq 0\}$ is a Zariski open set in $\mathbf{R}^d$. To show that W is not empty, and that $p \circ s|_W$ is a bijection of

W onto E/G, we need the following lemma.

LEMMA 3.2. *If $\gamma \in E$, then the map $\pi_r|_{O_\gamma}: O_\gamma \longrightarrow \mathbf{R}^r$ given by $\pi_r(\beta) = (\beta(v_{l_1}), \ldots, \beta(v_{l_r}))$ is bijective.* (*Here, $\{l_1, \ldots, l_r\}$ are the indices previously defined for the independent columns of M.*)

PROOF. The proof of Lemma 3.2 follows that of Pukańszky's orbit parametrization theorem [19, Theorem, pp. 50–54]. To show $\pi_r|_{O_\gamma}$ is bijective, we need suitable coordinates on G/G_γ. Recall from the proof of Lemma 3.1 that $M_j(\gamma)$ is the matrix for the mapping $dF_{P_j(\gamma)}(e): \mathfrak{g} \longrightarrow V'/V_j$ in terms of the basis $\{e_1, \ldots, e_s\}$ of $\mathfrak{g}$ and $\{v^j, \ldots, v^k\}$ of V'/V_j. Ker $M_j(\gamma)$ is the stability subalgebra

$$\mathfrak{g}_{P_j(\gamma)} = \left\{x = \sum_{i=1}^{s} x^i e_i \in \mathfrak{g} : P_j(\gamma) \cdot x = 0\right\}$$

$$= \{x : \gamma \cdot xv_j = \gamma \cdot xv_{j+1} = \cdots = \gamma \cdot xv_K = 0\}.$$

For $l_k < j \leq l_{(k+1)}$, $\operatorname{rank} M_j(\gamma) = \operatorname{rank} M_{l_{(k+1)}}(\gamma) = (\operatorname{rank} M_{l_k}(\gamma)) - 1$, $1 \leq k \leq r$ ($M_j = 0$ if $j > l_r$). Thus,

$$\begin{aligned}\dim \operatorname{Ker} M_j(\gamma) &= s - \operatorname{rank} M_j(\gamma) = s - \operatorname{rank} M_{l_{(k+1)}}(\gamma) \\ &= s - (\operatorname{rank} M_{l_k}(\gamma)) + 1 = (\dim \operatorname{Ker} M_{l_k}(\gamma)) + 1.\end{aligned}$$

Since $\operatorname{Ker} M_{l_k}(\gamma) \subset \operatorname{Ker} M_j(\gamma)$ whenever $j > l_k$, if $w_k \in \operatorname{Ker} M_{(l_k)+1}(\gamma)$, $w_k \notin \operatorname{Ker} M_{l_k}(\gamma)$, then $(\operatorname{Ker} M_{l_k}(\gamma)) \oplus (w_k) = \operatorname{Ker} M_j(\gamma)$ for $(l_k) + 1 \leq j \leq l_{(k+1)}$. For $1 \leq k \leq r$, choose $w_k = w_k(\gamma) \in \operatorname{Ker} M_{(l_k)+1}(\gamma), \notin \operatorname{Ker} M_{l_k}(\gamma)$, such that $(\gamma \cdot w_k)(v_{l_k}) = 1$. Then setting $\mathfrak{n}_0 = \operatorname{Ker} M_{l_1}(\gamma), \mathfrak{n}_k = \mathfrak{n}_{k-1} \oplus (w_k)$ for $1 \leq k \leq r$, we have an ascending sequence of subalgebras $\mathfrak{g}_\gamma = \mathfrak{n}_0 \subset \mathfrak{n}_1 \subset \cdots \subset \mathfrak{n}_r = \mathfrak{g}$ such that $\mathfrak{n}_k/\mathfrak{n}_{k-1} \simeq (w_k)$. Let $Q: \mathbf{R}^r \longrightarrow G$ be the map $Q(t) = Q(t^1, \ldots, t^r) = \exp t^1 w_1 \cdots \exp t^r w_r$. By Lemma 2.2, the map $t \longrightarrow G_\gamma \cdot Q(t): \mathbf{R}^r \longrightarrow G/G_\gamma$ is a homeomorphism. Thus, by Proposition 3.1(iii), the map $t \longrightarrow \gamma \cdot Q(t): \mathbf{R}^r \longrightarrow O_\gamma$ is a homeomorphism. The components of $\beta = \gamma \cdot Q(t)$ with respect to the basis $\{v^1, \ldots, v^K\}$ of V', $\beta_a = \gamma \cdot Q(t)(v_a)$, $1 \leq a \leq K$, have the following form:

$$\begin{aligned} &\beta_{l_r} = \gamma_{l_r} + t^r, \\ (3.11) \qquad &\beta_{l_k} = \gamma_{l_k} + t^k + \psi_k(t^{k+1}, \ldots, t^r; \gamma), \qquad 1 \leq k \leq r-1; \\ &\beta_j = \gamma_j + F_j(t^k, \ldots, t^r; \gamma), \end{aligned}$$

k the largest integer such that $j > l_{k-1}$ (setting $l_0 = 0$).

Hence $t^r, \ldots, t^1$ may be recursively determined from

$$\beta_{l_r}, \ldots, \beta_{l_1} (t^r = \beta_{l_r} - \gamma_{l_r}; t^{r-1} = \beta_{l_{r-1}} - \gamma_{l_{r-1}} - \psi_{r-1}(t^r; \gamma); \cdots ;$$
$$t^1 = \beta_{l_1} - \gamma_{l_1} - \psi_k(t^2, \ldots, t^r; \gamma)).$$

Thus, given $z = (z_1, \ldots, z_r) \in \mathbf{R}^r$, there is one and only one $t = (t^1, \ldots, t^r)$ such that $\gamma \cdot Q(t)(v_{l_k}) = z_k$, $1 \leq k \leq r$. This says there is one and only one point $\beta \in O_\gamma$ such that $\pi_r(\beta) = z$. Hence π_r is a bijection of O_γ onto $\mathbf{R}^r$.

To show that W is not empty, choose $\gamma \in E$. Then (Lemma 3.1) $O_\gamma \subset E$. By Lemma 3.2, there is a point $\beta \in O_\gamma$ such that $\pi_r(\beta) = 0$. Since $\{l_1, \ldots, l_r\}$, $\{j_1, \ldots, j_d\}$ is a partition of $\{1, \ldots, K\}$, $\beta = \Sigma_{k=1}^d \beta_{j_k} v^{j_k} = s(\beta_{j_1}, \ldots, \beta_{j_d}) \in E$, so that $(\beta_{j_1}, \ldots, \beta_{j_d}) \in W$. Since $\gamma \in E$ was arbitrary, this also shows that $ps(W) = E/G$ ($\beta \in s(W)$ and $p\beta = p\gamma$).

If $y, z \in W$, and if $ps(y) = ps(z)$, then $O_{s(y)} = O_{s(z)} \subset E$. By Lemma 3.2, $\pi_r|_{O_{s(y)}}$ is injective. $\pi_r(s(y)) = 0 = \pi_r(s(z)) \Longrightarrow s(y) = s(z) \Longrightarrow y = z$. Thus $p \cdot s|_W : W \longrightarrow E/G$ is bijective.

$s(W) = (\operatorname{span}\{v^{j_1}, \ldots, v^{j_d}\}) \cap E$ intersects each orbit in E in exactly one point, so that $\psi : E/G \longrightarrow V'$ defined by $\psi(p\gamma) = p^{-1}p\gamma \cap s(W)$ is a cross-section for E/G in V'.

$p \circ s|_W : W \longrightarrow E/G$ is continuous since both p and s are continuous. To show that $p \circ s|_W$ is open, we introduce the following map, which is also used in the proof of the disintegration formula. For $t = (t^1, \ldots, t^r) \in \mathbf{R}^r$, let $g(t) = \exp t^r e_{i_r} \cdots \exp t^1 e_{i_1} \in G$, where $i_1, \ldots, i_r$ are the indices previously defined for the independent rows of M. Let $H: \mathbf{R}^d \times \mathbf{R}^r \longrightarrow V'$ be the map $H(y, t) = s(y) \cdot g(t)$. $H(y, t)$ is linear in $(y_1, \ldots, y_d)$ and a polynomial in $(t^1, \ldots, t^r)$, so H is an analytic mapping of $\mathbf{R}^d \times \mathbf{R}^r$ into V'.

For $(y, t) \in \mathbf{R}^d \times \mathbf{R}^r$, let $J(y, t)$ be the absolute value of the determinant of the $K \times K$ matrix

$$\left[\begin{array}{ccc|ccc}
\partial H_{j_1}/\partial y_1 & \cdots & \partial H_{j_1}/\partial y_d & \partial H_{j_1}/\partial t^1 & \cdots & \partial H_{j_1}/\partial t^r \\
\vdots & & \vdots & \vdots & & \vdots \\
\partial H_{j_d}/\partial y_1 & \cdots & \partial H_{j_d}/\partial y_d & \partial H_{j_d}/\partial t^1 & \cdots & \partial H_{j_d}/\partial t^r \\
\hline
\partial H_{l_1}/\partial y_1 & \cdots & \partial H_{l_1}/\partial y_d & \partial H_{l_1}/\partial t^1 & \cdots & \partial H_{l_1}/\partial t^r \\
\vdots & & \vdots & \vdots & & \vdots \\
\partial H_{l_r}/\partial y_1 & \cdots & \partial H_{l_r}/\partial y_d & \partial H_{l_r}/\partial t^1 & \cdots & \partial H_{l_r}/\partial t^r
\end{array}\right]$$

evaluated at (y, t), where $H_a(y, t) = H(y, t)(v_a)$, $1 \leq a \leq K$. Then $J(y, t) = |\det dH(y, t)|$, where $dH(y, t) : \mathbf{R}^d \times \mathbf{R}^r \longrightarrow V'$ is the derivative of H at (y, t). Since each H_a is a polynomial in y and t, the partials are polynomials in y and t. Hence $\det dH(y, t)$ is a polynomial in y and t.

By calculation,

$$\begin{aligned}\frac{\partial H_{j_k}}{\partial y_m}(y,0) &= \lim_{h\to 0}\frac{1}{h}[H_{j_k}(y_1,\ldots,y_m+h,\ldots,y_d;0)\\ &\qquad\qquad - H_{j_k}(y_1,\ldots,y_m,\ldots,y_d;0)]\\ &= \lim_{h\to 0}\frac{1}{h}[s(y_1,\ldots,y_m+h,\ldots,y_d)-s(y_1,\ldots,y_m,\ldots,y_d)](v_{j_k})\\ &= v^{j_m}(v_{j_k}) = \delta_k^m, \qquad 1\leqslant k\leqslant d,\ 1\leqslant m\leqslant d.\end{aligned}$$

$H_{l_k}(y,0) = s(y)(v_{l_k}) = 0$, $1\leqslant k\leqslant r$, $\forall y\in\mathbf{R}^d$. Hence,

$$\frac{\partial H_{l_k}}{\partial y_m}(y,0) = 0, \qquad 1\leqslant k\leqslant r,\ 1\leqslant m\leqslant d.$$

$$\begin{aligned}\frac{\partial H_a}{\partial t^k}(y,0) &= \lim_{h\to 0}\frac{1}{h}[H_a(y;0\cdots 0,h,0\cdots 0)-H_a(y;0)]\\ &= \lim_{h\to 0}\frac{1}{h}[s(y)\cdot \exp he_{i_k} - s(y)](v_a)\\ &= \lim_{h\to 0}\frac{1}{h}\left[\left(s(y)+s(y)\cdot he_{i_k}+s(y)\cdot\frac{h^2}{2!}e_{i_k}^2+\cdots\right)-s(y)\right](v_a)\\ &= s(y)\cdot e_{i_k}(v_a), \qquad 1\leqslant a\leqslant K,\ 1\leqslant k\leqslant r.\end{aligned}$$

Therefore,

$$(3.12)\quad J(y,0) = \left|\det\left[\begin{array}{c|c}\begin{matrix}1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & & \ddots & \vdots\\ 0 & \cdots & & 1\end{matrix} & * \\ \hline 0 & \underbrace{\begin{matrix} s(y)e_{i_1}v_{l_1} & \cdots & s(y)e_{i_r}v_{l_1}\\ \vdots & & \vdots\\ s(y)e_{i_1}v_{l_r} & \cdots & s(y)e_{i_r}v_{l_r}\end{matrix}}_{={}^t(M^{(r)}(s(y)))}\end{array}\right]\right|$$

$$= |\det M^{(r)}(s(y))|.$$

If $y\in W$, then $J(y,0)\neq 0$. By the inverse function theorem [20, p. 35] there is an $\mathbf{R}^d\times\mathbf{R}^r$ open neighborhood $A\times B$ of $(y,0)$ and a V'-open neighborhood C of $H(y,0)$ such that $H|_{A\times B}: A\times B\longrightarrow C$ is a diffeomorphism of $A\times B$ onto C.

