TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 224, Number 1, 1976

THE PRIMITIVE LIFTING PROBLEM IN THE EQUIVALENCE PROBLEM FOR TRANSITIVE PSEUDOGROUP STRUCTURES: A COUNTEREXAMPLE

BY

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ABSTRACT. A transitive Lie pseudogroup Γ_M on M is a primitive extension of Γ_N if Γ_N is the quotient of Γ_M by an invariant fibration $\pi: M \longrightarrow N$ and if the pseudogroup induced by Γ_M on the fiber of π is primitive. In the present paper an example of this situation is given with the following property (counterexample to the primitive lifting property): the equivalence theorem is true for almost-Γ_N-structures but false for almost-Γ_M-structures.

1. We shall consider the following situation: Let $\pi: M \longrightarrow N$ be a fibration of smooth manifolds, Γ_M a transitive Lie pseudogroup on M respecting the fibration π. It is not always true that a quotient pseudogroup on N can be defined, but by projecting the equations of Γ_M (that is to say the associated structures) we obtain the equations of a transitive Lie pseudogroup Γ_N on N. Γ_N will be referred to as the pseudogroup defined by passing to the quotient. In the study of the equivalence problem for almost-Γ_M-structures (see [1], [3], [4]), it is a standard method to use such quotients. If the equivalence theorem is true for almost-Γ_N-structures, any almost-Γ_M-structure defines a quotient almost-Γ_N-structure, and the given equivalence problem reduces to a "lifting problem" from an equivalence for the quotient structure to an equivalence for the given structure.

Let K be a fiber of π, Γ_K the family of restrictions to K of those transformations in Γ_M which map K into K. We assume here that Γ_K is a flat irreducible transitive Lie pseudogroup on K. Then Γ_K is either an affine pseudogroup (Γ_M is an "affine extension" of Γ_N) or a primitive pseudogroup (Γ_M is a "primitive extension" of Γ_N).

The purpose of this paper is to give an example where Γ_M is a primitive extension of Γ_N and the equivalence theorem is true for Γ_N but false for Γ_M. In the terminology of A. Pollack (see [4]), the "primitive lifting theorem for $\Gamma_M \longrightarrow \Gamma_N$" is not true.

Received by the editors July 25, 1975.

AMS (MOS) subject classifications (1970). Primary 58A15; Secondary 53C10, 57D55.

Key words and phrases. Equivalence problem, almost-Γ-structures, primitive Lie pseudogroups, Lewy counterexample, equivalence theorem, flat pseudogroups, primitive lifting theorem.

In the paper just mentioned, A. Pollack asserts that, for Γ_M and Γ_N flat, "the primitive lifting theorem can be established with little difficulty", while "the affine lifting theorem requires deep results from partial differential equation theory" (essentially Ehrenpreis-Malgrange's theorem on partial differential equations with constant coefficients).

As a matter of fact, if Γ_K is a primitive simple pseudogroup, the lifting theorem can be easily proved by using the results of V. Guillemin (see [2]).

If Γ_K is primitive but not simple, the lifting theorem is true in the flat case (Γ_M and Γ_N flat) but the demonstration requires, as in the affine case, Ehrenpreis-Malgrange's results. The method has been indicated in [1].

2. Construction of the counterexample. The counterexample is constructed by a refinement of the Lewy-Guillemin-Sternberg counterexample to the equivalence problem for G-structures [3].

(a) First, recall the data of this counterexample: We consider on $\mathbf{R}^3$ a transitive algebra of vector fields isomorphic to $g = SO(3, \mathbf{R})$. $\{X_1, X_2, X_3\}$ is the standard basis of this algebra, that is to say, X_1, X_2, X_3 are globally defined vector fields on $\mathbf{R}^3$ satisfying the following relations:

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_1, \quad [X_2, X_3] = -X_2.$$

Let $\{Y_1, Y_2\}$ be the standard basis of vector fields on $\mathbf{R}^2$. $\hat{E}$ is the $\hat{G}$-structure on $\mathbf{R}^2 \times \mathbf{R}^3$ defined by the moving frame $\{Y_1, Y_2, X_1, X_2, X_3\}$, where $\hat{G}$ is the group of matrices of the form

$$\begin{bmatrix} 1 & 0 & a & b & c \\ 0 & 1 & d & e & f \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

E is the G-structure on $\mathbf{R}^2 \times \mathbf{R}^3$ defined by the same moving frame, where G is the group of matrices of the form

$$\begin{bmatrix} 1 & 0 & a & b & c \\ 0 & 1 & -b & a & d \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We denote by $\hat{\gamma}$ (resp. γ) the pseudogroup of automorphisms of $\hat{E}$ (resp. E) and by $\hat{l}$ (resp. l) the associated Lie algebra sheaf, in the terminology of [5].

$\hat{l}$ is a Lie algebra sheaf of vector fields of the following type:

$$X = \xi + \phi_1(x_1, x_2, x_3)Y_1 + \phi_2(x_1, x_2, x_3)Y_2 \tag{1}$$

where ξ is in the algebra of (local) right-invariant vectorfields on $\mathbf{R}^3 \subset SO(3, \mathbf{R})$.

