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LIE ALGEBRAS OF TYPE BC_1

BY

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ABSTRACT. Let L be a central simple Lie algebra of type BC_1 with highest root space of dimension greater than one over a field of characteristic zero. It is shown that either L is isomorphic to the simple Lie algebra associated with a skew hermitian form of index one or L can be constructed from the tensor product of two composition algebras. This result is obtained by completing the description (begun in [3]) of the corresponding class of ternary algebras.

1. Introduction and statement of results. Let k be a field of characteristic zero. In [13], Seligman has developed methods for "coordinatizing" central simple nonanisotropic Lie algebras over k. These methods give rise to a complete description of all such algebras except those of type BC_1 and BC_2. In [2] and [3], it is shown that every algebra of type BC_1 can be constructed from a ternary algebra V which has no zero divisors, which is a module over a central Jordan division algebra J, and which possesses a skew map $V \times V \xrightarrow{\langle\, , \,\rangle} J$. If the highest root space of the Lie algebra has dimension 1, the corresponding ternary algebras have been studied in [4], [5] and [6]. If this dimension is greater than 1 and the module is not irreducible, the ternary algebras (and hence the Lie algebras) have been completely described in [3]. This paper deals then with the irreducible case. We use the notation and terminology of [2] except for the action of J on V which is denoted by $(a, x) \longrightarrow a \circ x$.

One construction of a J-ternary algebra goes as follows: Let A be an algebra with involution J and identity u. Let $S = \{x \in A: x^J = -x\}$ and fix $t \neq 0 \in S$. Put $V = A$ and $J = S$. Define $(1/2)(x, y, \tilde{z}) = (1/2)(xy^J)(tz) + (1/2)(y(z^Jt))x + (1/2)(x(z^Jt))y$ and $\langle x, y\rangle = (1/2)xy^J - (1/2)yx^J$ for $x, y, z \in V$. Define a product on J by $a \cdot b = (1/2)a(tb) + (1/2)b(ta)$ and an action of J on V by $(a, x) \longrightarrow a \circ x = a(tx)$. Whenever this system forms a J-ternary algebra, we call it the *J-ternary algebra associated with* (A, J). A somewhat lengthy calculation shows that this is the case at least in the following situations:

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(a) (A, J) is a central associative division algebra with involution.

(b) $(A, J) = (O \otimes_k C, J_1 \otimes_k J_2)$, where O is a Cayley division algebra, C is a composition division algebra, and J_1 and J_2 are the canonical involutions.

(c) $(A, J) \otimes_k L = (O_1 \otimes_L O_2, J_1 \otimes_L J_2)$ for some quadratic field extension L/k, where (O_i, J_i) is a Cayley division algebra over L with canonical involution, $i = 1, 2$, and the corresponding Galois action interchanges O_1 and O_2.

We are now in a position to state our main results:

THEOREM A. *Let V be a finite dimensional $\mathfrak{J}$-irreducible $\mathfrak{J}$-ternary algebra without zero divisors, where $\mathfrak{J}$ is a central division algebra. Then, V is the ternary algebra associated with an algebra (A, J) with involution described by* (a), (b), *or* (c).

THEOREM B. *Let L be a central simple Lie algebra of type BC_1 over k with highest root space of dimension greater than* 1. *Then, L is isomorphic to one of the following*:

(i) *The derived algebra of the algebra of skew transformations of a skew hermitian form h of index* 1 *defined on a vector space W over a central associative division algebra (A, J) with involution.*

(ii) $L(\mathfrak{J}, V)$, *where V is the $\mathfrak{J}$-ternary algebra constructed from an algebra with involution (A, J) included in* (b) *or* (c).

REMARKS. (1) The Lie algebras described in (i) are all of type BC_1. However, those described in (ii) may have rank > 1. In fact the rank depends on the interplay between maximal subfields of the composition algebras (see §6).

(2) The basic tool used in the proof of Theorem A is the classification of central simple alternative algebras [12].

(3) The Lie algebras arising from case (b) can be obtained from Tits' second construction using O and a reduced Jordan algebra of degree 3 coordinatized by C [14].

(4) In [9], Kantor gives a classification over an algebraically closed field of a related class of ternary algebras. This classification involves a careful analysis of the root structure of the corresponding Lie algebra. Some of the ternary algebras that arise there do not occur in Theorem A as the Lie algebras have no forms of type BC_1.

In §2, we describe how the algebra (A, J) is constructed from the given ternary algebra and show how the ternary operations can be recovered from (A, J). In §3, we develop the properties of (A, J) that enable us in §§4 and 5 to give proofs of Theorems A and B. In §6, we note some further restrictions imposed on the algebras of (b) and (c) by the rank 1 assumption and we describe the algebras that can occur over some special fields.

Throughout the paper J will denote a finite dimensional central Jordan division algebra over k with identity e and V will denote a nonzero finite dimensional J-ternary algebra with product $\langle\ ,\ ,\ \rangle$ and skew mapping $V \times V \xrightarrow{\langle\ ,\ \rangle} J$. We also assume throughout that V has no nonzero zero divisors and that V is J-irreducible. We put $L = L(J, V) = \overline{J} \oplus \overline{V} \oplus L_0 \oplus J \oplus V$, where $L_0 = \mathrm{St}(J, V) = \mathrm{Inst}(J, V) = R_J \oplus \mathrm{Der}(J, V)$ (see Proposition 1 of [3]).