Now, to show $p\circ s|_W$ is open, let $U\subset W$ be open, and $\gamma\in p^{-1}(ps(U)) = s(U)\cdot G$. Then $\gamma = s(y_0)\cdot g_0$ for some $y_0\in U$, $g_0\in G$. Since y_0 is in $U\subset W$, by the preceding paragraph, there is an $\mathbf{R}^d$-open neighborhood A of y_0 (by taking $A\cap U$, we may assume $A\subset U$), an open neighborhood B of 0 in $\mathbf{R}^r$, and an open neighborhood

C of $H(y_0, 0) = s(y_0)$ in V' such that $H(A \times B) = C$. Since C is a V'-neighborhood of $s(y_0)$ and $g_0 \in G$ is a homeomorphism of V', $C \cdot g_0$ is a V'-neighborhood of $\gamma = s(y_0) \cdot g_0$. If $\beta \in C \cdot g_0$, then $\beta = H(y, t) \cdot g_0 = s(y) \cdot g(t) \cdot g_0$ for some $(y, t) \in A \times B \subset U \times B$. Therefore, $\beta \in s(U) \cdot G$. Thus γ is an interior point of $s(U) \cdot G$. Therefore, $s(U) \cdot G = p^{-1}(ps(U))$ is open in V', so $p \circ s(U)$ is open in E/G.

We have shown that the orbit space E/G is homeomorphic to a Zariski-open set in $\mathbf{R}^d$ (Theorem 3.2) and that each orbit in E is homeomorphic to $\mathbf{R}^r$ (Theorem 3.1). The following theorem uses the coordinate system $y \longrightarrow ps(y)$: $W \longrightarrow E/G$ just established for E/G and the coordinate system $y, y \longrightarrow s(y) \cdot g(t) = s(y) \cdot \exp t^r e_{i_r} \cdots \exp t^1 e_{i_1}$ for the orbits in E (Theorem 3.1) to decompose $m_{V'}$ relative to the action of G in V'.

Let x_T denote the characteristic function of the set T.

THEOREM 3.3. *The formula*

$$
\begin{aligned}
\int_{V'} f(\gamma)\, dm_{V'}(\gamma) &= \int_E f(\gamma)\, dm_{V'}(\gamma) \\
&= \int_W \int_{\mathbf{R}^r} f(s(y) \cdot g(t))\, dm_{\mathbf{R}^r}(t)\, |\det M^{(r)}(s(y))|\, dm_{\mathbf{R}^d}(y)
\end{aligned} \tag{3.13}
$$

is a disintegration of $m_{V'}$ by G, that is, $m_{V'}(p^{-1}(V/G - E/G)) = m_{V'}(V' - E) = 0$. The image of the measure $|\det M^{(r)}(s)|x_W m_{\mathbf{R}^d}$ under the homeomorphism $p \circ s$: $W \longrightarrow E/G$ is a measure on E/G which is a pseudo-image of $x_E m_{V'}$ by $p|_E$, the projection of E onto E/G. If, for $y \in W$, $\nu_{s(y)}$ is the measure on E given by

$$\langle \nu_{s(y)}, f \rangle = \int_{\mathbf{R}^r} f(s(y) \cdot g(t))\, dm_{\mathbf{R}^r}(t),$$

then $y \longrightarrow \nu_{s(y)}: W \longrightarrow M_+(E)$ (positive measures on E) has the following properties:

(i) $\nu_{s(y)} \neq 0$ *for* $y \in W$;

(ii) $\nu_{s(y)}$ *is concentrated in* $O_{s(y)}$ *for all* $y \in W$;

(iii) *if* $f \in L^1(x_E m_{V'})$, *then* $y \longrightarrow \langle \nu_{s(y)}, f \rangle \in L^1(|\det M^{(r)}(s)|x_W m_{\mathbf{R}^d})$,

and

$$\langle x_E m_{V'}, f \rangle = \int_W \langle \nu_{s(y)}, f \rangle\, |\det M^{(r)}(s(y))|\, dm_{\mathbf{R}^d}(y).$$

PROOF. By Lemma 3.1, E is a nonempty, G-invariant Zariski open set in V'. Therefore, $p^{-1}(V'/G - E/G) = V' - E$ is $m_{V'}$-null. That $\nu_{s(y)} \in M_+(E)$ and properties (i) and (ii) follow from Theorem 3.1 and the fact that G orbits are closed in V' (Proposition 3.1(a)). The proof of (iii) and formula (3.13) consists in (1) showing that $p \circ s(x_W m_{\mathbf{R}^d})$ is a pseudo-image of $x_E m_{V'}$ by p; (2) using Bourbaki's theorem [6, Chapitre 6, Théorème 2, p. 64] on the disintegration of a measure relative to a pseudo-image to get a disintegration of $x_E m_{V'}$ relative to $p \circ s(x_W m_{\mathbf{R}^d})$; and (3) showing that the orbit measures provided by Bourbaki's theorem are $|\det M^{(r)}(s(y))|\nu_{s(y)}$.

The following three lemmas show that the measure $p \circ s(x_W m_{\mathbf{R}^d})$ on E/G is a pseudo-image of the measure $x_E m_{V'}$ on E. Equation (3.14) in Lemma 3.3 would be the disintegration formula (3.13) if we knew that $|\det dH(y, t)| = J(y, t) = J(y, 0) m_{\mathbf{R}^d \times \mathbf{R}^r}$ a. a. (y, t). This is proved in Lemma 3.7.

LEMMA 3.3. *If $f: V' \longrightarrow R$ is $m_{V'}$-integrable, then*

$$\int_{V'} f(\gamma)\, dm_{V'}(\gamma) = \int_{\mathbf{R}^d \times \mathbf{R}^r} f(H(y, t))\, |\det dH(y, t)|\, dm_{\mathbf{R}^d \times \mathbf{R}^r}(y, t). \tag{3.14}$$

PROOF. Let $A = \{(y, t) \in \mathbf{R}^d \times \mathbf{R}^r : \det dH(y, t) \neq 0\}$. A is a Zariski open set in $\mathbf{R}^d \times \mathbf{R}^r$. $A \supset W \times \{0\}$, so A is nonempty. Suppose $H(y_1, t_1) = H(y_2, t_2)$ for $(y_1, t_1) \in W \times \mathbf{R}^r$. Then $s(y_2) \in O_{s(y_1)} \subset E$. By Lemma 3.2, $\pi_r(s(y_2)) = 0 = \pi_r(s(y_1)) \Longrightarrow s(y_2) = s(y_1) \Longrightarrow y_2 = y_1$. By Theorem 3.1(b), $s(y_1) \cdot g(t_1) = s(y_1) \cdot g(t_2) \Longrightarrow t_1 = t_2$. Therefore $H|_{A \cap (W \times \mathbf{R}^r)}: A \cap (W \times \mathbf{R}^r) \longrightarrow V'$ is a 1-1, continuously differentiable function such that $\det dH(y, t) \neq 0$ for all $(y, t) \in A \cap (W \times \mathbf{R}^r)$. By the change of variable theorem for integrals on $\mathbf{R}^K$ [20, p. 67], if $f: H(A \cap (W \times \mathbf{R}^r)) \longrightarrow \mathbf{R}$ is integrable, then

$$\begin{aligned} &\int_{H(A \cap (W \times \mathbf{R}^r))} f(\gamma)\, dm_{V'}(\gamma) \\ &\quad = \int_{A \cap (W \times \mathbf{R}^r)} f \circ H(y, t)\, |\det dH(y, t)|\, dm_{\mathbf{R}^d \times \mathbf{R}^r}(y, t). \end{aligned} \tag{3.15}$$

Since $A \cap (W \times \mathbf{R}^r)$ is a nonempty Zariski open set in $\mathbf{R}^d \times \mathbf{R}^r$, it is conull. Hence the integral on the right-hand side of (3.15) is

$$\int_{\mathbf{R}^d \times \mathbf{R}^r} f \circ H(y, t)\, J(y, t)\, dm_{\mathbf{R}^d \times \mathbf{R}^r}(y, t).$$

Let $B = \{(y, t) \in \mathbf{R}^d \times \mathbf{R}^r : \det dH(y, t) = 0\}$. By Sard's theorem [20, p. 72], $H(B)$ is an $m_{V'}$-null set in V'. Since H is 1-1 on $W \times \mathbf{R}^r$, $H(W \times \mathbf{R}^r)$ is the disjoint union of $H(A \cap (W \times \mathbf{R}^r))$ and $H(B \cap (W \times \mathbf{R}^r)) \subset H(B)$. Hence, the integral on the left-hand side of (3.15) is $\int_{H(W \times \mathbf{R}^r)} f(\gamma)\, dm_{V'}(\gamma)$. By Theorem 3.2, $H(W \times \mathbf{R}^r) = E$, which is $m_{V'}$-conull. This proves (3.14).

COROLLARY. *$f: V' \longrightarrow \mathbf{R}$ is $m_{V'}$-measurable $\Longleftrightarrow f \circ H: \mathbf{R}^d \times \mathbf{R}^r \longrightarrow \mathbf{R}$ is $m_{\mathbf{R}^d \times \mathbf{R}^r}$-measurable.*

PROOF. Lemma 3.3 says that

$$m_{V'} = \int_{\mathbf{R}^d \times \mathbf{R}^r} \epsilon_{H(y,t)} J(y, t)\, dm_{\mathbf{R}^d \times \mathbf{R}^r}(y, t)$$

(where $\langle \epsilon_{H(y,t)}, f \rangle = f(H(y, t))$. By [5, Chapitre 5, Proposition 3, p. 39],

$$f : V' \longrightarrow R \text{ is } m_{V'}\text{-measurable} \Longleftrightarrow (f \circ H) \circ J \text{ is } m_{\mathbf{R}^d \times \mathbf{R}^r}\text{-measurable}$$
$$\Longleftrightarrow (f \circ H)|_A \text{ is } m_{\mathbf{R}^d \times \mathbf{R}^r}\text{-measurable}$$

(where $A = \{(y, t) \in \mathbf{R}^d \times \mathbf{R}^r : J(y, t) \neq 0\}$). Since A is conull in $\mathbf{R}^d \times \mathbf{R}^r, f: V' \longrightarrow \mathbf{R}$ is $m_{V'}$-measurable $\Longleftrightarrow f \circ H : \mathbf{R}^d \times \mathbf{R}^r \longrightarrow \mathbf{R}$ is $m_{\mathbf{R}^d \times \mathbf{R}^r}$-measurable.

LEMMA 3.4. *Suppose $f: V'/G \longrightarrow \mathbf{R}$ is nonnegative. Then $f \circ p : V' \longrightarrow \mathbf{R}$ is $m_{V'}$-measurable $\Longleftrightarrow f \circ p \circ s : \mathbf{R}^d \longrightarrow \mathbf{R}$ is $m_{\mathbf{R}^d}$-measurable.*

PROOF. By the above corollary, $f \circ p : V' \longrightarrow R$ is $m_{V'}$-measurable $\Longleftrightarrow$ $f \circ p \circ H : \mathbf{R}^d \times \mathbf{R}^r \longrightarrow \mathbf{R}$ is $m_{\mathbf{R}^d \times \mathbf{R}^r}$-measurable.

Suppose $f \circ p \circ s : \mathbf{R}^d \longrightarrow \mathbf{R}$ is $m_{\mathbf{R}^d}$-measurable. Then $f \circ p \circ H(y, t) = f \circ p(s(y) \cdot g(t)) = f(p(s(y)))$ for all $(y, t) \in \mathbf{R}^d \times \mathbf{R}^r \Rightarrow f \circ p \circ H$ is $m_{\mathbf{R}^d \times \mathbf{R}^r}$-measurable. $(\{(y, t) : f \circ p \circ H(y, t) > a\} = \{y : f \circ p \circ s(y) > a\} \times \mathbf{R}^r.)$

Suppose $f \circ p : V' \longrightarrow \mathbf{R}$ is $m_{V'}$-measurable. Let $\beta \simeq m_{\mathbf{R}^r}$ be a finite measure on $\mathbf{R}^r$. By Tonnelli's theorem, $y \longrightarrow \int_{\mathbf{R}^r} f \circ p \circ H(y, t)\, d\beta(t) : \mathbf{R}^d \longrightarrow \mathbf{R}$ is $m_{\mathbf{R}^d}$-measurable. ($f \circ p \circ H$ is $(m_{\mathbf{R}^d \times \mathbf{R}^r} = m_{\mathbf{R}^d} \times m_{\mathbf{R}^r})$-measurable $\Longleftrightarrow f \circ p \circ H$ is $(m_{\mathbf{R}^d} \times \beta)$-measurable.) Since $f \circ p \circ H(y, t) = f(p(s(y)))$, this implies $y \longrightarrow f(p(s(y)))\, \beta(\mathbf{R}^r) : \mathbf{R}^d \longrightarrow \mathbf{R}$ is $m_{\mathbf{R}^d}$-measurable, so $f \circ p \circ s$ is $m_{\mathbf{R}^d}$-measurable.

Let $\Omega = \{U \subset V'/G : p^{-1}(U) \text{ is } m_{V'}\text{-measurable}\}$. Lemma 3.4 shows that $\Omega = \{U \subset V'/G : (p \circ s)^{-1}(U) \text{ is } m_{\mathbf{R}^d}\text{-measurable}\}$. (Take $f = \chi_U$, the characteristic function of U.)

