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QUASI-SIMILAR MODELS FOR NILPOTENT OPERATORS(1)

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ABSTRACT. Every nilpotent operator on a complex Hilbert space is shown to be quasi-similar to a canonical Jordan model. Further, the para-reflexive operators are characterized generalizing a result of Deddens and Fillmore.

A familiar result states that each nilpotent operator on a finite dimensional complex Hilbert space is similar to its adjoint. One proof proceeds by showing that both a nilpotent operator and its adjoint have the same canonical form. In this note we show that although this result does not extend to infinite dimensional spaces, the weaker quasi-similarity version of it, together with the proof indicated above, still holds on any Hilbert space. This yields an affirmative answer to a question raised by P. Rosenthal in connection with the content of [3].

The canonical form exhibited provides positive evidence that the theory of Jordan models might be extended to cover operators of class C_0 of infinite multiplicity and indeed, considerable progress [2] has been made recently in this direction. Although the Jordan model for nilpotent operators on infinite dimensional Hilbert spaces is no longer unique, we single out a "canonical" model. A similar result has been obtained independently by Berkovici [1]. We conclude with an application of our results to extend to infinite dimensional spaces a theorem of Deddens and Fillmore [4] which characterizes reflexive operators on finite dimensional spaces.

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1. In this note, a nilpotent operator T will be called a Jordan operator if $T = \bigoplus_\alpha T_\alpha$, where each T_α operates on some $\mathbf{C}^{l_\alpha}$ for $0 < l_\alpha < \infty$ by the Jordan one-cell matrix

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$$\begin{pmatrix} 0 & 1 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & & 0 & \cdot & & & \cdot \\ \cdot & & & \cdot & \cdot & & \cdot \\ \cdot & & & & \cdot & 1 & 0 \\ 0 & & & & & 0 & 1 \\ 0 & 0 & \cdot & \cdot & & \cdot & 0 \end{pmatrix}.$$

Recall that an operator X between Hilbert spaces H and K is said to be a quasi-affinity if $\ker X = (0)$ and $\operatorname{clos}(XH) = K$. An operator A on H is said to be a quasi-affine transform of an operator B on K if there exists a quasi-affinity X such that $XA = BX$. Finally, two operators A and B are quasi-similar if each is a quasi-affine transform of the other. For further information on these concepts see the monograph [8, Chapter II, No. 3.2], or [7].

Our main result is given by the following

THEOREM 1. *Every nilpotent operator T is quasi-similar to a Jordan operator T_0.*

Since for any Jordan operator T_0, the operators T_0 and T_0^* are obviously unitarily equivalent, we can infer

THEOREM 2. *If T is a nilpotent operator, then T and T^* are quasi-similar.*

Before starting the proof of Theorem 1, we give an example to show that quasi-similarity cannot be replaced by similarity.

Let X be any compact quasi-affinity on an infinite dimensional Hilbert space H (for example, the Volterra operator on $L^2(0, 1)$) and consider the operator T defined by

$$T = \begin{pmatrix} 0 & X & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{pmatrix} \quad \text{on } H \oplus H \oplus H.$$

Clearly $T^3 = 0$ and thus T and T^* are quasi-similar by Theorem 2, but T and T^* are not similar.(2) The proof of this is straightforward.

If S were an invertible operator on $H \oplus H \oplus H$ with matrix

(2) The same example was found independently by H. Radjavi (see [3, §6]).

$$S = \begin{pmatrix} A_0 & B_0 & C_0 \\ A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix}$$

which satisfied $ST^* = TS$, then a simple computation shows that $B_2 = C_2 = C_1 = 0$, $C_0 = XB_1$ and $A_2 = B_1X^*$. Thus the operator

$$S_0 = \begin{pmatrix} A_0 & B_0 & 0 \\ A_1 & B_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is a compact perturbation of S, and hence a Fredholm operator, which is contradicted by the fact that $\ker S_0 = (0) \oplus (0) \oplus \mathcal{H}$ is not finite dimensional (cf. [5, Chapter 5]).

2. We start the proof of Theorem 1 with the following

LEMMA 1. *If T_0 and T_1 are two Jordan operators and T_0 is a quasi-affine transform of T_1, then T_0 and T_1 are quasi-similar.*

PROOF. The fact that T_0 is a quasi-affine transform of T_1 means that there exists a quasi-affinity X such that $XT_0 = T_1X$ and thus $T_0^*X^* = X^*T_1^*$. Since T_j is a Jordan operator, there exists a unitary operator U_j such that $T_j^* = U_j^*T_jU_j$ $(j = 0, 1)$. Therefore, $T_0(U_0X^*U_1^*) = (U_0X^*U_1^*)T_1$, where $U_0X^*U_1^*$ is a quasi-affinity, and T_1 is also a quasi-affine transform of T_0. Consequently, T_0 and T_1 are quasi-similar.