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2. **Identification of the operations.** We fix an element $u \neq 0 \in V$. As in [3], choose $v \in V$ such that $[u, \overline{v}] = 4R_e$ and define $U = \mathrm{ad}(u)$, $V = \mathrm{ad}(\overline{v})$, and $\phi = \exp(V)\exp(U)\exp(V)$. We recall that L is the direct sum of $ku \oplus kR_e \oplus k\overline{v}$ irreducibles of dimension 1, 3, or 5. These irreducibles have bases of the form $x, xV, \ldots, xV^{i-1}$ and the corresponding matrices of U and ϕ are given by

$$(1)\qquad \begin{array}{c|cc} i & U & \phi \\ \hline 1 & [0] & [1] \\ 3 & \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 1/2 \\ 0 & -1 & 0 \\ 2 & 0 & 0 \end{bmatrix} \\ 5 & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 & 0 \\ 0 & 0 & -6 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 1/24 \\ 0 & 0 & 0 & -1/6 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -6 & 0 & 0 & 0 \\ 24 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Define $V \xrightarrow{J} V$ by $x^J = x + (1/2)\langle x, u\rangle \circ v$. Then, by (1),

$$x^J = \begin{cases} x & \text{if } \langle x, u\rangle = 0, \\ -x & \text{if } x \in J \circ v. \end{cases}$$

Therefore, $J^2 = \mathrm{id}$.

Define a product on V by $xy = -(1/4)\langle u, x, y^{J\phi\epsilon}\rangle - (1/4)\langle u, y^J, x^{\phi\epsilon}\rangle - (1/4)\langle x, y^J\rangle \circ v$. Then, for $x, y \in V$,

(2) $(1/2)(xy^J - yx^J) = -(1/4)\langle x, y\rangle \circ v$ and $(1/2)(xy^J + yx^J) = [A_{x,y}, u]$

where $A_{x,y} = -(1/4)[y, x^{\phi}] - (1/4)[x, y^{\phi}]$. Now, $A^{\phi}_{x,y} = -A_{x,y}$ and hence $[[A_{x,y}, u], u] = 0$. This implies $[A_{x,y}, u]$ is fixed by J. But the first expression in (2) is J-skew. It follows that $(xy^J)^J = yx^J$ and J *is an involution.*

LEMMA 1. *Let $x, y \in V$. Then, $xy = (1/2)\langle y, x, v\rangle$ if $x^J = x$ and $xy = (1/4)\langle x, u\rangle \circ y^{\phi\epsilon}$ if $x^J = -x$.*

PROOF. $-4xy = [x, y^{J\phi}]U + [y^J, x^{\phi}]U + [x, y^J]V$. But using (1), it follows that $x^{\phi}U = (xV)^{\phi} = -x^J V$ and hence

(3) $$-4xy = [xU, y^{J\phi}] - [x, yV] + [y^J U, x^{\phi}] - [y^J, x^J V] + [x, y^J]V.$$

If $y^J = y$, the result follows from (3). Suppose $y^J = -y$. Then,

$$0 = [yU, x]V^2 = -4[yV, x] - 8[y, xV] + [yU, xV^2]$$

using Leibniz' rule and (1). Therefore,

(4) $$[yU, xV^2] = 4[yV, x] + 8[y, xV].$$

If $x^J = x$, we have $[yU, x^{\phi}] = (1/2)[yU, xV^2] = 2[yV, x] + 4[y, xV]$ and substituting this into (3) gives the result. Finally, if $x^J = -x$, $[yU, x^{\phi}] = -(1/6)[yU, xV^2] = -(2/3)[yV, x] - (4/3)[y, xV]$. Substituting this into (3) and simplifying gives $-4xy = -[xU, y^{\phi}] - (8/3)[x, yV] - (4/3)[xV, y] = -[xU, y^{\phi}] - (1/3)[xU, yV^2]$ (applying (4) with x and y interchanged). Thus, $-4xy = [xU, y^{\phi}]$. □

For $x \in V$, $ux = (1/2)\langle x, u, v\rangle = x$ and hence u is a left identity. Since J is an involution, *u is an identity element.*

Put $t = (2/3)\langle u, u, u\rangle$. Then, $\langle t, u\rangle \circ v = -(2/3)\bar{e}U^4V = (8/3)\bar{e}U^3 = -4t$. Thus, $t^J = -t$ and for $x \in V$, $tx = (1/4)\langle t, u\rangle \circ x^{\phi\epsilon} = (1/6)[\bar{e}U^4, x^{\phi}] = (1/6)[(\bar{e}U^4)^{\phi}, x]^{\phi} = 4[\bar{e}, x]^{\phi} = -4x^{\epsilon\phi}$. Also, for $x \in V$, $vx = (1/4)\langle v, u\rangle \circ x^{\phi\epsilon} = -x^{\phi\epsilon}$. Hence,

(5) $$x^{\epsilon\phi} = -(1/4)tx \quad \text{and} \quad x^{\phi\epsilon} = -vx$$

for $x \in V$.