LEMMA 3.5. *Let $N \subset V'/G$, $N \in \Omega$. Then $m_{V'}(p^{-1}(N)) = 0 \Longleftrightarrow$ $m_{\mathbf{R}^d}((p \circ s)^{-1}(N)) = 0$.*

PROOF. $m_{V'}(p^{-1}(N)) = 0 \Longleftrightarrow \chi_N \circ p = 0\ m_{V'}$ a.e. $\Longleftrightarrow (\chi_N \circ p \circ H) \cdot J = 0\ m_{\mathbf{R}^d \times \mathbf{R}^r}$ a.e. (by Lemma 3.3) $\Longleftrightarrow \chi_N \circ p \circ H = 0\ m_{\mathbf{R}^d \times \mathbf{R}^r}$ a.e. (since A is conull).

Suppose $\chi_N \circ p \circ H = 0\ m_{\mathbf{R}^d \times \mathbf{R}^r}$ a.e. By Fubini's theorem, for $m_{\mathbf{R}^d}$ almost all y, $\chi_N \circ p \circ H(y, t) = \chi_N(p(s(y))) = 0$ for $m_{\mathbf{R}^r}$ a.a. t. Hence $m_{\mathbf{R}^d}((p \circ s)^{-1}(N)) = 0$.

Conversely, suppose $\chi_N \circ p \circ s = 0$, $m_{\mathbf{R}^d}$ a.e. Then by Tonnelli's theorem ($\chi_N \circ p \circ H$ is $m_{\mathbf{R}^d \times \mathbf{R}^r}$-measurable by the corollary to Lemma 3.3),

$$\begin{aligned}
&\int_{\mathbf{R}^d \times \mathbf{R}^r} \chi_N \circ p \circ H(y, t)\, dm_{\mathbf{R}^d \times \mathbf{R}^r}(y, t) \\
&\qquad = \int_{\mathbf{R}^r} \left(\int_{\mathbf{R}^d} \chi_N(p(H(y, t)))\, dm_{\mathbf{R}^d}(y) \right) dm_{\mathbf{R}^r}(t) \\
&\qquad = \int_{\mathbf{R}^r} \left(\int_{\mathbf{R}^d} \chi_N(p(s(y)))\, dm_{\mathbf{R}^d}(y) \right) dm_{\mathbf{R}^r}(t) \\
&\qquad = \int_{\mathbf{R}^r} 0\, dm_{\mathbf{R}^r}(t) = 0.
\end{aligned}$$

Thus $\chi_N \circ p \circ H = 0\ m_{\mathbf{R}^d \times \mathbf{R}^r}$ a.e.

The following argument uses Bourbaki's theorem on the disintegration of a measure relative to a pseudo-image [6, Chapitre 6, Théorème 2, p. 64] to get a disintegration of $x_E m_{V'}$ relative to $(p \circ s)(x_W m_{\mathbf{R}^d})$. The rest of the proof of Theorem 3.3 consists of showing that the orbit measures λ_b $(b = psy \in E/G)$ from [6, Chapitre 6, Théorème 2, p. 64] are equal to $|\det M^{(r)}(s(y))|\,\nu_{s(y)}$.

Since E is an open set in V', E is a locally compact topological space with a countable basis. By Theorem 3.2, $p \circ s|_W$ is a homeomorphism of the Zariski open set $W \subset \mathbf{R}^d$ onto E/G. Therefore E/G is a locally compact space with a countable basis. Since W is $m_{\mathbf{R}^d}$-conull, and E is $m_{V'}$-conull, Lemma 3.5 shows that the measure on E/G, $(p \circ s)(x_W m_{\mathbf{R}^d})$, is a pseudo-image of $x_E m_{V'}$ by $p|_E$, i.e., $N \subset E/G$ is $(p \circ s)(x_W m_{\mathbf{R}^d})$-null $\Longleftrightarrow p^{-1}(N)$ is $(x_E m_{V'})$-null. By [6, Chapitre 6, Théorème 2, p. 64] there exists a $(p \circ s)(x_W m_{\mathbf{R}^d})$-adequate family [5, Chapitre 5, Définition 1, p. 19] $b \longrightarrow \lambda_b$ $(b \in E/G)$ of positive measures on E having the following properties:

(a) $\lambda_b \neq 0$ for $b \in p(E) = E/G$;

(b) λ_b is concentrated in $p^{-1}(b)$ for all $b \in E/G$;

(c) $x_E m_{V'} = \int_{E/G} \lambda_b \, d(p \circ s)(x_W m_{\mathbf{R}^d})(b)$.

Thus, if $f: E \longrightarrow \mathbf{R}$ is $(x_E m_{V'})$-integrable (f is $(x_E m_{V'})$-measurable, and $\int_E |f(\gamma)|\,dm_{V'}(\gamma) < \infty$), then $b \longrightarrow \langle \lambda_b, f\rangle = \int_{p^{-1}(b)} f(\gamma)\,d\lambda_b(\gamma): E/G \longrightarrow \mathbf{R}$ is $(p \circ s)(x_W m_{\mathbf{R}^d})$-integrable; $y \longrightarrow \langle \lambda_{ps(y)}, f\rangle = \int_{p^{-1}(ps(y))} f(\gamma)\,d\lambda_{ps(y)}(\gamma)$: $W \longrightarrow \mathbf{R}$ is $(x_W m_{\mathbf{R}^d})$-integrable; and

$$\begin{aligned} \int_E f(\gamma)\,dm_{V'}(\gamma) &= \int_{E/G}\left(\int_{p^{-1}(b)} f(\gamma)\,d\lambda_b(\gamma)\right) d(p \circ s)(x_W m_{\mathbf{R}^d})(b) \\ &= \int_W \left(\int_{p^{-1}(ps(y))} f(\gamma)\,d\lambda_{ps(y)}(\gamma)\right) dm_{\mathbf{R}^d}(y). \end{aligned} \tag{3.16}$$

To complete the proof of Theorem 3.3, we show that for $(x_W m_{\mathbf{R}^d})$ a.a. y, $\lambda_{ps(y)} = |\det M^{(r)}(s(y))|\,\nu_{s(y)}$.

Since $x_E m_{V'}$ is G-invariant, $(x_W m_{\mathbf{R}^d})$ almost all the $\lambda_{ps(y)}$ are G-invariant [16, Lemma 11.5, p. 126]. Let $N \subset W$ be a null set such that $y \in W - N \Rightarrow \lambda_{ps(y)}$ is G-invariant. Then $\lambda_{ps(y)}$ and $\nu_{s(y)}$ are both G-invariant measures on $O_{s(y)} \simeq G/G_{s(y)}$. Therefore, if $y \in W - N$, there is a positive number $c(y)$ such that

$$\lambda_{ps(y)} = c(y)\nu_{s(y)}. \tag{3.17}$$

Put $c(y) = 1$ if $y \in N \cup (\mathbf{R}^d - W)$.

LEMMA 3.6. *$c: \mathbf{R}^d \longrightarrow \mathbf{R}$ is $m_{\mathbf{R}^d}$-measurable.*

PROOF. Let $f: V' \longrightarrow \mathbf{R}$ be an everywhere positive, continuous, $m_{V'}$-integrable function. By the corollary to Lemma 3.3, $f \circ H$ is $m_{\mathbf{R}^d \times \mathbf{R}^r}$-measurable, nonnegative. By Tonnelli's theorem $y \longrightarrow \int_{\mathbf{R}^r} f(H(y, t))\,dm_{\mathbf{R}^r}(t)$ is $m_{\mathbf{R}^d}$-measurable.

If $y \in W$, then by Theorem 3.1(b), $\langle \nu_{s(y)}, f\rangle = \int_{\mathbf{R}^r} f(s(y)\cdot g(t))\, dm_{\mathbf{R}^r}(t) = \int_{\mathbf{R}^r} f(H(y, t))\, dm_{\mathbf{R}^r}(t) > 0$ (since $f(\gamma) > 0$ for all γ). Therefore $y \longrightarrow \langle \nu_{s(y)}, f\rangle$: $W \longrightarrow \mathbf{R} \cup \{\infty\}$ is an everywhere positive, $(x_W m_{\mathbf{R}^d})$-measurable function. Hence $y \longrightarrow 1/\langle \nu_{s(y)}, f\rangle : W \longrightarrow \mathbf{R}$ is $m_{\mathbf{R}^d}$-measurable. Since f is $m_{V'}$-integrable, $y \longrightarrow \langle \lambda_{ps(y)}, f\rangle = c(y)\langle \nu_{s(y)}, f\rangle$ a.e. is $(x_W m_{\mathbf{R}^d})$-integrable, hence measurable. Therefore $y \longrightarrow \langle \lambda_{ps(y)}, f\rangle / \langle \nu_{s(y)}, f\rangle = c(y)$ is $m_{\mathbf{R}^d}$-measurable on $W - N$. Hence $y \longrightarrow c(y)$ is $m_{\mathbf{R}^d}$-measurable on W, hence on $\mathbf{R}^d$.

LEMMA 3.7. *For $m_{\mathbf{R}^d}$ almost all $y \in \mathbf{R}^d$,*

$$c(y) = |\det M^{(r)}(s(y))|. \tag{3.18}$$

PROOF. We substitute $c(y)\nu_{s(y)}$ for $\lambda_{ps(y)}$ in (3.16), write $\nu_{s(y)}$ in terms of the coordinates $t = (t^1, \ldots, t^r) \longrightarrow s(y)\cdot g(t) = H(y, t)$, and compare the resulting equation with (3.14). The result is

$$\begin{aligned} &\int_W \left(\int_{\mathbf{R}^r} f(H(y, t))\, dm_{\mathbf{R}^r}(t)\right) c(y)\, dm_{\mathbf{R}^d}(y) \\ &\qquad = \int_{W\times\mathbf{R}^r} f(H(y, t)) J(y, t)\, dm_{\mathbf{R}^d\times\mathbf{R}^r}(y, t), \qquad f \in L^1(m_{V'}). \end{aligned} \tag{3.19}$$

Suppose $f \in L^1(m_{V'})$ is nonnegative. By the corollary to Lemma 3.3, $f\circ H$: $\mathbf{R}^d \times \mathbf{R}^r \longrightarrow \mathbf{R}$ is $m_{\mathbf{R}^d\times\mathbf{R}^r}$-measurable. By Lemma 3.6, $c : \mathbf{R}^d \longrightarrow \mathbf{R}$ is $m_{\mathbf{R}^d}$-measurable. Hence $(f\circ H)\circ c$ is $m_{\mathbf{R}^d\times\mathbf{R}^r}$-measurable, nonnegative. By Tonnelli's theorem, the left-hand side of (3.19) is equal to

$$\int_{W\times\mathbf{R}^r} f(H(y, t)) c(y)\, dm_{\mathbf{R}^d\times\mathbf{R}^r}(y, t).$$

Therefore, whenever $f \geqslant 0$ is $m_{V'}$-integrable,

$$0 = \int_{W\times\mathbf{R}^r} f(H(y, t))\,(J(y, t) - c(y))\, dm_{\mathbf{R}^d\times\mathbf{R}^r}(y, t). \tag{3.20}$$

Let $D = \{(y, t) \in W \times \mathbf{R}^r : J(y, t) > c(y)\}$. $x_D = (x_D \circ H^{-1})\circ H$ is $m_{\mathbf{R}^d\times\mathbf{R}^r}$-measurable so (by the corollary to Lemma 3.3) $x_D \circ H^{-1}$ is $m_{V'}$-measurable. Let $f: V' \longrightarrow \mathbf{R}$ be an everywhere positive, integrable function. $(x_D \circ H^{-1})\circ f \leqslant f$, so $(x_D \circ H^{-1})\circ f$ is $m_{V'}$-integrable, nonnegative. By (3.20),

$$0 = \int_{W\times\mathbf{R}^r} x_D(y, t) f(H(y, t))\,(J(y, t) - c(y))\, dm_{\mathbf{R}^d\times\mathbf{R}^r}(y, t).$$

Hence $x_D(y, t)(J(y, t) - c(y)) = 0$ for $(m_{\mathbf{R}^d\times\mathbf{R}^r})$ a.a. (y, t). Since $J(y, t) - c(y) > 0$ on D, $(m_{\mathbf{R}^d\times\mathbf{R}^r})(D) = 0$. Similarly,

$$m_{\mathbf{R}^d\times\mathbf{R}^r}(\{(y, t) \in W \times \mathbf{R}^r : J(y, t) < c(y)\}) = 0.$$

Therefore, $J(y, t) = c(y)$ for $m_{\mathbf{R}^d\times\mathbf{R}^r}$ a.a. (y, t) in $W \times \mathbf{R}^r$, hence for $m_{\mathbf{R}^d\times\mathbf{R}^r}$ a.a. (y, t). By Fubini's theorem, for almost all $y \in \mathbf{R}^d$, $J(y, t) = c(y)$ for almost all $t \in \mathbf{R}^r$. Since $t \longrightarrow J(y, t)$ is continuous on $\mathbf{R}^r$, $J(y, t) = c(y)$ for all $t \in \mathbf{R}^r$.