LEMMA 2. *Any nilpotent operator T has a quasi-affine transform T_0 which is a Jordan operator.*

PROOF. Suppose that $T^n = 0$, $T^{n-1} \neq 0$ for some $n \geqslant 1$. If we set

$$\mathcal{X}_j = \ker T^j \ominus \ker T^{j-1} \quad \text{for } j = 1, 2, \ldots, n,$$

$$\mathcal{Y}_n = \mathcal{X}_n, \qquad \mathcal{Y}_{n-1} = \mathcal{X}_{n-1} \cap (T\mathcal{Y}_n)^\perp, \ldots,$$

$$\mathcal{Y}_1 = \mathcal{X}_1 \cap (T^{n-1}\mathcal{Y}_n + \cdots + T\mathcal{Y}_2)^\perp \text{ and}$$

$$\mathcal{H}_0 = \underbrace{(\mathcal{Y}_n \oplus \cdots \oplus \mathcal{Y}_n)}_{n \text{ times}} \oplus \underbrace{(\mathcal{Y}_{n-1} \oplus \cdots \oplus \mathcal{Y}_{n-1})}_{(n-1) \text{ times}} \oplus \cdots \oplus (\mathcal{Y}_2 \oplus \mathcal{Y}_2) \oplus \mathcal{Y}_1$$

we can define the bounded operators T_0 on $\mathcal{H}_0$ and A: $\mathcal{H}_0 \longrightarrow \mathcal{H}$ by the equations

$$T_0(y_n^1 \oplus \cdots \oplus y_n^n \oplus \cdots \oplus y_2^1 \oplus y_2^2 \oplus y_1^1)$$
$$= 0 \oplus y_n^1 \oplus \cdots \oplus y_n^{n-1} \oplus \cdots \oplus 0 \oplus y_2^1 \oplus 0,$$

and

$$A(y_n^1 \oplus \cdots \oplus y_n^n \oplus \cdots \oplus y_2^1 \oplus y_2^2 \oplus y_1^1)$$
$$= y_n^1 + \cdots + T^{n-1}y_n^n + \cdots + y_2^1 + Ty_2^2 + y_1^1.$$

It is easy to see that T_0 is a Jordan operator and that $AT_0 = TA$. Using the fact that

$$(\mathcal{Y}_1 + \cdots + T^{n-1}\mathcal{Y}_n + \mathcal{Y}_2 + \cdots + T^{n-2}\mathcal{Y}_n + \cdots + \mathcal{Y}_k + \cdots + T^{n-k}\mathcal{Y}_n)^- = \ker T^k,$$

which is proved by induction on k, we conclude that $\operatorname{clos}(A\mathcal{H}_0) = \mathcal{H}$. To complete the proof we must show that A is injective.

If A is not injective, there must exist y_j^k in $\mathcal{Y}_j$, $1 \leq j \leq n$, $1 \leq k \leq j$, such that

$$\sum_{j=1}^{n} \sum_{k=1}^{j} T^{k-1}y_{n-j+k}^k = 0 \quad \text{but} \quad \sum_{j=1}^{n} \sum_{k=1}^{j} \|y_{n-j+k}^k\| \neq 0.$$

Let m be the smallest integer such that $\Sigma_{k=1}^m \|y_{n-m+k}^k\| \neq 0$ and let p be the smallest integer such that $y_{n-m+p}^p \neq 0$. Because we have

$$\sum_{k=p}^{m} T^{k-1}y_{n-m+k}^k = -\sum_{j=m+1}^{n} \sum_{k=1}^{j} T^{k-1}y_{n-j+k}^k \quad \text{in } \ker T^{n-m},$$

it follows that $y_{n-m+p}^p + \cdots + T^{m-p}y_n^m$ is in $\ker T^{n-m+p-1}$. If we let P denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{X}_{n-m+p}$, then

$$y_{n-m+p}^p + P(Ty_{n-m+p+1}^{p+1} + \cdots + T^{m-p}y_n^m)$$
$$= P(y_{n-m+p}^p + \cdots + T^{m-p}y_n^m) = 0$$

since $\mathcal{X}_{n-m+p}$ is orthogonal to $\ker T^{n-m+p-1}$ and y_{n-m+p}^p is in $\mathcal{X}_{n-m+p}$. Moreover, since y_{n-m+p}^p is orthogonal to $T^{m-p}\mathcal{Y}_n + T^{m-p-1}\mathcal{Y}_{n-1} + \cdots + T\mathcal{Y}_{n-p+m+1}$, it follows that

$$y_{n-m+p}^p \perp P(Ty_{n-m+p+1}^{p+1} + \cdots + T^{m-p}y_n^m)$$

and hence that $y_{n-m+p}^p = 0$ which is a contradiction.