Now, for $x, y, z \in V$, we have $(xy^J)z = (1/2)(xy^J - yx^J)z + (1/2)(xy^J + yx^J)z = -(1/16)(\langle x, y\rangle VU) \circ z^{\phi\epsilon} + (1/2)[z, A_{x,y}UV] = (1/4)\langle x, y\rangle \circ z^{\phi\epsilon} - (1/4)\langle z, x, y^{\phi\epsilon}\rangle - (1/4)\langle z, y, x^{\phi\epsilon}\rangle$.

Using this equation it is straightforward to check that

(6) $$\langle x, y, z^{\phi\epsilon}\rangle = +2(xy^J)z - 2(yz^J)x - 2(xz^J)y$$

and thus by (5)

(7) $$\langle x, y, z\rangle = (1/2)(xy^J)(tz) + (1/2)(y(z^Jt))x + (1/2)(x(z^Jt))y$$

for $x, y, z \in V$.

Let $S = \{s \in V : s^J = -s\}$. Then, the map $a \to \tilde{a} = -(1/4)a \circ v$ is a linear bijection of J onto S. (We would identify $\tilde{a}$ with a and hence J with S

except for the resulting ambiguities involving the multiplication in L.) Define $K = \{x \in V : x^J = x\}$ and $A = V$.

Now, $a = \langle \tilde{a}, u \rangle$ and $a \circ x = -\langle \tilde{a}, u \rangle \circ x^{\epsilon\phi\phi\epsilon} = -4\tilde{a}x^{\epsilon\phi} = \tilde{a}(tx)$ for $a \in J, x \in V$. Thus, $\widetilde{-4a \cdot b} = (a \cdot b) \circ v = (1/2)a \circ (b \circ v) + (1/2)b \circ (a \circ v) = (1/2)\tilde{a}(t(\tilde{b}(tv))) + (1/2)\tilde{b}(t(\tilde{a}(tv))) = -2\tilde{a}(t\tilde{b}) - 2\tilde{b}(t\tilde{a})$ for $a, b \in J$, since $tv = -4v^{\epsilon\phi} = -4u$. Combining these equations with the first equation of (2) gives

$$a \circ x = \tilde{a}(tx), \quad \widetilde{a \cdot b} = (1/2)\tilde{a}(t\tilde{b}) + (1/2)\tilde{b}(t\tilde{a}), \tag{8}$$
$$\widetilde{\langle x, y \rangle} = (1/2)(xy^J - yx^J)$$

for $a, b \in J$, $x, y \in V$.

3. Properties of (V, J). Having identified the operations in terms of the multiplication on V, we now proceed to develop the properties of this algebra that will enable us to describe the possibilities.

Let $[x, y, z] = (xy)z - x(yz)$, $x, y, z \in V$. We then have the following alternative property:

LEMMA 2. *$[s, x, y] = -[x, s, y] = [x, y, s]$ for $x, y \in V$, $s \in S$.*

PROOF. Since $[x, y, z]^J = -[z^J, y^J, x^J]$, it suffices to prove the first equation. Therefore, it suffices to show $[s, s, x] = 0$ and $[s, y, x] = -[y, s, x]$ for $s \in S$, $x \in V$, $y \in K$.

Now, $[s, s^\phi] = -(1/2)[s, s^\phi]UV = 2[s, s^\phi] - (1/2)[sU, s^\phi V] - (1/2)[sV, s^\phi U] + 3[s, s^\phi]$ using (1). Then, since sV is a multiple of $s^\phi U$, we have

$$[s, s^\phi] = (1/8)[sU, s^\phi V]. \tag{9}$$

Thus,

$$\begin{aligned}[s, s, x] &= -(1/4)[x, [s, s^\phi]UV] + (1/16)[sU, [sU, x^\phi]^\phi] \\ &= (1/2)[x, [s, s^\phi]] + (1/16)[sU, [s^\phi V, x]] \\ &= (1/2)[x, [s, s^\phi]] - (1/16)[x, [sU, s^\phi V]] = 0.\end{aligned}$$

On the other hand,

$$\begin{aligned}s(yx) + y(sx) &= -(1/8)[sU, [x, yV]^\phi] - (1/8)[[sU, x^\phi], yV] \\ &= (1/8)[sU, [x^\phi, yV]] - (1/8)[[sU, x^\phi], yV] \\ &= -(1/8)[[sU, yV], x^\phi] = -(1/8)[[y, sUV], x^\phi] \\ &= (1/2)[[y, s], x^\phi] = -(1/2)\langle y, s, x^{\phi\epsilon}\rangle + (1/2)\langle s, y, x^{\phi\epsilon}\rangle \\ &= (sy)x + (yz)x\end{aligned}$$

by (6). □

COROLLARY 3. *Let $r, s \in S$ and $x \in V$. Then, $r(s(rx)) = (rsr)x$, $r[s, r, x] = -[r, sr, x]$, $((xr)s)r = x(rsr)$, and $[x, r, s]r = -[x, rs, r]$.*