Hence, $c(y) = J(y, 0)$ for almost all $y \in \mathbf{R}^d$. By (3.12), $J(y, 0) = |\det M^{(r)}(s(y))|$ for $y \in W$. Thus $c(y) = |\det M^{(r)}(s(y))|$ for almost all $y \in \mathbf{R}^d$.

Substituting $c(y)\nu_{s(y)} = |\det M^{(r)}(s(y))|\nu_{s(y)}$ for $\lambda_{ps(y)}$ in (3.16), we obtain (3.13). This completes the proof of Theorem 3.3. The above proof also gives the following fact.

THEOREM 3.4. *$H: W \times \mathbf{R}^r \longrightarrow E: (y, t) \longrightarrow s(y) \cdot g(t)$ is a diffeomorphism.*

PROOF. H is a polynomial in y and t so it is differentiable. The proof of Lemma 3.7 shows that the continuous function $(y, t) \longrightarrow J(y, t) - |\det M^{(r)}(s(y))|$ is zero for $m_{\mathbf{R}^d \times \mathbf{R}^r}$ almost all (y, t). Hence $J(y, t) = |\det dH(y, t)| = |\det M^{(r)}(s(y))|$ for all $(y, t) \in \mathbf{R}^d \times \mathbf{R}^r$. Thus $\{(y, t) \in \mathbf{R}^d \times \mathbf{R}^r: |\det dH(y, t)| \neq 0\} = W \times \mathbf{R}^r$. From the proof of Lemma 3.3, H is a bijection of $W \times \mathbf{R}^r$ onto E. Therefore, the inverse function theorem shows H is a diffeomorphism.

4. A Plancherel formula for idyllic nilpotent Lie groups. In §4 we bring together the results of §§1–3 to obtain a procedure for computing Plancherel measure for the following class of nilpotent Lie groups.

Suppose G is a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. $\mathfrak{g}$ will be called "idyllic" if $\mathfrak{g}$ has an abelian ideal $\mathfrak{n}$ such that for Lebesgue almost all γ in $\mathfrak{n}'$, $\mathfrak{g}_\gamma/\mathfrak{n}$ is abelian, where $\mathfrak{g}_\gamma = \{x \in \mathfrak{g}: \langle \gamma, [x, n] \rangle = 0 \ \forall n \in \mathfrak{n}\}$. Such an ideal $\mathfrak{n}$ will be called an "idyll" of $\mathfrak{g}$. G is called idyllic if its Lie algebra $\mathfrak{g}$ is idyllic. If $\mathfrak{n}$ is an idyll of $\mathfrak{g}$, then $N = \exp \mathfrak{n}$ is called an idyll of G.

To compute Plancherel measure for idyllic G with idyll N, we combine the projective Plancherel formula from §1 with the disintegration theorem of §3 (Theorem 3.3) via Kleppner and Lipsman's Plancherel formula for group extensions [15, Theorem 2.3, p. 108]

$$(4.1) \quad \int_G |f(x)|^2 \, dm_G(x) = \int_{\hat{N}/G} \int_{(G_\gamma/N, \bar{\omega}_\gamma)^\wedge} \operatorname{tr} \pi_{\gamma,\sigma}(f * f^*) \, d\mu_\gamma(\sigma) \, d\bar{\mu}_N(\bar{\gamma}),$$

which expresses Plancherel measure on $\hat{G}$ corresponding to a given Haar measure m_G on G as a fibered measure with base $\hat{N}/G$ and fibers $(G_\gamma/N, \bar{\omega}_\gamma)^\wedge$, where G_γ is the stability subgroup at $\gamma \in \hat{N}$. μ_N is Plancherel measure on $\hat{N}$ corresponding to a given Haar measure m_N on N. $\bar{\mu}_N$ is a pseudo-image of μ_N by the projection $p: \hat{N} \longrightarrow \hat{N}/G$. Since $\hat{N}/G$ is countably separated, there are orbit measures ν_γ which provide a disintegration of Plancherel measure μ_N on $\hat{N}$ relative to the pseudo-image $\bar{\mu}_N$ on $\hat{N}/G$, i.e.,

$$(4.2) \qquad \mu_N = \int_{\hat{N}/G} \nu_\gamma \, d\bar{\mu}_N(\bar{\gamma}),$$

ν_γ concentrated on $\gamma \cdot G \simeq G/G_\gamma$. The projective Plancherel measure μ_γ on $(G_\gamma/N, \bar{\omega}_\gamma)^\wedge$ corresponds to the Haar measure $m_{G_\gamma/N}$ on G_γ/N which satisfies

(4.3) $\int_G f(x)\,dm_G(x) = \int_{G/G_\gamma}\int_{G_\gamma/N}\int_N f(nzx)\,dm_N(n)\,dm_{G_\gamma/N}(Nz)\,d\nu_\gamma(\bar{x})$.

For $\gamma \in \hat{N}$, $\pi_{\gamma,\sigma} = \operatorname{ind}_{G_\gamma}^G \gamma' \otimes \sigma''$ is an irreducible representation of G. γ' is the extension of γ to an ω_γ-representation of G_γ, where ω_γ is a multiplier on G_γ/N. σ is an irreducible $\bar{\omega}_\gamma$-representation of G_γ/N, and σ'' denotes the lift of σ to G_γ.

If μ_γ is the projective Plancherel measure on $(G_\gamma/N, \bar{\omega}_\gamma)^\wedge$ corresponding to $m_{G_\gamma/N}$ satisfying (4.3), then [15, (2.10), p. 109], for $f \in C_0(G)$ (= continuous functions with compact support),

$$\int_{(G_\gamma/N,\bar{\omega}_\gamma)^\wedge} \operatorname{tr}[\pi_{\gamma,\sigma}(f*f^*)]\,d\mu_\gamma(\sigma) = \int_{G/G_\gamma} \operatorname{tr}[\gamma\cdot A(f*f^*|_N)]\,d\nu_\gamma(\bar{A}),$$

so that

$$\begin{aligned}\int_{\hat{N}/G}\int_{(G_\gamma/N,\bar{\omega}_\gamma)^\wedge} &\operatorname{tr}[\pi_{\gamma,\sigma}(f*f^*)]\,d\mu_\gamma(\sigma)\,d\bar{\mu}_N(\bar{\gamma})\\ &= \int_{\hat{N}/G}\int_{G/G_\gamma}\operatorname{tr}[\gamma\cdot A(f*f^*|_N)]\,d\nu_\gamma(\bar{A})\,d\bar{\mu}_N(\bar{\gamma})\\ &= \int_{\hat{N}}\operatorname{tr}[\gamma(f*f^*|_N)]\,d\mu_N(\gamma) = f*f^*(e) = \int_G |f(x)|^2\,dm_G(x).\end{aligned}$$

(This implies the validity of (4.1) for $f \in L^1(G)\cap L^2(G)$ since $C_0(G)$ is dense in the C^*-algebra of G.)

The Plancherel measure for idyllic $G = \exp \mathfrak{g}$ with idyll $N = \exp \mathfrak{n}$ computed via (4.1) is given in terms of coordinates on $\hat{N}/G$ and on the fibers $(G_\gamma/N, \bar{\omega}_\gamma)^\wedge$. We start by making an explicit choice of Haar measures m_G and m_N in terms of coordinates on G and N, respectively. Then we compute Plancherel measure μ_N, in terms of coordinates on $\hat{N}$, corresponding to m_N. Next, we use Theorem 3.3 to obtain a disintegration of μ_N by G,

$$\mu_N = \int_{\hat{N}/G} \nu_\gamma\, d\bar{\mu}_N(\bar{\gamma}),$$

in which the pseudo-image $\bar{\mu}_N$ is given in terms of coordinates on almost all of $\hat{N}/G$, and the orbit measures ν_γ are expressed in terms of coordinates on the orbit of γ. Then we use Theorem 3.1 to find the Haar measure $m_{G_\gamma/N}$ on G_γ/N which satisfies (4.3). Then we use §1 to compute the projective Plancherel measure μ_γ corresponding to $m_{G_\gamma/N}$ in terms of coordinates on $(G_\gamma/N, \bar{\omega}_\gamma)^\wedge$. Finally, we combine $\bar{\mu}_N$ and the μ_γ to obtain a Plancherel formula for G. The steps involved in the computational process and the resulting Plancherel formula are described in the following theorem.

THEOREM 4.1. *A Plancherel-measure-computing procedure for idyllic* $G = \exp \mathfrak{g}$ *with idyll* $N = \exp \mathfrak{n}$ *consists of the following steps*:

(1) *Take a basis* $\{v_1 < \cdots < v_K\}$ *of* $\mathfrak{n}$ *in Jordan-Hölder order relative to*

the adjoint action of $\mathfrak{g}$ *on* $\mathfrak{n}$, *and a Jordan-Hölder basis* $\{\bar{e}_1 < \cdots < \bar{e}_s\}$ *of* $\mathfrak{g}/\mathfrak{n}$. *Let* $\{v^1, \ldots, v^K\}$ *be the basis of* $\mathfrak{n}'$ *such that* $\langle v^i, v_j\rangle = \delta^i_j$.

(2) *Compute* $M = (e_i v_j)_{1 \leq i \leq s, 1 \leq j \leq K}$, *where* $(e_i v_j) = [e_i, v_j]$.

(3) *Find the partitions defined in* §3 (p. 13)

$$\{1, \ldots, K\} = \{l_1, \ldots, l_r\} \cup \{j_1, \ldots, j_d\},$$
$$\{1, \ldots, s\} = \{i_1, \ldots, i_r\} \cup \{m_1, \ldots, m_q\},$$

i. e., determine the independent columns of M from the right and the independent rows of M from below.

(4) *For* $y = (y_1, \ldots, y_d) \in \mathbf{R}^d$, *let* $sy = \Sigma_{k=1}^d y_k v^{j_k}$, *and compute* $\det M^{(r)}(sy) = |sy(e_{i_a} v_{l_b})_{1 \leq a,b \leq r}|$.

(5) *For* $y \in W = \{y \in \mathbf{R}^d : \det M^{(r)}(sy) \neq 0\}$, *compute, for* $1 \leq b \leq q$, $u_b(sy) = e_{m_b} - \Sigma_{s=1}^r \lambda^{i_s}_{m_b}(sy) e_{i_s}$;

$$\lambda^{i_s}_{m_b}(sy) = \frac{\begin{vmatrix} sy(e_{i_1} v_{l_1}) & \cdots & sy(e_{i_r} v_{l_r}) \\ \vdots & & \vdots \\ sy(e_{i_{(s-1)}} v_{l_1}) & \cdots & sy(e_{i_{(s-1)}} v_{l_r}) \\ sy(e_{m_b} v_{l_1}) & \cdots & sy(e_{m_b} v_{l_r}) \\ sy(e_{i_{(s+1)}} v_{l_1}) & \cdots & sy(e_{i_{(s+1)}} v_{l_r}) \\ \vdots & & \vdots \\ sy(e_{i_r} v_{l_1}) & \cdots & sy(e_{i_r} v_{l_r}) \end{vmatrix}}{\det M^{(r)}(sy)}, \qquad 1 \leq s \leq r.$$

(6) *For* $y \in W$, *compute the matrix* $(\langle sy, [u_i(sy), u_j(sy)]\rangle)_{1 \leq i,j \leq q}$.

(7) *For* $y \in W_1 = \{y \in W : (\langle sy, [u_i(sy), u_j(sy)]\rangle)_{1 \leq i,j \leq q}$ *has maximal rank,* $2l\}$, *find a nonsingular* $q \times q$ *matrix* P_{sy} *such that*

$$P_{sy}(\langle sy, [u_i(sy), u_j(sy)]\rangle)_{1 \leq i,j \leq q}\, {}^tP_{sy} = \left[\begin{array}{cc|c} 0 & I_l & \\ -I_l & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right].$$

Let $m = q - 2l$, *and let*

$$\mu_{sy} = |\det P_{sy}|^{-1} \frac{1}{(2\pi)^{l+m}} \psi_{P_{sy}}(m_{\mathbf{R}^m}), \tag{4.4}$$

where $\psi_{P_{sy}}$ *is defined in* §1.

Then

$$(4.5) \qquad \mu_G = \frac{1}{(2\pi)^K} \int_{W_1} \mu_{sy} \, |\det M^{(r)}(sy)| \, dm_{\mathbf{R}^d}(y)$$

is Plancherel measure on $\hat{G}$ corresponding to m_G, Haar measure on G defined in terms of the basis $\{e_1, \ldots, e_s, v_1, \ldots, v_K\}$ of $\mathfrak{g}$.