This completes the proof of the lemma.

3. **Proof of Theorem 1.** By applying Lemma 2 to T and T^* we obtain quasi-affinities X and X_* together with Jordan operators T_0 and T_1 such that

$TX = XT_0$, and $T^*X_* = X_*T_1$. Hence, $X_*^*T = T_1^*X_*^*$ and T is a quasi-affine transform of the Jordan operator T_1^*. Thus T_0 is a quasi-affine transform of T_1^* and hence T_0 and T_1^* are quasi-similar by Lemma 1. Consequently, T is a quasi-affine transform of T_0 and we have established that T and T_0 are quasi-similar.

4. We make several remarks before continuing.

Since there exist quasi-nilpotent operators T such that $\ker T = (0) \neq \ker T^*$ (for example, take T to be the weighted shift with weights $1, 1/2, 1/3, \ldots$), Theorem 2 is not valid for quasi-nilpotent operators.

As a consequence of Theorem 2, observe that Lemma 1 holds for all nilpotent operators, that is, if one nilpotent operator is a quasi-affine transform of another, then the two operators are actually quasi-similar.

Lastly, by using the Dunford-Riesz spectral decomposition Theorem 2 can be shown to hold for algebraic operators with *real* spectrum.

5. Theorem 1 provides a Jordan model for every nilpotent operator on Hilbert space. However, in contrast with the finite dimensional case, distinct Jordan models may be quasi-similar. Fortunately, the situation is not as complicated as it might first appear. We obtain a canonical choice and hence a complete set of quasi-similarity invariants for nilpotent operators after introducing some terminology.

For each integer m $(1 \leqslant m < \infty)$ and each infinite cardinal $\aleph$, let $J_m^{\aleph}$ denote the Jordan operator defined by the $m \times m$ operator matrix

$$\begin{pmatrix} 0 & I_H & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & I_H & & & & \cdot \\ \cdot & & \cdot & \cdot & & & \cdot \\ \cdot & & & \cdot & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot & 0 \\ 0 & & & \cdot & & 0 & I_H \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \end{pmatrix}$$

$$\text{on } mH = \underbrace{H \oplus \cdots \oplus H}_{m \text{ times}},$$

where H is a Hilbert space of dimension $\aleph$.

THEOREM 3. *Every nilpotent operator is quasi-similar to a unique Jordan* *model of the form* $\bigoplus J_{m_i}^{\aleph_i} \oplus N$, *where* $1 \leqslant m_1 < m_2 < \cdots < m_k < \infty$, $\aleph_1 > \aleph_2 > \cdots > \aleph_k$, *and* N *is a finite rank Jordan model* $\bigoplus_{j=1}^n T_j$ *on* $\bigoplus_{j=1}^n \mathbf{C}^{l_j}$ *with* $m_k < l_j$ *for* $j = 1, 2, \ldots, n$

PROOF. By Theorem 1 we need only consider Jordan models and easy arguments reduce the result to proving that $J_k^\aleph \oplus J_{k-1}^\aleph$ and $J_k^\aleph$ are quasi-similar for each $1 \leqslant k < \infty$ and infinite cardinal $\aleph$. Moreover, since $J_k^{\aleph *}$ and $J_{k-1}^{\aleph *}$ are unitarily equivalent to $J_k^\aleph$ and $J_{k-1}^\aleph$, respectively, it is sufficient to show that $J_k^\aleph$ is a quasi-affine transform of $J_k^\aleph \oplus J_{k-1}^\aleph$. Let H be a Hilbert space of dimension $\aleph$ and suppose A and B are operators on H which satisfy

(1) $\ker A = (0)$,

(2) $\operatorname{clos}(AH) = H$, and

(3) $\operatorname{clos}\{Ax \oplus Bx : x \in H\} = H \oplus H$.

Then the identity

$$\begin{pmatrix} \begin{matrix} A & & 0 \\ & \ddots & \\ 0 & & A \end{matrix} \\ \hline \begin{array}{c|ccc} 0 & B & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & B \end{array} \end{pmatrix} (J_k^\aleph) = \left(\begin{array}{c|c} J_k^\aleph & 0 \\ \hline 0 & J_{k-1}^\aleph \end{array}\right) \begin{pmatrix} \begin{matrix} A & & 0 \\ & \ddots & \\ 0 & & A \end{matrix} \\ \hline \begin{array}{c|ccc} 0 & B & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & B \end{array} \end{pmatrix}$$

would complete the proof since (1), (2) and (3) imply that the matrix

$$\begin{pmatrix} \begin{matrix} A & & 0 \\ & \ddots & \\ 0 & & A \end{matrix} \\ \hline \begin{array}{c|ccc} 0 & B & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & B \end{array} \end{pmatrix}$$

defines a quasi-affinity from kH to $(2k-1)H$.