PROOF. The standard proofs go through (see for example [12]). □

COROLLARY 4. *For every $s \neq 0 \in S$, there exists $s^{-1} \in S$ such that $s^{-1}s = ss^{-1} = u$ and $[s, s^{-1}, x] = 0$ for $x \in V$.*

PROOF. $s = \widetilde{a}$ for some $a \neq 0 \in J$. Putting $s^{-1} = t\widetilde{a^{-1}}t$, we have $ss^{-1} = \widetilde{a}(t\widetilde{a^{-1}}t) = \widetilde{a}(t(\widetilde{a^{-1}}t)) = a \circ (a^{-1} \circ u) = u$. Applying J to this equation gives $s^{-1}s = u$. Moreover, $[s^{-1}, s, x] = x - \widetilde{a}((t\widetilde{a^{-1}}t)x) = x - \widetilde{a}(t(\widetilde{a^{-1}}(tx))) = x - a \circ (a^{-1} \circ x) = x$. □

We define $*$ and $\#$ on V by $x * y = xy - yx$ and $x \# y = (1/2)xy + (1/2)yx$. We then have the following well-known identities:

$$(10)\qquad \begin{aligned}(x * y) * z = (x * z) * y + x * (y * z) + [x, y, z] - [y, x, z] \\ + [z, x, y] - [z, y, x] - [x, z, y] + [y, z, x],\end{aligned}$$

$$(11)\qquad \begin{aligned}(x \# y) * z = (x * z) \# y + x \# (y * z) + [x, y, z] + [y, x, z] \\ + [z, x, y] + [z, y, x] - [x, z, y] - [y, x, z].\end{aligned}$$

for $x, y, z \in V$.

Define $G(x, y, z) = (x * y) * z + (y * z) * x + (z * x) * y$ for $x, y, z \in V$. Then by Lemma 2 and (10)

$$(12)\qquad G(r, s, x) = 6[r, s, x]$$

for $r, s \in S$, $x \in V$. By Corollary 3, $G(p, q, r) * p = G(p, q, p * r)$ for $p, q, r \in S$. Thus, S is a Malcev algebra with respect to $*$ [11, Lemma 2.3].

Define $(r, s)_S = (r, s^{\phi})$ for $r, s \in S$, where $(,)$ is the Killing form on L. Then

$$(13)\qquad (\widetilde{a}, \widetilde{b})_S = (1/16)(aV, b^{\phi}U) = -(1/16)(aVU, b^{\phi}) = (1/4)(a, b^{\phi})$$

for $a, b \in J$.

LEMMA 5. *Let $q, r, s \in S$ and $x \in K$. Then,*

$$(r * q, s)_S = (r, q * s)_S \quad \text{and} \quad (r \# x, s)_S = (q, s \# x)_S.$$

PROOF. Suppose $r = \widetilde{a}$ and $s = \widetilde{b}$. Then, for $y \in V$, we have $(\widetilde{\langle r, y\rangle}, s)_S = -(1/16)([aV, y], b^{\phi})$ (by (13)) $= (1/16)(a, [yV, b^{\phi}]) = -(1/16)([a, b^{\phi}], yV)$. Similarly, $(\widetilde{\langle s, y^J\rangle}, r)_S = -(1/16)([b, a^{\phi}], y^J V)$. Hence, since ϕ preserves the Killing form and $(yV)^{\phi} = -y^J V$, we have $(\widetilde{\langle r, y\rangle}, s)_S = (r, \widetilde{\langle s, y^J\rangle})_S$. Applying (8) then gives the lemma. □

If W is a subspace of V, it follows from (8) and the fact that V is an irreducible $\mathcal{J}$-module that

(14) $$S\mathcal{W} \subseteq \mathcal{W} \Rightarrow \mathcal{W} = (0) \quad \text{or} \quad \mathcal{V}.$$

In particular, $\mathcal{V}$ *is a simple algebra* and, if $\mathcal{V}$ is associative, $\mathcal{V}$ must be a division algebra. Moreover, by (10) and (11), $S * (S \# S \oplus S)$ and $S \# (S \# S \oplus S)$ are contained in $S \# S \oplus S$. Thus, (14) implies

(15) $$\mathcal{V} = (S \# S) \oplus S \quad \text{and} \quad K = S \# S.$$

Let $C(\mathcal{V})$ denote the center of $\mathcal{V}$, i.e. the set of elements of $\mathcal{V}$ that associate and commute with all other elements of $\mathcal{V}$. Now, by (8) the product $r \cdot s = (1/2)r(ts) + (1/2)s(tr)$ gives S the structure of a central Jordan division algebra with identity t^{-1}. But then if $x \in C(\mathcal{V}) \cap K$, the element xt^{-1} lies in the center of S (as a Jordan algebra) and hence $xt^{-1} \in kt^{-1}$. Thus,

(16) $$C(\mathcal{V}) \cap K = ku,$$

Now, $C(\mathcal{V})$ *is a field* since $\mathcal{V}$ is simple. Moreover, $C(\mathcal{V}) = C(\mathcal{V}) \cap K \oplus C(\mathcal{V}) \cap S = ku \oplus C(\mathcal{V}) \cap S$. But if $r, s \in C(\mathcal{V}) \cap S$, we have $r^2, rs \in ku$ by (16) and hence r is a multiple of s. Thus, either $C(\mathcal{V}) = ku$ *or* $C(\mathcal{V})/ku$ *is a quadratic extension generated by an element* $s \in S$ *such that* $s^2 \in ku$.