The Plancherel formula is

$$(4.6) \qquad \int_G |f(x)|^2 \, dm_G(x) = \frac{1}{(2\pi)^{K+l+m}} \int_{W_1} \int_{\mathbf{R}^m} \operatorname{tr} \pi_{y,t}(f * f^*) \, dm_{\mathbf{R}^m}(t) \, |\det P_{sy}|^{-1} \, |\det M^{(r)}(sy)| \, dm_{\mathbf{R}^d}(y).$$

For $(y, t) \in W_1 \times \mathbf{R}^m$, $\pi_{y,t} = \operatorname{ind}_{G_{sy}}^G (\chi_{sy})' \otimes (\psi_{P_{sy}}(t))''$ is an irreducible representation of G, where G_{sy} is the stability subgroup at sy for the coadjoint representation of G in $\mathfrak{n}'$. χ_{sy} is the character of $N = \exp \mathfrak{n}$ defined by

$$\chi_{sy}(\exp n) = e^{i\langle sy, n\rangle}, \qquad n \in \mathfrak{n}.$$

$(\chi_{sy})'$ is the extension of χ_{sy} to an ω_{sy}-representation of G_{sy}, where

$$\omega_{sy}(\exp x, \exp z) = e^{-i/2\langle sy, [x,z]\rangle}, \qquad x, z \in \mathfrak{g}_{sy},$$

the stability subalgebra at sy for the coadjoint representation of $\mathfrak{g}$ in $\mathfrak{n}'$. $\psi_{P_{sy}}(t)$ is an irreducible $\bar{\omega}_{sy}$-representation of G_{sy}/N, and $(\psi_{P_{sy}}(t))''$ denotes the lift of $\psi_{P_{sy}}(t)$ to G_{sy}.

PROOF. To prove Theorem 4.1, we relate steps (1)–(7) to $\hat{N}$, μ_N (step (1)); the disintegration of μ_N by G (steps (2)–(4)); equation (4.3) (step (5)); and $(G_\gamma/N, \bar{\omega}_\gamma)^\wedge$, μ_γ (steps (6) and (7)). Then we use (4.1).

Since $\mathfrak{n}$ is abelian, $\exp : \mathfrak{n} \longrightarrow N$ is an isomorphism ($\exp(x + y) = \exp x \exp y$), and may be used to identify $\hat{N}$ with $\mathfrak{n}'$. If $\gamma \in \mathfrak{n}'$, let χ_γ be the character of N defined by

$$\chi_\gamma(\exp x) = e^{i\langle\gamma, x\rangle}, \qquad x \in \mathfrak{n}.$$

The map $\gamma \longrightarrow \chi_\gamma : \mathfrak{n}' \longrightarrow \hat{N}$ is an isomorphism. Let m_N be the Haar measure on N defined in terms of the basis $\{v_1, \ldots, v_K\}$ of $\mathfrak{n}$. Let $m_{\mathfrak{n}'}$ be the measure on $\mathfrak{n}'$ defined by

$$\langle m_{\mathfrak{n}'}, f\rangle = \int_{\mathbf{R}^K} f\left(\sum_{j=1}^K \gamma_j v^j\right) dm_{\mathbf{R}^K}(\gamma_1, \ldots, \gamma_K).$$

LEMMA 4.1. *Plancherel measure μ_N on $\hat{N}$ corresponding to m_N is the image of $(2\pi)^{-K} m_{\mathfrak{n}'}$ under the map $\gamma \longrightarrow \chi_\gamma : \mathfrak{n}' \longrightarrow \hat{N}$.*

PROOF. If $f \in C_0(N)$, let $f_1 \in C_0(\mathbf{R}^K)$ be

$$f_1(x^1, \ldots, x^K) = f\left(\exp \sum_{j=1}^K x^j v_j\right), \qquad (x^1, \ldots, x^K) \in \mathbf{R}^K.$$

Then, for $\gamma = \Sigma_{j=1}^K \gamma_j v^j \in \mathfrak{n}'$,

$$\begin{aligned}\chi_\gamma(f) &= \int_N f(n)\chi_\gamma(n)\, dm_N(n)\\ &= \int_{\mathbf{R}^K} f\left(\exp \sum_{j=1}^K x^j v_j\right) \chi_\gamma\left(\exp \sum_{j=1}^K x^j v_j\right) dm_{\mathbf{R}^K}(x^1, \ldots, x^K)\\ &= \int_{\mathbf{R}^K} f_1(x^1, \ldots, x^K) e^{i\langle\gamma, \Sigma_{j=1}^K x^j v_j\rangle}\, dm_{\mathbf{R}^K}(x^1, \ldots, x^K)\\ &= \int_{\mathbf{R}^K} f_1(x^1, \ldots, x^K) e^{i\Sigma_{j=1}^K \gamma_j x^j}\, dm_{\mathbf{R}^K}(x^1, \ldots, x^K)\\ &= \hat{f}_1(\gamma_1, \ldots, \gamma_K).\end{aligned}$$

Hence

$$\begin{aligned}\int_{\hat N} |\chi(f)|^2\, d\mu_N(\chi) &= (2\pi)^{-K}\int_{\mathfrak{n}'} |\chi_\gamma(f)|^2\, dm_{\mathfrak{n}'}(\gamma)\\ &= (2\pi)^{-K}\int_{\mathbf{R}^K} |\hat{f}_1(\gamma_1, \ldots, \gamma_K)|^2\, dm_{\mathbf{R}^K}(\gamma_1, \ldots, \gamma_K)\\ &= \int_{\mathbf{R}^K} |f_1(x^1, \ldots, x^K)|^2\, dm_{\mathbf{R}^K}(x^1, \ldots, x^K)\end{aligned}$$

by the Plancherel formula for $\mathbf{R}^K$. By definition of f_1, the latter integral is

$$\int_{\mathbf{R}^K} \left| f\left(\exp \sum_{j=1}^K x^j v_j\right)\right|^2 dm_{\mathbf{R}^K}(x^1, \ldots, x^K) = \int_N |f(n)|^2\, dm_N(n),$$

by definition of m_N.

The action of G on $\hat N$ corresponds to the coadjoint action of G on $\mathfrak{g}'$ restricted to $\mathfrak{n}'$. If $\gamma \in \mathfrak{n}'$, $A \in G$ and $x \in \mathfrak{n}$, then

$$\begin{aligned}(\chi_\gamma \cdot A)(\exp x) &= \chi_\gamma(A \exp x A^{-1})\\ &= \chi_\gamma(\exp \operatorname{Ad} A(x)) = e^{i\langle\gamma, \operatorname{Ad} A(x)\rangle} = e^{i\langle\gamma\cdot A, x\rangle} = \chi_{\gamma\cdot A}(\exp x).\end{aligned}$$

Hence the map $\bar\gamma \longrightarrow \bar\chi_\gamma : \mathfrak{n}'/G \longrightarrow \hat N/G$ identifies $\hat N/G$ with $\mathfrak{n}'/G$. We apply §3 to the adjoint action of G on $\mathfrak{n}: G \times \mathfrak{n} \longrightarrow \mathfrak{n} : (A, x) \longrightarrow A \cdot x$, where $A \cdot x = \operatorname{Ad} A(x) = (d/dt)A \exp tx A^{-1}|_{t=0}$, $A \in G$, $x \in \mathfrak{n}$. The contragredient action of G on $\mathfrak{n}'$: $\mathfrak{n}' \times G \longrightarrow \mathfrak{n}' : (\gamma, A) \longrightarrow \gamma \cdot A$, where $\langle \gamma\cdot A, x\rangle = \langle \gamma, A\cdot x\rangle$, $\gamma \in \mathfrak{n}'$, $A \in G$, $x \in \mathfrak{n}$, is the coadjoint action of G on $\mathfrak{g}'$ restricted to $\mathfrak{n}'$.

The derivative of the adjoint action of G on $\mathfrak{n}$ is the adjoint action of $\mathfrak{g}$ on $\mathfrak{n}: \mathfrak{g} \times \mathfrak{n} \longrightarrow \mathfrak{n} : (x, n) \longrightarrow x \cdot n = [x, n]$. The contragredient action of $\mathfrak{g}$ on $\mathfrak{n}'$ is the coadjoint action of $\mathfrak{g}$ on $\mathfrak{n}' : \mathfrak{n}' \times \mathfrak{g} \longrightarrow \mathfrak{n} : (\gamma, x) \longrightarrow \gamma \cdot x$, where $\langle \gamma\cdot x, n\rangle = \langle \gamma, [x, n]\rangle$, $\gamma \in \mathfrak{n}'$, $x \in \mathfrak{g}$, $n \in \mathfrak{n}$.

Since $\{v_1 < \cdots < v_K\}$ is a basis of $\mathfrak{n}$ in Jordan-Hölder order relative to $\mathfrak{g}$, and $\{\bar e_1 < \cdots < \bar e_s\}$ is a Jordan-Hölder basis of $\mathfrak{g}/\mathfrak{n}$, $\{e_1 < \cdots < e_s < v_1 <$

$\cdots < v_K\}$ is a Jordan-Hölder basis of $\mathfrak{g}$. We take Haar measure on G to be the measure m_G defined in terms of this basis.

Define $e_{s+j} = v_j$, $1 \leq j \leq K$. Since $\mathfrak{n}$ is abelian, $[e_{s+j}, v_k] = 0, 1 \leq j, k \leq K$. Thus, the matrix $M = (e_i v_j)_{1 \leq i \leq s+K, 1 \leq j \leq K}$ defined in §3 has the form

$$M = \begin{bmatrix} e_1 v_1 & \cdots & e_1 v_K \\ e_s v_1 & \cdots & e_s v_K \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{bmatrix}$$

Disregarding the last K rows, we have $M = (e_i v_j)_{1 \leq i \leq s, 1 \leq j \leq K}$ as in step (2). As in §3, $E = \{\gamma \in \mathfrak{n}' : \det M^{(r)}(\gamma) \neq 0\}$.

By Theorem 3.2, for $sy = \Sigma_{k=1}^d y_k v^{j_k}$, $W = \{y \in \mathbf{R}^d : sy \in E\}$, and $p : \mathfrak{n}' \longrightarrow \mathfrak{n}'/G$ the projection $p \circ s|_W : W \longrightarrow E/G$ is a homeomorphism. By Theorem 3.3,

$$m_{\mathfrak{n}'} = \int_W \nu_{sy} \, |\det M^{(r)}(sy)| \, dm_{\mathbf{R}^d}(y)$$

is a disintegration of $m_{\mathfrak{n}'}$ by G. By Lemma 4.1, $\mu_N = (2\pi)^{-K} m_{\mathfrak{n}'}$. Since $\hat{N}/G = \mathfrak{n}'/G$,

$$\mu_N = (2\pi)^{-K} \int_W \nu_{sy} \, |\det M^{(r)}(sy)| \, dm_{\mathbf{R}^d}(y) \tag{4.7}$$

is a disintegration of μ_N by G, in which the pseudo-image $\overline{\mu_N}$ is given in terms of coordinates on E/G.

By Theorem 3.1, if $u_b(sy)$, $1 \leq b \leq q$, are computed as in step (5), then $\{u_1(sy), \ldots, u_q(sy)\}$ is a basis of $\mathfrak{g}_{sy}/\mathfrak{n}$, and Haar measure $m_{G_{sy}/N}$ on G_{sy}/N defined in terms of this basis satisfies (4.3) relative to the orbit measure ν_{sy} and m_N.

As stated, Theorem 3.1 gives a basis of the stability subalgebra $\mathfrak{g}_{sy}$ such that Haar measure $m_{G_{sy}}$ on $G_{sy} = \exp \mathfrak{g}_{sy}$ computed in terms of this basis satisfies

$$\int_G f(x) \, dm_G(x) = \int_{G/G_{sy}} \int_{G_{sy}} f(zx) \, dm_{G_{sy}}(z) \, d\nu_{sy}(\bar{x}).$$

In the present situation, $\mathfrak{g}_{sy} = \operatorname{span}\{u_1(sy), \ldots, u_q(sy)\} \oplus \mathfrak{n}$, and the basis of $\mathfrak{g}_{sy}$ computed in Theorem 3.1 is $\{u_1(sy), \ldots, u_q(sy), v_1, \ldots, v_K\}$. By definition of $m_{G_{sy}}$ (§2),

$$\int_{G_{sy}} f(z) \, dm_{G_{sy}}(z) = \int_{\mathbf{R}^q \times \mathbf{R}^K} f\left(\exp\left(\sum_{i=1}^q z^i u_i(sy) + \sum_{i=1}^K n^i v_i\right)\right) (dm_{\mathbf{R}^q \times \mathbf{R}^K})(z^1, \ldots, z^q, n^1, \ldots, n^K).$$

By Lemma 2.1 applied to the Jordan-Hölder basis $\{u_1(sy) < \cdots < u_q(sy) < v_1 <$

$\cdots < v_K\}$ of $\mathfrak{g}_{sy}$, the second integral is equal to

$$\int_{\mathbf{R}^q}\left(\int_{\mathbf{R}^K} f\left(\exp\left(\sum_{i=1}^{K} n^i v_i\right)\cdot\exp\left(\sum_{i=1}^{q} z^i u_i(sy)\right)\right)\right.$$

$$\left. dm_{\mathbf{R}^K}(n^1, \ldots, n^K)\right) dm_{\mathbf{R}^q}(z^1, \ldots, z^q)$$

$$= \int_{G_{sy}/N}\int_N f(nz)\, dm_N(n)\, dm_{G_{sy}/N}(Nz)$$

by definition of m_N and $m_{G_{sy}/N}$ (§3).