X is in l if and only if φ_1, φ_2 satisfy a system (Σ_l) of linear partial differential equations.

(b) Let us consider now, for $p \geqslant 2$, the product $\mathbf{C}^p \times \mathbf{R}^3$ endowed with the $\tilde{H}$-structure $\tilde{E}$ obtained from the moving frame $\{\partial/\partial z_1, \ldots, \partial/\partial z_p, X_1, X_2, X_3\}$ by linear transformations in the group $\tilde{H}$ of matrices of the following type:

$$\left[\begin{array}{ccc|ccc} & \alpha_j^i & & \times & \times & \times \\ & & & \cdot & \cdot & \cdot \\ & & & \times & \times & \times \\ \hline 0 & \ldots & 0 & 1 & 0 & 0 \\ \cdot & \cdots & \cdot & 0 & 1 & 0 \\ 0 & \ldots & 0 & 0 & 0 & 1 \end{array}\right], \quad \text{with } [\alpha_j^i] \in GL(p, \mathbf{C}).$$

Let $\tilde{\Gamma}$ be the pseudogroup of automorphisms of this structure, $\tilde{L}$ the associated *LAS*. We consider the projection $\tilde{\pi}$: $\tilde{E} \longrightarrow \mathbf{R}^2 \times \mathbf{R}^3$ defined by:

$$\tilde{\pi}(\tilde{z}(z, x)) = (\det(\alpha_j^i), x) \quad \text{where } \tilde{z} = \left\{ \sum_i \alpha_j^i \frac{\partial}{\partial z_j}, \tilde{X}_1(x), \tilde{X}_2(x), \tilde{X}_3(x) \right\}.$$

If $\tilde{Y}_1 = \Sigma_{k=1}^p z_k \partial/\partial z_k$ and $\tilde{Y}_2 = \Sigma_{k=1}^p iz_k\partial/\partial z_k$, let $\hat{L}$ be the *LAS* of vector fields on $\mathbf{C}^p \times \mathbf{R}^3$ of the following type:

$$\tilde{X} = \sum_{k=1}^p X_k(x, z)\frac{\partial}{\partial z_k} + \varphi_1(x_1, x_2, x_3)\tilde{Y}_1 + \varphi_2(x_1, x_2, x_3)\tilde{Y}_2 + \xi \tag{2}$$

where X_k is holomorphic with respect to z, with the sole condition $\Sigma_k \partial X_k/\partial z_k = 0$.

$\hat{L}$ defines a transitive Lie pseudogroup $\hat{\Gamma}$ on $\mathbf{C}^p \times \mathbf{R}^3$, with $\hat{\Gamma} \subset \tilde{\Gamma}$. By the projection $\tilde{\pi}$, $\hat{\Gamma}$ acting on $\tilde{E}$ projects onto $\hat{\gamma}$. It is in this case a true quotient! The vector field (2) projects onto (1).

Let Γ be the preimage of γ by this projection $\hat{\Gamma} \longrightarrow \hat{\gamma}$. The associated *LAS* L is the *LAS* of vector fields of the type (2) with the condition (Σ_l) on φ_1, φ_2.

The fibration π: $\mathbf{C}^p \times \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ allows us to consider Γ as a primitive extension of $SO(3, \mathbf{R})$. The counterexample follows now from the following proposition:

PROPOSITION. *The equivalence theorem is not true for the pseudogroup* Γ.

PROOF. Let $\hat{E}^1$ be the second order structure associated to $\hat{\Gamma}$. We have a natural projection $\hat{\pi}^1$: $\hat{E}^1 \longrightarrow \hat{E}$. The second order structure E^1 associated to Γ is the preimage of E by $\hat{\pi}^1$.

If E' is an arbitrary G-subbundle of $\hat{E}$, let $E^{1\prime} = [\hat{\pi}^1]^{-1}(E')$. By [3], E' defines an almost-γ-structure on $\mathbf{R}^2 \times \mathbf{R}^3$. Then $E^{1\prime}$ will define an almost-Γ-structure on $\mathbf{C}^p \times \mathbf{R}^3$. If E' is not a γ-structure, $E^{1\prime}$ does not define a Γ-structure. Q. E. D.

3. **The primitive lifting theorem in the flat case.** Recall the principle of the demonstration in the flat case (following the method indicated in [1]): If Γ_K is not simple, we consider the subpseudogroup Γ'_K, the *LAS* of which is the derived algebra of the *LAS* of Γ_K. Then we can obtain a Γ'_K-extension Γ'_M of Γ_N, with $\Gamma'_M \subset \Gamma_M$.

In the first order structure associated to Γ_M, we have an invariant foliation such that the first order structure associated to Γ'_M is foliated. If π^1 is the (local) projection onto a (local) quotient, we resolve the equivalence problem for the pseudogroup defined by passing to the quotient. This allows us to obtain for any almost-Γ_M-structure a subordinate almost-Γ'_M-structure. Now, the primitive lifting theorem reduces to the simple case, which is elementary.

But the equivalence problem for the pseudogroup obtained by passing to the quotient by π^1 requires Ehrenpreis-Malgrange's theorem as an essential tool: in fact it is equivalent to the equivalence problem for a flat abelian pseudogroup.

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