Now, if $q, r, s \in S$ and $x, y \in \mathcal{V}$, we have

(17) $$[q^2, x, y] = [q, qx, y] + q[q, x, y]$$

and

(18) $$[q^2, r, s] = [q, r, s]q + q[q, r, s].$$

We use these to prove:

LEMMA 6. $C(\mathcal{V}) = \{x \in \mathcal{V} : x * S = (0)\}$.

PROOF. Suppose $x \in \mathcal{V}$ and $x * S = (0)$. By (12), $[S, S, x] = (0)$ and hence by (11), $(\mathcal{J} \# \mathcal{J}) * x = (0)$. Thus, $\mathcal{V} * x = (0)$. Therefore, putting $z = s$ in (10), we have $[x, y, s] - [y, x, s] + [s, x, y] - [s, y, x] - [x, s, y] + [y, s, x] = 0$ and hence $[s, x, y] = [s, y, x]$ for $y \in \mathcal{V}$, $s \in S$. But then for $q, r \in S$, $[q^2, r, x] = [q, qr, x] + q[q, r, x] = [q, qr, x] = [q, x, qr] = [x, qr, q] = -[x, q, r]q = 0$. Hence, $[S \# S, S, x] = (0)$ and therefore $[\mathcal{V}, S, x] = (0)$. Thus, $[S, \mathcal{V}, x] = (0)$ and hence by (17), $[S \# S, \mathcal{V}, x] = (0)$. Therefore, $[\mathcal{V}, \mathcal{V}, x] = (0)$. Similarly, $[x, \mathcal{V}, \mathcal{V}] = (0)$. Then, for $y, z \in \mathcal{V}$, $[y, x, z] = (yx)z - y(xz) = (xy)z - y(zx) = x(yz) - (yz)x = 0$. Thus, $[\mathcal{V}, x, \mathcal{V}] = (0)$ and $x \in C(\mathcal{V})$. □

COROLLARY 7. *If* $C(\mathcal{V}) = ku$, *then* S *is a semisimple Malcev algebra.*

PROOF. By Dieudonné's theorem [12, Theorem 2.6] and Lemma 5, it suffices to prove that if $\mathcal{R}$ is an ideal of $\mathcal{S}$ such that $\mathcal{R} * \mathcal{R} = (0)$, then $\mathcal{R} = (0)$. But

$$\tilde{a} * \tilde{b} = (1/8)\widetilde{aA(v, b \circ v)} \tag{19}$$

for $a, b \in \mathcal{J}$. Hence, $A(v, b \circ v)^2 \mid \mathcal{J} = 0$ for $\tilde{b} \in \mathcal{R}$. But $A(v, b \circ v)$ is semisimple and hence $A(v, b \circ v) \mid \mathcal{J} = 0$ for $\tilde{b} \in \mathcal{R}$. (19) and Lemma 6 give $\mathcal{R} = (0)$. □

4. **Proof of Theorem A.** If $[\mathcal{S}, \mathcal{S}, \mathcal{S}] = (0)$, then $[\mathcal{V}, \mathcal{S}, \mathcal{S}] = (0)$ (by (18)) and hence $[\mathcal{S}, \mathcal{V}, \mathcal{S}] = (0)$. This implies $[\mathcal{V}, \mathcal{V}, \mathcal{S}] = (0)$ (by (17)) and hence $[\mathcal{S}, \mathcal{V}, \mathcal{V}] = (0)$. Thus $[\mathcal{V}, \mathcal{V}, \mathcal{V}] = (0)$ (again by (17)) and we have conclusion (a). Thus, we may assume $[\mathcal{S}, \mathcal{S}, \mathcal{S}] \neq (0)$. We now separate the proof into two cases:

Case 1. $\mathcal{C}(\mathcal{V}) = ku \oplus ks$, $s \neq 0 \in \mathcal{S}$, $s^2 \in ku$. Now, $\mathcal{V} = s\mathcal{S} \oplus \mathcal{S}$. But $[sq, x, y] = s[q, x, y] = -s[x, q, y] = -[x, sq, y]$ for $x, y \in \mathcal{V}$ and $q \in \mathcal{S}$. Thus, $[z, x, y] = -[x, z, y]$ for $x, y, z \in \mathcal{V}$ and therefore $\mathcal{V}$ is alternative. But $\mathcal{V}$ is central simple over $C = \mathcal{C}(\mathcal{V})$ and hence $\mathcal{V}/C$ is an 8-dimensional Cayley algebra over C [12, Theorem 3.17]. Then, $\mathcal{V} = C \oplus \mathcal{V}_0$ and $\mathcal{V}_0 = \mathcal{O}_0 \oplus s\mathcal{O}_0$, where $\mathcal{V}_0$ is the set of trace zero elements of $\mathcal{V}$ and $\mathcal{O}_0 = \mathcal{V}_0 \cap \mathcal{S}$. If $a \in \mathcal{O}_0$, $a^2 \in C$ (since $\mathcal{V}/C$ is a Cayley algebra) and $a^2 \in \mathcal{K}$. Thus, $a^2 \in ku$ for $a \in \mathcal{O}_0$ and hence $\mathcal{O} = ku \oplus \mathcal{O}_0$ is an 8-dimensional composition algebra over k. Thus, $\mathcal{O}$ is a Cayley algebra over k, $\mathcal{V} = \mathcal{O} \otimes_k C$ and $J = J_1 \otimes J_2$, where J_1 is the canonical involution on $\mathcal{O}$ and $J_2 = J \mid C$. $\mathcal{O}$ is a division algebra since its skew elements are invertible (see for example, Corollary 3.24 of [12]).