Steps (6) and (7) are the projective Plancherel measure parts of the procedure. Using the Campbell-Baker-Hausdorff formula, we write, for $x, y \in \mathfrak{g}$,

$$\exp x \exp y = \exp(x + y + B(x, y)),$$

where

$$B(x, y) = (1/2)[x, y] + (1/12)([x, [x, y]] - [y, [x, y]])$$
$$+ (\text{terms of the form } [x, [, \ldots, [x, y] \cdots]]$$
$$\text{and } [y, [, \ldots, [x, y] \cdots]]).$$

Since $\mathfrak{g}$ is nilpotent, $B(x, y)$ has only finitely many terms.

LEMMA 4.2. *Suppose $G = \exp \mathfrak{g}$ is a nilpotent Lie group. If $f \in \mathfrak{g}'$, let*

$$\omega_f(\exp x, \exp y) = e^{-i\langle f, B(x,y)\rangle}.$$

Then ω_f is a normalized, trivial multiplier on G.

PROOF. Since $(\exp x)^{-1} = \exp(-x)$, $B(x, -x) = 0$, so $\omega_f(\exp x, (\exp x)^{-1}) = 1$. The cocycle identity follows from associativity of multiplication on G.

$$\begin{aligned}(\exp x \exp y)\exp z &= \exp(x + y + B(x, y))\exp z \\ &= \exp((x + y + B(x, y)) + z + B(x + y + B(x, y), z)) \\ &= \exp x(\exp y \exp z) = \exp x \exp(y + z + B(y, z)) \\ &= \exp(x + (y + z + B(y, z))) + B(x, y + z + B(y, z)).\end{aligned}$$

Since exp is injective,

$$B(x, y) + B(x + y + B(x, y), z) = B(y, z) + B(x, y + z + B(y, z)).$$

Thus,

$$\begin{aligned}&\omega_f(\exp x, \exp y)\omega_f(\exp x \exp y, \exp z) \\ &\quad = e^{-i\langle f, B(x,y)\rangle} e^{-i\langle f, B(x+y+B(x,y), z)\rangle} \\ &\quad = e^{-i\langle f, B(y,z)\rangle} e^{-i\langle f, B(x, y+z+B(y,z))\rangle} \\ &\quad = \omega_f(\exp y, \exp z)\omega_f(\exp x, \exp y \exp z).\end{aligned}$$

To see that ω_f is trivial, let $\chi_f : G \longrightarrow T$ be defined by $\chi_f(\exp x) = e^{i\langle f,x\rangle}$, $x \in \mathfrak{g}$. Then

$$\begin{aligned}\chi_f(\exp x \exp y) &= \chi_f(\exp(x + y + B(x, y))) \\ &= e^{i\langle f,x+y+B(x,y)\rangle} = \chi_f(\exp x)\chi_f(\exp y)\bar{\omega}_f(\exp x, \exp y),\end{aligned}$$

so that

$$\omega_f(\exp x, \exp y) = \frac{\chi_f(\exp x)\chi_f(\exp y)}{\chi_f(\exp x \exp y)}.$$

The above proof shows that if $\gamma \in \mathfrak{n}'$, then χ_γ may be extended to a multiplier representation of G as follows. Let γ' in $\mathfrak{g}'$ be any extension of γ to $\mathfrak{g}$. Then $(\chi_\gamma)' = \chi_{\gamma'}$ is an $\omega_{\gamma'}$-representation of G, where $\chi_{\gamma'}$ and $\omega_{\gamma'}$ are defined above. $\omega_{\gamma'}|_{G_\gamma \times G_\gamma}$ is a multiplier on G_γ/N because, if $x \in \mathfrak{g}_\gamma$, then $\langle \gamma, [x, \mathfrak{n}] \rangle = 0$. This implies that $\langle \gamma', B(x + n + B(n, x), y)\rangle = \langle \gamma', B(x, y)\rangle$ for $x, y \in \mathfrak{g}_\gamma$, $n \in \mathfrak{n}$, which says that $\omega_{\gamma'}(\exp n \exp x, \exp y) = \omega_{\gamma'}(\exp x, \exp y)$. Although $\omega_{\gamma'}$ is a trivial multiplier on G_γ, it is not, in general, trivial on G_γ/N (unless $\gamma = 0$), because $\chi_{\gamma'}(\exp n) = e^{i\langle \gamma, n\rangle}$ is not one on N.

Now suppose G_γ/N is abelian. Then $[\mathfrak{g}_\gamma, \mathfrak{g}_\gamma] \subset \mathfrak{n}$. If $x, y \in \mathfrak{g}_\gamma$, $[x, y] \in \mathfrak{n}$, $\mathfrak{n}$ is an ideal, so

$$B(x, y) = \tfrac{1}{2}[x, y] + (\text{terms of the form } [x, \text{an element of } \mathfrak{n}] \text{ or } [y, \text{an element of } \mathfrak{n}]).$$

Since $\langle \gamma, [x, \mathfrak{n}]\rangle = \langle \gamma, [y, \mathfrak{n}]\rangle = 0$, $\langle \gamma, B(x, y)\rangle = \tfrac{1}{2}\langle \gamma, [x, y]\rangle$. Therefore $\bar{\omega}_\gamma(\exp x, \exp y) = e^{i\frac{1}{2}\langle \gamma, [x,y]\rangle}$. Since $\mathfrak{g}_\gamma/\mathfrak{n}$ is abelian, $\exp : \mathfrak{g}_\gamma/\mathfrak{n} \longrightarrow G_\gamma/N$ is an isomorphism. Define $A_\gamma : \mathfrak{g}_\gamma/\mathfrak{n} \times \mathfrak{g}_\gamma/\mathfrak{n} \longrightarrow \mathbf{R}$ by $A_\gamma(x, y) = \langle \gamma, [x, y]\rangle$. Then A_γ is bilinear and skew symmetric, and $\bar{\omega}_\gamma$ has the form of the multiplier in §1, $\bar{\omega}_\gamma(x, y) = e^{i\frac{1}{2}A_\gamma(x,y)}$, $x, y \in G_\gamma/N$ (identified with $\mathfrak{g}_\gamma/\mathfrak{n}$).

By definition of idyllic, $\mathfrak{g}_\gamma/\mathfrak{n}$ is abelian for $m_{\mathfrak{n}'}$ almost all γ in $\mathfrak{n}'$. The following lemma shows that $\mathfrak{g}_\gamma/\mathfrak{n}$ is abelian for all γ in E.

LEMMA 4.3. *If there is a γ in E such that $\mathfrak{g}_\gamma/\mathfrak{n}$ is not abelian, then $\mathfrak{g}_\gamma/\mathfrak{n}$ is not abelian for all γ in a nonempty Zariski open subset of E.*

PROOF. Let γ be in E, and $\{u_a(\gamma) : 1 \leqslant a \leqslant q\}$ be the basis of $\mathfrak{g}_\gamma/\mathfrak{n}$ defined in step (5). Then

$$[\mathfrak{g}_\gamma, \mathfrak{g}_\gamma] \subset \mathfrak{n} \Longleftrightarrow [u_a(\gamma), u_b(\gamma)] \in \mathfrak{n},$$

for $1 \leqslant a, b \leqslant q$. This requirement, when written out in terms of the definition of $u_a(\gamma)$, determines a family of rational functions of the form

$$R^l_{ab}(\gamma) = \Gamma^l_{m_a m_b} + \frac{P^l_{ab}(\gamma)}{\det M^{(r)}(\gamma)} + \frac{Q^l_{ab}(\gamma)}{(\det M^{(r)}(\gamma))^2}$$

(where $\Gamma^l_{m_a m_b} \in \mathbf{R}$, and P^l_{ab}, Q^l_{ab} are polynomials in $\gamma_1, \ldots, \gamma_K$), which must vanish for $1 \leqslant l \leqslant s$, $1 \leqslant a, b \leqslant q$. Each $R^l_{ab}(\gamma) = 0 \Longleftrightarrow$ the family of polynomials $F^l_{ab}(\gamma) = (\det M^{(r)}(\gamma))^2 R^l_{ab}(\gamma) = 0$ for $1 \leqslant l \leqslant s$, $1 \leqslant a, b \leqslant q$. Therefore, $\{\gamma \in E : [\mathfrak{g}_\gamma, \mathfrak{g}_\gamma] \subset \mathfrak{n}\} = \{\gamma \in E : F^l_{ab}(\gamma) = 0, 1 \leqslant l \leqslant s, 1 \leqslant a, b \leqslant q\} = F$, a Zariski closed set in E.

The projective Plancherel measure determined in §1 for the multiplier on a vector space H arising from a bilinear skew-symmetric mapping $A : H \times H \longrightarrow \mathbf{R}$ depends on the rank of the form A, where $\operatorname{rank} A$ is the rank of the matrix $(A(u_i, u_j))_{1 < i,j < \dim H}$, for any basis $\{u_i\}$ of H. The following lemma shows that the rank of the form $A_\gamma : \mathfrak{g}_\gamma/\mathfrak{n} \times \mathfrak{g}_\gamma/\mathfrak{n} \longrightarrow \mathbf{R}$, $A_\gamma(x, y) = \langle \gamma, [x, y] \rangle$, is constant on a nonempty, G-invariant Zariski open set E_1 of E. By passing to E_1, we obtain a Plancherel measure for G in which the dimension of the coordinate space of the fibers $(G_\gamma/N, \overline{\omega}_\gamma)^\wedge$ is constant.

LEMMA 4.4. *There is an integer l, $0 \leqslant l \leqslant q/2$, such that* $\operatorname{rank} A_\gamma = 2l$ *for all γ in a nonempty, G-invariant Zariski open set $E_1 \subset E$.*

PROOF. Let $\gamma \in E$. For $0 \leqslant k \leqslant q$, let $T_k(\gamma)$ be the set of all $k \times k$ minors of the matrix $(A_\gamma(u_a(\gamma), u_b(\gamma)))_{1 < a,b < q}$. From the definition of the $u_a(\gamma)$, each element of $T_k(\gamma)$ is a rational function of the form

$$R(\gamma) = (\det M^{(r)}(\gamma))^{-2K} P(\gamma).$$

where $P(\gamma)$ is a polynomial in $\gamma_1, \ldots, \gamma_K$. Since $R(\gamma) = 0 \Longleftrightarrow P(\gamma) = 0$, there is a family B_k of polynomial functions on E such that $\operatorname{rank} A_\gamma \geqslant k \Longleftrightarrow P(\gamma) \neq 0$ for some $P \in B_k$. Therefore, the set $Z_k = \{\gamma \in E : \operatorname{rank} A_\gamma \geqslant k\}$ is a Zariski open set in E. Let l_1 be the largest integer, $0 \leqslant l_1 \leqslant q$, such that Z_{l_1} is not empty. If $l_1 < k \leqslant q$, then Z_k is empty; so $\operatorname{rank} A_\gamma < k$ for all γ in E. But $\gamma \in Z_{l_1} \Rightarrow \operatorname{rank} A_\gamma \geqslant l_1$. Therefore, $\gamma \in Z_{l_1} \Longleftrightarrow \operatorname{rank} A_\gamma = l_1$. Since A_γ is skew-symmetric, $l_1 = 2l$. Let $E_1 = Z_{l_1} = \{\gamma \in E : \operatorname{rank} A_\gamma = 2l\}$.

To show that E_1 is G-invariant, let $\gamma \in E$ and $x \in G$. Since $\mathfrak{g}_{\gamma \cdot x} = \operatorname{Ad} x^{-1}(\mathfrak{g}_\gamma)$, $\{\operatorname{Ad} x^{-1}(u_a(\gamma)) : 1 \leqslant a \leqslant q\}$ is a basis of $\mathfrak{g}_{\gamma \cdot x}/\mathfrak{n}$. The following calculation shows that $\operatorname{rank} A_{\gamma \cdot x} = \operatorname{rank} A_\gamma$:

$$\begin{aligned} &A_{\gamma \cdot x}(\operatorname{Ad} x^{-1}(u_a(\gamma)), \operatorname{Ad} x^{-1}(u_b(\gamma))) \\ &\quad = \langle \gamma \cdot x, [\operatorname{Ad} x^{-1}(u_a(\gamma)), \operatorname{Ad} x^{-1}(u_b(\gamma))] \rangle \\ &\quad = \langle \gamma \cdot x, \operatorname{Ad} x^{-1}([u_a(\gamma), u_b(\gamma)]) \rangle \\ &\quad = \langle \gamma, [u_a(\gamma), u_b(\gamma)] \rangle = A_\gamma(u_a(\gamma), u_b(\gamma)). \end{aligned}$$

Thus E_1 is a nonempty, G-invariant, Zariski open subset of E.

Let $W_1 = s^{-1}(E_1)$. W_1 is a nonempty Zariski open subset of W, and $\operatorname{rank} A_{sy} = 2l$ for all $y \in W_1$. Since $\chi_W m_{\mathbf{R}^d} = \chi_{W_1} m_{\mathbf{R}^d}$, the disintegration formula (4.7) may be written as

$$\mu_N = (2\pi)^{-K} \int_{W_1} \nu_{sy} \, |\det M^{(r)}(sy)| \, dm_{\mathbf{R}^d}(y). \tag{4.8}$$

By §1 for $y \in W_1$, the map $\psi_{P_{sy}} : \mathbf{R}^m \longrightarrow (G_{sy}/N, \bar{\omega}_{sy})^\wedge$ is a homeomorphism, where $m = q - 2l$; and (4.4) in step (7),

$$\mu_{sy} = |\det P_{sy}|^{-1} (2\pi)^{-(l+m)} \psi_{P_{sy}}(m_{\mathbf{R}^m}),$$

is the projective Plancherel measure on $(G_{sy}/N, \bar{\omega}_{sy})^\wedge$ corresponding to the Haar measures $m_{G_{sy}/N}$ on G_{sy}/N.

Since $m_{G_{sy}/N}$ satisfies (4.3) with respect to the orbit measure ν_{sy} in the disintegration formula (4.8), Kleppner and Lipsman's Plancherel formula for group extensions (4.1) [15, Theorem 2.3, p. 108] says that (4.5),

$$\mu_G = (2\pi)^{-K} \int_{W_1} \mu_{sy} \, |\det M^{(r)}(sy)| \, dm_{\mathbf{R}^d}(y),$$

is Plancherel measure on $\hat{G}$ corresponding to Haar measure m_G on G, and that formula (4.6) is a Plancherel formula for G.

Table I: Plancherel formulas. Plancherel formulas computed in [23] are summarized here. For each group $G = \exp \mathfrak{g}$, data are listed in the following order

(1) A Jordan-Hölder basis $B = \{e_i : 1 \leqslant i \leqslant \dim \mathfrak{g}\}$. (The basis of $\mathfrak{g}'$ dual to B is denoted $\{e^i : 1 \leqslant i \leqslant \dim \mathfrak{g}\}$.)

(2) Nonzero vectors in the set $\{[x, y] : x, y \in B\}$.

(3) A basis of $\mathfrak{n}$, the idyll of $\mathfrak{g}$ used to compute μ_G. ($N = \exp \mathfrak{n} \lhd G$.)

(4) A basis of $\mathfrak{g}_\gamma/\mathfrak{n}$, where $\mathfrak{g}_\gamma = \{x \in \mathfrak{g} : \langle \gamma, [x, n] \rangle = 0 \ \forall n \in \mathfrak{n}\}$ for $\gamma \in E$ ($E = \{\gamma \in \mathfrak{n}' : \det M^{(r)}(\gamma) \neq 0\}$ as in §3 and Theorem 4.1.)

(5) The Plancherel formula,

$$\int_G |f|^2 = \int_{W_1} \int_{\mathbf{R}^m} \operatorname{tr}[\pi_{sy,t}(f * f^*)] \, dm_{\mathbf{R}^m}(t) R(y) \, dm_{\mathbf{R}^d}(y),$$

$f \in L^1(G) \cap L^2(G)$.

In each case, $\int_G |f|^2$ denotes the $\int_G |f(x)|^2 \, dm_G(x)$, where m_G is the Haar measure on G defined in terms of the basis B of $\mathfrak{g}$ (as in §2). $R(y)$ is the rational function of y defined in Theorem 4.1. d is the codimension of a maximal dimension orbit in $\mathfrak{n}'$ under the coadjoint representation of G in $\mathfrak{n}'$. $s : \mathbf{R}^d \longrightarrow \mathfrak{n}'$ is the section for the orbits of G in $\mathfrak{n}'$ used to compute μ_G. $W = \{y \in \mathbf{R}^d : \det M^{(r)}(sy) \neq 0\}$. For $y \in W_1 \subset W$, $\pi_{sy,t} = \operatorname{ind}_{G_{sy}}^G (\chi_{sy})' \otimes (\psi_{P_{sy}}(t))''$ (Theorem 4.1) is an irreducible representation of $\mathfrak{g}$ for $t \in \mathbf{R}^m$.

The following procedure gives most of the idylls listed below. Let $\mathfrak{z}_1 \subset \cdots \subset \mathfrak{z}_n = \mathfrak{g}$ be the ascending central series of $\mathfrak{g}$. Let $\mathfrak{n}_1 = \mathfrak{z}_1$. Having chosen $\mathfrak{n}_i$, let $\mathfrak{n}_{i+1}$ be a maximal dimensional abelian subalgebra of $\mathfrak{z}_{i+1}$ containing $\mathfrak{n}_i$.

Then $\mathfrak{n} = \mathfrak{n}_n$. It is a conjecture that if $\mathfrak{g}$ is idyllic, then the maximal abelian ideal $\mathfrak{n}$ of $\mathfrak{g}$ obtained in this way is an idyll.

A. Heisenberg groups, H_n

(1) $\{e_1, \ldots, e_{2n}, e_{2n+1}\}$

(2) $[e_i, e_{n+i}] = -[e_{n+i}, e_i] = e_{2n+1}, 1 \leqslant i \leqslant n$

(3) $\{e_{n+1}, \ldots, e_{2n}, e_{2n+1}\}$

(4) $\{0\}$

(5) $\int_G |f|^2 = (2\pi)^{-(n+1)} \int_W \operatorname{tr}[\pi_{sy}(f * f^*)] \, |y|^n \, dm_{\mathbf{R}}(y)$

$s : \mathbf{R} \longrightarrow \mathfrak{n}', sy = ye^{2n+1}$

$W = \mathbf{R} - \{0\}$

$\pi_{sy} = \operatorname{ind}_N^G \chi_{sy}$

$\chi_{sy}(\exp(\Sigma_{i=n+1}^{2n+1} x^i e_i)) = e^{iyx^{2n+1}}$

B. Kirillov's second example [12, p. 102]

(1) $\{e_0, \ldots, e_n\}$

(2) $[e_0, e_i] = -[e_i, e_0] = e_{i+1}, 1 \leqslant i \leqslant n-1$

(3) $\{e_1, \ldots, e_n\}$

(4) $\{0\}$

(5) $\int_G |f|^2 = (2\pi)^{-n} \int_W \operatorname{tr}[\pi_{sy}(f * f^*)] \, |y_{n-1}| \, dm_{\mathbf{R}^{n-1}}(y)$

$s : \mathbf{R}^{n-1} \longrightarrow \mathfrak{n}'$,

$s(y_1, \ldots, y_{n-1}) = y_1 e^1 + \cdots + y_{n-2} e^{n-2} + y_{n-1} e^n$

$W = \{y = (y_1, \ldots, y_{n-1}) : y_{n-1} \neq 0\}$

$\pi_{sy} = \operatorname{ind}_N^G \chi_{sy}$

$\chi_{sy}(\exp(\Sigma_{i=1}^n x^i e_i)) = e^{i(y_1 x^1 + \cdots + y_{n-2} x^{n-2} + y_{n-1} x^n)}$

C. Groups of dimension $\leqslant 5$

These are the groups $\Gamma = \exp \mathfrak{g}$, where $\mathfrak{g}$ is one of the algebras listed by Dixmier [9, Proposition 1, p. 323]. The Plancherel formula is given here for those groups which are not products.

$\Gamma_1 = \mathbf{R}$.

$$\int_{\Gamma_1} |f|^2 = \frac{1}{2\pi} \int_{\mathbf{R}} \chi_y(f * f^*) \, dm_{\mathbf{R}}(y).$$

$$\chi_y(x) = e^{iy \cdot x} x, y \in \mathbf{R}.$$

$\Gamma_3 = H_1$.

(1) $\{e_1, e_2, e_3\}$

(2) $[e_1, e_2] = -[e_2, e_1] = e_3$

(3) $\{e_2, e_3\}$

(4) $\{0\}$

(5) $\int_{\Gamma_3} |f|^2 = (2\pi)^{-2} \int_W \operatorname{tr}[\pi_{sy}(f * f^*)] \, |y| \, dm_{\mathbf{R}}(y)$

$s : \mathbf{R} \longrightarrow \mathfrak{n}', sy = ye^3$
$W = \mathbf{R} - \{0\}$
$\pi_{sy} = \operatorname{ind}_N^G \chi_{sy}$
$\chi_{sy}(\exp(x^2 e_2 + x^3 e_3)) = e^{iy \cdot x^3}$

Dimension 4: Γ_4

(1) $\{e_1, e_2, e_3, e_4\}$
(2) $[e_1, e_2] = -[e_2, e_1] = e_3$
$[e_1, e_3] = -[e_3, e_1] = e_4$
(3) $\{e_2, e_3, e_4\}$
(4) $\{0\}$
(5) $\int_{\Gamma_4} |f|^2 = (2\pi)^{-3} \int_W \operatorname{tr}[\pi_{sy}(f * f^*)] \, |y_2| \, dm_{\mathbf{R}^2}(y_2, y_4)$
$s : \mathbf{R}^2 \longrightarrow \mathfrak{n}', s(y) = y_2 e^2 + y_4 e^4$
$W = \{y = (y_2, y_4) : |y_4| \neq 0\}$
$\pi_{sy} = \operatorname{ind}_N^G \chi_{sy}$
$\chi_{sy}(\exp(x^2 e_2 + x^3 e_3 + x^4 e_4)) = e^{i(y_2 x^2 + y_4 x^4)}$

$\Gamma_{5,1}$

(1) $\{e_1, e_2, e_3, e_4, e_5\}$
(2) $[e_1, e_2] = -[e_2, e_1] = e_5$
$[e_3, e_4] = -[e_4, e_3] = e_5$
(3) $\{e_2, e_4, e_5\}$
(4) $\{0\}$
(5) $\int_{\Gamma_{5,1}} |f|^2 = (2\pi)^{-3} \int_W \operatorname{tr}[\pi_{sy}(f * f^*)] \, y^2 \, dm_{\mathbf{R}}(y)$
$s : \mathbf{R} \longrightarrow \mathfrak{n}', sy = ye^5$
$W = \mathbf{R} - \{0\}$
$\pi_{sy} = \operatorname{ind}_N^G \chi_{sy}$
$\chi_{sy}(\exp(x^2 e_2 + x^4 e_4 + x^5 e_5)) = e^{iy \cdot x^5}$

$\Gamma_{5,2}$

(1) $\{e_1, e_2, e_3, e_4, e_5\}$
(2) $[e_1, e_2] = -[e_2, e_1] = e_4$
$[e_1, e_3] = -[e_3, e_1] = e_5$
(3) $\{e_2, e_3, e_4, e_5\}$
(4) $\{0\}$
(5) $\int_{\Gamma_{5,2}} |f|^2 = (2\pi)^{-4} \int_W \operatorname{tr}[\pi_{sy}(f * f^*)] \, |y_5| \, dm_{\mathbf{R}^3}(y_2, y_4, y_5)$
$s : \mathbf{R}^3 \longrightarrow \mathfrak{n}', s(y_2, y_4, y_5) = y_2 e^2 + y_4 e^4 + y_5 e^5$
$W = \{y = (y_2, y_4, y_5) : y_5 \neq 0\}$
$\pi_{sy} = \operatorname{ind}_N^G \chi_{sy}$
$\chi_{sy}(\exp(x^2 e_2 + x^3 e_3 + x^4 e_4 + x^5 e_5)) = e^{i(y_2 x^2 + y_4 x^4 + y_5 x^5)}$

$\Gamma_{5,3}$

(1) $\{e_1, e_2, e_3, e_4, e_5\}$

(2) $[e_1, e_2] = -[e_2, e_1] = e_4$
$[e_1, e_4] = -[e_4, e_1] = e_5$
$[e_2, e_3] = -[e_3, e_2] = e_5$

(3) $\{e_3, e_4, e_5\}$

(4) $\{0\}$

(5) $\int_{\Gamma_{5,3}} |f|^2 = (2\pi)^{-3} \int_W \operatorname{tr}[\pi_{sy}(f * f^*)]\, |y|^2\, dm_{\mathbf{R}}(y)$
$s : \mathbf{R} \longrightarrow \mathfrak{n}', sy = ye^5$
$W = \mathbf{R} - \{0\}$
$\pi_{sy} = \operatorname{ind}_N^G \chi_{sy}$
$\chi_{sy}(\exp(x^3 e_3 + x^4 e_4 + x^5 e_5)) = e^{iy \cdot x^5}$

$\Gamma_{5,4}$

(1) $\{e_1, e_2, e_3, e_4, e_5\}$

(2) $[e_1, e_2] = -[e_2, e_1] = e_3$
$[e_1, e_3] = -[e_3, e_1] = e_4$
$[e_2, e_3] = -[e_3, e_2] = e_5$

(3) $\{e_3, e_4, e_5\}$

(4) $\operatorname{span}_{\mathbf{R}}\{e_1 - (\langle \gamma, e_4 \rangle / \langle \gamma, e_5 \rangle) e_2\}$