Case 2. $\mathcal{C}(\mathcal{V}) = ku$. Let F be the algebraic closure of k. Then, $\mathcal{S}_F$ is the direct sum of simple Malcev ideals by Corollary 7.

Assume first of all that $\mathcal{S}_F$ is simple. Then $\mathcal{S}$ is 7-dimensional [10, Theorem B] and therefore $\mathcal{J}$ is a 7-dimensional central Jordan division algebra. Thus, $\mathcal{J}$ is isomorphic to the Jordan algebra of a 6-dimensional quadratic form [8, Example 1, p. 210]. For $L \in \mathrm{End}_k(\mathcal{J})$, define $\tilde{L} \in \mathrm{End}_k(\mathcal{S})$ by $\tilde{a}\tilde{L} = \widetilde{aL}$. For $b \in \mathcal{J}$, $\mathrm{tr}(\langle v, b \circ v\rangle) = 0$ [3, Lemma 4] and hence $A(v, b \circ v) \mid \mathcal{J} \in \mathrm{Str}(\mathcal{J})'$, where $\mathrm{Str}(\mathcal{J})'$ denotes the derived algebra of the structure Lie algebra of $\mathcal{J}$. Thus, by (19), $M(\mathcal{S}) \subseteq \widetilde{\mathrm{Str}(\mathcal{J})'}$, where $M(\mathcal{S})$ is the Lie transformation algebra of $\mathcal{S}$. But both of these Lie algebras are 21 dimensional [11, §8] and hence $M(\mathcal{S}) = \widetilde{\mathrm{Str}(\mathcal{J})'}$. Therefore, if $a, b \in \mathcal{J}$ and b has trace zero, we have $([a, a^{\Phi}], R_b) = -(a \circ b, a^{\Phi}) = -4(\widetilde{a \circ b}, \tilde{a})_{\mathcal{S}}$ (by (13)) $= -4(\tilde{a}M, \tilde{a})_{\mathcal{S}}$ for some $M \in M(\mathcal{S})$ and hence $([a, a^{\Phi}], R_b) = 0$ (by Lemma 5). Thus, if $s = \tilde{a} \in \mathcal{S}$, we have $[s, s^{\Phi}] = (1/8)[a, a^{\Phi}] \in kR_e$ (by (9)) and therefore $s^2 = -(1/2)[u, [s, s^{\Phi}]] \in ku$ (by (6)). Hence $\mathcal{V} = ku \oplus \mathcal{S}$ is a composition algebra and, since it is not associative and has invertible skew elements, it is a Cayley division algebra.

Suppose next that $\mathcal{S}_F$ is not simple. Thus, $\mathcal{S}_F = \mathcal{S}_1 \oplus \mathcal{S}_2$, where $\mathcal{S}_1$,

S_2 are nonzero ideals of S_F such that $S_1 * S_1 = S_1$ and $S_2 * S_2 = S_2$. Define $C_i = S_i \# S_i \oplus S_i$, $i = 1, 2$. It follows from (18) that $[S_i \# S_i, S_i, S_i] \subseteq S_i \# S_i$ and hence from (11) that $C_i * C_i \subseteq C_i$, $i = 1, 2$. Now, $[S_F, S_1, S_2] = (0)$ and therefore (by (18)) $[V_F, S_1, S_2] = (0)$. Then, $[S_1, V_F, S_2] = (0)$ and (by (17)) $[C_1, V_F, S_2] = (0)$. Applying J to this equation gives $[S_2, V_F, C_1] = (0)$ and therefore any associator involving 3 elements from S_2, V_F and C_1 respectively is zero. A similar remark applies to elements from S_1, V_F and C_2. In particular, (11) then implies that $C_1 * C_2 = (0)$.

Now, if we put $[x, y, z]_\# = (x \# y) \# z - x \# (y \# z)$, we have the following well known identity for $x, y, z \in V_F$:

$$(20)\qquad \begin{aligned} 4[x, y, z]_\# = {} & [x, y, z] - [z, y, x] + [y, x, z] \\ & - [z, x, y] + [x, z, y] - [y, z, x] + y * (x * z). \end{aligned}$$

Therefore, $[S_i, S_j, S_k]_\# = (0)$ unless $i = j = k$. In all cases therefore, the notation $S_i \# S_j \# S_k$ is unambiguous. But $S_1 \# S_2 \# S_2 = (S_1 * S_1) \# (S_2 \# S_2) = S_1 * (S_1 \# S_2 \# S_2)$ (by (11)) $\subseteq S_1 * S_F \subseteq S_1$. But then $(S_2 \# S_2 \# S_2, S_1)_S = (S_2, S_1 \# S_2 \# S_2)_S = (0)$ (by Lemma 5) and hence $S_2 \# S_2 \# S_2 \subseteq S_2$. Similarly, $S_2 \# S_1 \# S_1 \subseteq S_2$ and $S_1 \# S_1 \# S_1 \subseteq S_1$. But $u \in S_1 \# S_1 + S_1 \# S_2 + S_2 \# S_2$ and therefore

$$(21)\qquad \begin{aligned} S_1 &= S_1 \# S_1 \# S_1 + S_1 \# S_2 \# S_2 \quad \text{and} \\ S_2 &= S_2 \# S_2 \# S_2 + S_2 \# S_1 \# S_1. \end{aligned}$$

Moreover, $(S_1 \# S_1) \# (S_1 \# S_1) \subseteq [S_1 \# S_1, S_1, S_1] + S_1 \# (S_1 \# (S_1 \# S_1)) + S_1 * (S_1 * (S_1 \# S_1))$ (by (20)) $\subseteq S_1 \# S_1$ and hence $C_1 \# C_1 \subseteq C_1$. But $C_1 * C_1 \subseteq C_1$ and therefore C_1 and C_2 are subalgebras of V_F.

Suppose for contradiction that $S_1 \# S_2 \# S_2 = (0)$. Then by (21), $S_1 = S_1 \# S_1 \# S_1$. But $(S_2 \# S_1 \# S_1, S_2)_S = (S_2, S_2 \# S_2 \# S_1)_S = (0)$ and hence $S_2 \# S_1 \# S_1 = (0)$ and $S_2 = S_2 \# S_2 \# S_2$. But then $S_1 \# S_2 = (S_1 \# S_1 \# S_1) \# S_2 = S_1 \# (S_1 \# S_1 \# S_2)$ (by (20)) and hence $S_1 \# S_2 = (0)$. Therefore, $V_F = C_1 + C_2$ and $C_2 C_1 = C_2 \# C_1 = (S_1 \# S_1) \# (S_2 \# S_2) \subseteq S_1 \# (S_1 \# S_2 \# S_2) = (0)$. Thus, C_1 and C_2 are ideals of V_F. But V is central simple and hence V_F is simple. Therefore, $C_1 = C_2 = V_F$ and we have a contradiction. Hence, $S_1 \# S_2 \# S_2 \neq (0)$ and similarly $S_2 \# S_1 \# S_1 \neq (0)$.

Suppose now that we have chosen S_1 simple. Then, $S_1 = S_1 \# S_2 \# S_2$ and $(S_1 \# S_1) * S_1 = (S_1 \# S_1) * (S_1 \# S_2 \# S_2) \subseteq ((S_1 \# S_1) * S_1) \# (S_2 \# S_2)$ (by (11)) $\subseteq (((S_1 \# S_1) * S_1) \# S_2) \# S_2$ (by (19)) $\subseteq ((S_1 \# S_1 \# S_2) * S_1) \# S_2$ (by (11)) $= (0)$. But certainly $(S_1 \# S_1) * S_2 = (0)$. Therefore, $S_1 \# S_1$ centralizes S_F and by Lemma 6, $S_1 \# S_1 \subseteq Fu$. But S_1

$\# S_1 \neq (0)$ and hence $S_1 \# S_1 = Fu$. Therefore, $C_1 = Fu \oplus S_1$ is a composition algebra.

Now, $S_2 = S_2 \# S_1 \# S_1$ and the argument of the previous paragraph gives the result that $C_2 = Fu \oplus S_2$ is a composition algebra. The map $C_1 \otimes_F C_2 \longrightarrow V_F$ is an algebra epimorphism. The kernel is an ideal stabilized by $J_1 \otimes J_2$, where J_1 and J_2 are the canonical involutions. It then follows easily that the map is an isomorphism and hence $(V_F, J) = (C_1 \otimes_F C_2, J_1 \otimes J_2)$.