(5) $\int_{\Gamma_{5,4}} |f|^2 = (2\pi)^{-4} \int_W \int_{\mathbf{R}} \operatorname{tr}[\pi_{sy,t}(f * f^*)]\, dt\, |y_5|\, dm_{\mathbf{R}^2}(y_4, y_5)$
$s : \mathbf{R}^2 \longrightarrow \mathfrak{n}', s(y_4, y_5) = y_4 e^4 + y_5 e^5$
$W = \{y = (y_4, y_5) : y_5 \neq 0\}$
$\pi_{sy,t} = \operatorname{ind}_{G_{sy}}^G \chi'_{sy} \otimes (\chi_t)''$
$\chi_{sy}(\exp(x^3 e_3 + x^4 e_4 + x^5 e_5)) = e^{i(y_4 x^4 + y_5 x^5)}$
$\chi_t(\exp \alpha(e_1 - (y_4/y_5) e_2)) = e^{it \cdot \alpha}, \alpha, t \in \mathbf{R}$

$\Gamma_{5,5}$

(1) $\{e_1, e_2, e_3, e_4, e_5\}$

(2) $[e_1, e_2] = -[e_2, e_1] = e_3$
$[e_1, e_3] = -[e_3, e_1] = e_4$
$[e_1, e_4] = -[e_4, e_1] = e_5$

(3) $\{e_2, e_3, e_4, e_5\}$

(4) $\{0\}$

(5) $\int_{\Gamma_{5,5}} |f|^2 = (2\pi)^{-4} \int_W \operatorname{tr}[\pi_{sy}(f * f^*)]\, |y_5|\, dm_{\mathbf{R}^3}(y_2, y_3, y_5)$
$s : \mathbf{R}^3 \longrightarrow \mathfrak{n}' s(y_2, y_3, y_5) = y_2 e^2 + y_3 e^3 + y_5 e^5$
$W = \{y = (y_2, y_3, y_5) : |y_5| \neq 0\}$
$\pi_{sy} = \operatorname{ind}_N^G \chi_{sy}$
$\chi_{sy}(\exp(x^2 e_2 + x^3 e_3 + x^4 e_4 + x^5 e_5)) = e^{i(y_2 x^2 y_3 x^3 + y_5 x^5)}$

$\Gamma_{5,6}$

(1) $\{e_1, e_2, e_3, e_4, e_5\}$

(2) $[e_1, e_2] = -[e_2, e_1] = e_3$
$[e_1, e_3] = -[e_3, e_1] = e_4$
$[e_1, e_4] = -[e_4, e_1] = e_5$
$[e_2, e_3] = -[e_3, e_2] = e_5$

(3) $\{e_3, e_4, e_5\}$

(4) $\{0\}$

(5) $\int_{\Gamma_{5,6}} |f|^2 = (2\pi)^{-3} \int_W \operatorname{tr}[\pi_{sy}(f * f^*)]\, |y|^2\, dm_{\mathbf{R}}(y)$
$s : \mathbf{R} \longrightarrow \mathfrak{n}', sy = ye^5$
$W = \mathbf{R} - \{0\}$
$\pi_{sy} = \operatorname{ind}_N^G \chi_{sy}$
$\chi_{sy}(\exp(x^3 e_3 + x^4 e_4 + x^5 e_5)) = e^{iy \cdot x^5}$

D. Two-step groups

(1) $\{e_1, \ldots, e_S\} \cup \{v_1, \ldots, v_K\}$

(2) $[e_i, e_j] = -[e_j, e_i] \subset \operatorname{span}\{v_1, \ldots, v_K\}, 1 \leqslant i < j \leqslant S$

(3) $\{v_1, \ldots, v_K\} =$ center of $\mathfrak{g}$

(4) $\{e_1, \ldots, e_S\}$

(5) $$\int_G |f|^2 = (2\pi)^{-(l+m+K)} \int_{W_1} \int_{\mathbf{R}^m} \operatorname{tr}[\pi_{sy,t}(f * f^*)]\, dm_{\mathbf{R}^m}(t)\, |\det P_{sy}|^{-1}\, dm_{\mathbf{R}^K}(y).$$

$s : \mathbf{R}^K \longrightarrow \mathfrak{n}', s(y_1, \ldots, y_K) = \Sigma_{j=1}^K y_j v^j$
$2l = \max\{\operatorname{rank}_{\mathbf{R}}(\langle \gamma, [e_i, e_j]\rangle)_{1 \leqslant i,j \leqslant S} : \gamma \in \mathfrak{n}'\}$
$m = s - 2l$
$W_1 = \{y = (y_1, \ldots, y_K) \in \mathbf{R}^K : \operatorname{rank}_{\mathbf{R}}(\langle sy, [e_i, e_j]\rangle)_{1 \leqslant i,j \leqslant S} = 2l\}$

For $y \in W_1$, P_{sy} is a nonsingular $S \times S$ matrix such that

$$P_{sy}(\langle sy, [e_i, e_j]\rangle)_{1 \leqslant i,j \leqslant S}\, {}^t P_{sy} = \begin{array}{c c} \left[\begin{array}{cc|c} 0 & I_l & \\ -I_l & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right] & \begin{array}{l} \}\, 2l \\ \}\, m \end{array} \\ \underbrace{\qquad\quad}_{2l}\ \underbrace{\quad}_{m} & \end{array}.$$

For $(y, t) \in W_1 \times \mathbf{R}^m$,

$$\pi_{sy,t} = (\chi_{sy})' \otimes (\psi_{P_{sy_i}}(t))''; \quad \chi_{sy}\left(\exp\left(\sum_{j=1}^K u^j v_j\right)\right) = e^{i \Sigma_{j=1}^K y_j u^j};$$

$$\psi_{P_{sy}}(t)\left(\exp\left(\sum_{i=1}^{S} x^i e_i\right)\right)$$

$$= \sigma_1\left(\left(\sum_{i=1}^{S} x^i Q_i^1, \ldots, \sum_{i=1}^{S} x^i Q_i^l\right),\right.$$

$$\left.\left(\sum_{i=1}^{S} x^i Q_i^{l+1}, \ldots, \sum_{i=1}^{S} x^i Q_i^{2l}\right)\right) e^{i\Sigma_{a=1}^{m}\Sigma_{i=1}^{S} x^i Q_i^{2l+a} t_a},$$

where $(Q_i^j)_{1<i,j<S} = P_{sy}^{-1}$.

E1. Nilpotent part of G_2I (see [10, [11], [21])

(1) $\{e_1, e_2, e_3, e_4, e_5, e_6\}$

(2) $[e_1, e_2] = -[e_2, e_1] = e_3$
$[e_1, e_3] = -[e_3, e_1] = e_4$
$[e_1, e_4] = -[e_4, e_1] = e_5$
$[e_2, e_5] = -[e_5, e_2] = e_6$
$[e_3, e_4] = -[e_4, e_3] = -e_6$

(3) $\{e_4, e_5, e_6\}$

(4) $\{e_1 + \langle\gamma, e_5\rangle/\langle\gamma, e_6\rangle e_3\}$

(5) $\int_G |f|^2 = (2\pi)^{-4}\int_W\int_{\mathbf{R}} \operatorname{tr}[\pi_{sy,t}(f * f^*)]\, dm_{\mathbf{R}}(t)\, |y|^2\, dm_{\mathbf{R}}(y)$
$s:\mathbf{R} \longrightarrow \mathfrak{n}'$, $sy = ye^6$
$W = \mathbf{R} - \{0\}$
$\pi_{sy,t} = \operatorname{ind}_{G_{sy}}^{G}(\chi_{sy})' \otimes (\chi_t)''$
$\chi_{sy}(\exp(x^4 e_4 + x^5 e_5 + x^6 e_6)) = e^{iyx_6}$
$\chi_t(\exp \lambda e_1) = e^{i\lambda t}$, $\lambda, t \in \mathbf{R}$

E2a. Nilpotent part of A_lI, $l + 1 = 2m$

(1) $\{e_{ij} : 1 \leq i < j \leq 2m\}$

(2) $[e_{ir}, e_{rj}] = -[e_{rj}, e_{ir}] = e_{ij}$, $1 \leq i < r < j \leq 2m$

(3) $\{e_{ij} : 1 \leq i \leq m,\ m+1 \leq j \leq 2m\}$

(4) $\{0\}$

(5) $\int_G |f|^2 = (2\pi)^{-(m^2)}\int_W \operatorname{tr}[\pi_{sy}(f * f^*)]\, \Pi_{k=1}^{m-1} |y_{k,2m+1-k}|^{2(m-k)}$
$dm_{\mathbf{R}^m}(y_{1,2m}, \ldots, y_{m-1,m+2}, y_{m,m+1})$
$s:\mathbf{R}^m \longrightarrow \mathfrak{n}'$, $s(y_{1,2m}, \ldots, y_{m,m+1}) = \Sigma_{k=1}^{m} y_{k,2m+1-k} e^{k,2m+1-k}$
$W = \{y = (y_{1,2m}, \ldots, y_{m,m+1}) : \Pi_{k=1}^{m-1} |y_{k,2m+1-k}| \neq 0\}$
$\pi_{sy} = \operatorname{ind}_N^G \chi_{sy}$
$\chi_{sy}(\exp(\Sigma_{1<i<m;m+1<j<2m} x^{ij} e_{ij})) = e^{i\Sigma_{k=1}^{m} y_{k,2m+1-k} x^{k,2m+1-k}}$

E2b. NILPOTENT PART OF $A_l I$, $l+1 = 2m-1$

(1) $\{e_{ij} : 1 \leqslant i < j \leqslant 2m-1\}$

(2) $[e_{ir}, e_{rj}] = -[e_{rj}, e_{ir}] = e_{ij}$, $1 \leqslant i < r < j \leqslant 2m-1$

(3) $\{e_{ij} : 1 \leqslant i \leqslant m,\ m+1 \leqslant j \leqslant 2m-1\}$

(4) $\{0\}$

(5) $$\int_G |f|^2 = (2\pi)^{-m(m-1)} \int_W \operatorname{tr}[\pi_{sy}(f * f^*)] \prod_{k=1}^{m-1} |y_{k,2m-k}|^{2(m-k)-1} \, dm_{\mathbf{R}^{m-1}}(y_{1,2m-1}, \ldots, y_{m-1,m+1})$$

$s : \mathbf{R}^{m-1} \longrightarrow \mathfrak{n}'$, $s(y_{1,2m-1}, \ldots, y_{m-1,m+1}) = \sum_{k=1}^{m-1} y_{k,2m-k} e^{k,2m-k}$

$W = \{y = (y_{1,2m-1}, \ldots, y_{m-1,m+1}) : \prod_{k=1}^{m-1} |y_{k,2m-k}| \neq 0\}$.

$\pi_{sy} = \operatorname{ind}_N^G \chi_{sy}$,

$\chi_{sy}(\exp(\sum_{1 \leqslant i \leqslant m; m+1 \leqslant j \leqslant 2m-1} x^{ij} e_{ij})) = e^{i \sum_{k=1}^{m-1} y_{k,2m-k} x^{k,2m-k}}$

E3. NILPOTENT PART OF $C_l I$

(1) $\{a_{ij} : 1 \leqslant i < j \leqslant l\} \cup \{b_{ij} : 1 \leqslant i \leqslant l,\ 2l+1-i \leqslant j \leqslant 2l\}$

(2) $[a_{ij}, a_{jk}] = -[a_{jk}, a_{ij}] = a_{ik}$, $1 \leqslant i < j < k \leqslant l$

For $1 \leqslant i < j \leqslant l$; $1 \leqslant t \leqslant l$, $2l+1-t \leqslant s \leqslant 2l$,

$$[a_{ij}, b_{ts}] = -[b_{ts}, a_{ij}] = \begin{cases} 2b_{t,2l+1-i} & \text{if } t = j = 2l+1-s \\ b_{2l+1-s,2l+1-i} & \text{if } t = j > 2l+1-s \text{ and } 2l+1-s \geqslant i \\ b_{is} & \text{if } t = j > 2l+1-s \text{ and } 2l+1-s < i \\ b_{t,2l+1-i} & \text{if } t > j = 2l+1-s \end{cases}$$

(3) $\{b_{ij} : 1 \leqslant i \leqslant l,\ 2l+1-i \leqslant j \leqslant 2l\}$

(4) $\{0\}$

(5) $$\int_G |f|^2 = (2\pi)^{-l(l+1)/2} \int_W \operatorname{tr}[\pi_{sy}(f * f^*)] \prod_{k=1}^{l-1} |y_{k,2l+1-k}|^{l-k} \, dm_{\mathbf{R}^l}(y_{1,2l}, \ldots, y_{l,l+1})$$

$s : \mathbf{R}^l \longrightarrow \mathfrak{n}'$, $s(y_{1,2l}, \ldots, y_{l,l+1}) = \sum_{k=1}^{l} y_{k,2l+1-k} b^{k,2l+1-k}$

$W = \{y = (y_{1,2l}, \ldots, y_{l,l+1}) : \prod_{k=1}^{l-1} |y_{k,2l+1-k}| \neq 0$

$\pi_{sy} = \operatorname{ind}_N^G \chi_{sy}$

$\chi_{sy}(\exp(\sum_{1 \leqslant i \leqslant l; 2l+1-i \leqslant j \leqslant 2l} x^{ij} b_{ij})) = e^{i \sum_{k=1}^{l} y_{k,2l+1-k} x^{k,2l+1-k}}$

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