Now, V is not associative and hence one of the C_i's is a Cayley algebra. The Galois action on V_F corresponding to the form V must either stabilize C_1 and C_2 or interchange them. In the first case, we have conclusion (b). Assume then that the Galois action interchanges C_1 and C_2. Then, the subgroup of $G = \mathrm{Gal}(F/k)$ that stabilizes C_1 and C_2 has index 2 in G and hence C_1 and C_2 are defined over a quadratic extension L/k. Thus, $V_L = O_1 \otimes_L O_2$, where O_i is a Cayley algebra over L such that $C_i = O_{iF}$, $i = 1, 2$. Let $O_{i,0}$ be the set of elements of O_i of trace zero, $i = 1, 2$, and suppose for contradiction that O_1 is split. Then, there exists $s_1 \neq 0 \in O_{1,0}$ such that $s_1^2 = 0$. Then $s_1^\sigma \in O_{2,0}$ and $s = s_1 + s_1^\sigma \in S$, where σ is the generator of $\mathrm{Gal}(L/k)$. Thus, $s^{-1} = r_1 + r_1^\sigma$ for some $r_1 \in O_{1,0}$. Now, $0 = s * s^{-1} = s_1 * r_1 + s_1^\sigma * r_1^\sigma$ and hence $s_1 * r_1 = 0$. Thus, $(s_1 r_1)^2 = s_1^2 r_1^2 = 0$ and therefore, since $s_1 r_1 \in Lu$, $s_1 r_1 = 0$. Therefore, $u = ss^{-1} = s_1 r_1 + s_1^\sigma r_1^\sigma + s_1 r_1^\sigma + s_1^\sigma r_1 = s_1 r_1^\sigma + s_1^\sigma r_1 \in O_{1,0} \otimes_L O_{2,0}$ and we have our contradiction. Thus, O_1 and O_2 are division algebras.

5. **Proof of Theorem B.** $L = L(\mathcal{J}, V)$ for some $\mathcal{J}$-ternary algebra V without zero divisors [3, Theorem 2]. By Theorem 17 of [3] and its following remark, we may assume we are in case (a) of §1. Let M be the derived algebra of the Lie algebra of skew transformations of the hermitian form over (A, J) with matrix

$$\begin{bmatrix} 0 & 0 & u \\ 0 & u & 0 \\ u & 0 & 0 \end{bmatrix}$$

If we choose $k\ \mathrm{diag}[u, 0, -u]$ as our maximal split toral subalgebra and

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2t^{-1} & 0 & 0 \end{bmatrix}$$

as the identity of our highest root space, an easy calculation shows that we obtain V as the associated ternary algebra. Thus, we have conclusion (i) with skew hermitian replaced by hermitian. The usual argument shows that this replacement is immaterial (see §10.6 of [7] for example).

6. **Further restrictions on (A, J).** Suppose (A, J) is an algebra described in (b) or (c) of §1. The assumption that the ternary algebra V has no zero divisors

(i.e. $L(J, V)$ has rank 1) imposes further restrictions on (A, J). We consider the cases separately.

Suppose (A, J) is as in (b). Then *O and C cannot contain k-isomorphic subfields of degree* 2 *over k.* For otherwise there exist skew elements $s_1 \neq 0 \in O$ and $s_2 \neq 0 \in C$ such that $s_1^2 = s_2^2$. Then, $(s_1 + s_2)(s_1 - s_2) = 0$ contradicting Corollary 4.

Suppose (A, J) is as in (c). Let $\langle\sigma\rangle = \mathrm{Gal}(L/k)$. If $s_1 \neq 0$ is skew in O_1, then $s = s_1 + s_1^\sigma$ is skew in A and $(s_1 + s_1^\sigma)(s_1 - s_1^\sigma) = s_1^2 - (s_1^2)^\sigma$. Thus, by Corollary 4, we cannot have $(s_1^2)^\sigma = s_1^2$. Therefore, *$s_i^2 \notin ku$ for $s_i \neq 0$ skew in O_i, $i = 1, 2$.*

If k is the real field, it follows from the above remarks that the only algebra of type BC_1 covered by Theorem B (ii) occurs when A is the Cayley division algebra. If k is the p-adic field, every Cayley algebra is split and hence no algebras of type BC_1 are covered by Theorem B (ii).

Suppose k is an algebraic number field. Now every Cayley algebra over an algebraic number field contains a skew element of norm 1 [1, §10]. The above remarks imply then that we may restrict our attention to algebras $(A, J) = (O \otimes_k C, J_1 \otimes J_2)$, where C has dimension 1, 2 or 4 over k. Suppose C is a quaternion algebra. Then, there exist $\alpha, \beta, \gamma \in k$ such that O and C can be constructed by the Cayley-Dickson process using constants $-1, -1, \alpha$ and β, γ respectively [1, §10]. By the Hasse-Minkowski principle [1, Lemma 8], the form $X_1^2 + X_2^2 + X_3^2 + \beta X_4^2 + \gamma X_5^2 - \beta\gamma X_6^2$ must be isotropic (since given an isomorphism ρ of k into the reals, one of $\beta^\rho, \gamma^\rho, -(\beta\gamma)^\rho$ must be negative). Thus, we may choose $\alpha_1, \alpha_2, \ldots, \alpha_6$ not all zero in k such that $-\alpha_1^2 - \alpha_2^2 - \alpha_3^2 = \beta\alpha_4^2 + \gamma\alpha_5^2 - \beta\gamma\alpha_6^2$. Hence, O and C contain nonzero skew elements with equal squares, contradicting the above. Therefore, *over an algebraic number field the only algebras of type BC_1 covered by Theorem* B (ii) *occur when $(A, J) = (O \otimes_k C, J_1 \otimes J_2)$, where O is a Cayley division algebra and $C = k$ or C is a field extension of degree* 2 *over k not k-isomorphic to a subfield of O.* It is not difficult to verify that all such algebras (A, J) in fact give rise to ternary algebras without zero divisors and hence to Lie algebras of type BC_1.

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