There are various ways of exhibiting operators satisfying (1), (2) and (3). For example, let M_1 and M_2 denote multiplication by the characteristic functions of the first and second quarters Q_1 and Q_2 of the unit circle respectively, defined from the Hardy space H^2 to the L^2 spaces $L^2(Q_1)$ and $L^2(Q_2)$ respectively. If V_1, V_2, and V_3 are unitary maps from H onto $H^2 \otimes H$, $L^2(Q_1) \otimes H$, and $L^2(Q_2) \otimes H$, then $A = V_2^*(M_1 \otimes I_H)V_1$ and $B = V_3^*(M_2 \otimes I_H)V_1$ have the desired properties.

This theorem is probably indicative of the kind of uniqueness one can expect for Jordan models for C_0-operators of infinite multiplicity.

We conclude this section with a corollary which completes the classification of nilpotents up to quasi-similarity.

COROLLARY. *If T_1 and T_2 are nilpotent operators on the Hilbert spaces H_1 and H_2, respectively, then T_1 and T_2 are quasi-similar if and only if* $\dim \operatorname{clos}[T_1^l H_1] = \dim \operatorname{clos}[T_2^l H_2]$ *for* $l = 0, 1, \ldots$.

PROOF. If X is a quasi-affinity from H_1 to H_2 such that $T_2X = XT_1$, then

$$\operatorname{clos}[XT_1^l H_1] = \operatorname{clos}[T_2^l X H_1] = \operatorname{clos}[T_2^l H_2]$$

which implies that $\dim \operatorname{clos}[T_1^l H_1] = \dim \operatorname{clos}[T_2^l H_2]$ for $l = 0, 1, 2, \ldots$. Conversely, an easy argument shows that the Jordan model given in the theorem is uniquely determined by these dimensions.

6. The results of this note enable us to extend a characterization of reflexive operator of Deddens and Fillmore [4] to infinite dimensional spaces. Recall that a linear subspace M of the Hilbert space H is said to be para-closed for the operator T on H if M is the range of some bounded operator on H. Let us call an operator T on H *para-reflexive* if any operator U on H leaving invariant the para-closed-invariant spaces of T is an entire function of T. The definition is one of the possible natural extensions to infinite dimensional spaces of the concept of reflexive operators on a finite dimensional space.

We begin this section with a result which may have some independent interest.

PROPOSITION 1. *Para-reflexivity is preserved under quasi-similarity.*

PROOF. If T and S are quasi-similar and S is para-reflexive we must show that T is also para-reflexive. If T is not algebraic, then by virtue of Theorem 2 [6], T is para-reflexive. Thus we can assume that T (and consequently S also) is algebraic. Suppose $TA = AS$, $BT = SB$ where A, B are quasi-affinities, and let Z be an operator leaving invariant every finite dimensional subspace invariant for T, that is, for every h in H there exists some polynomial p_h such that $Zh = p_h(T)h$. If we set $Z_0 = BZA$, then

$$Z_0 h_0 = BZAh_0 = Bp_{Ah_0}(T)Ah_0 = BAp_{Ah}(S)h_0 \quad \text{is in } BAH$$

for every h_0 in H. Thus $X = (BA)^{-1}Z_0$ is, by the closed graph theorem, an operator on H such that

$$Xh_0 \text{ is in the finite dimensional space } \bigvee_{j \geq 0} S^j h_0$$

for every h_0 in H. It follows that X leaves invariant every finite dimensional subspace of H invariant under S. Thus, since S is para-reflexive, we infer from Corollary 2 [6] that $X = q(S)$, where q is a suitable polynomial. Consequently, $BZA = Z_0 = BAq(S) = Bq(T)A$ and hence $Z = q(T)$. Using Corollary 2 [6] once again, we conclude that T is para-reflexive.

A nilpotent operator on H is said to satisfy the Deddens-Fillmore condition [4], if either dim $H \leqslant 1$ or its Jordan model $\bigoplus_{\alpha\in A} J_\alpha$ has the following property: If n_α denotes the order of the matrix of J_α ($\alpha \in A$) and α_0 is chosen in A such that

(0) $$n_{\alpha_0} = \max\{n_\alpha | \alpha \in A\},$$

then

(1) $$\max\{n_\alpha | \alpha \in A\backslash\{\alpha_0\}\} \geqslant n_{\alpha_0} - 1.$$

Proposition 2. *A nilpotent operator T on H is para-reflexive if and only if it satisfies the Deddens-Fillmore condition.*

Proof. By virtue of Proposition 1 and Theorem 1, it is sufficient to prove the statement in case $T = \bigoplus_{\alpha\in A} J_\alpha$. Exactly as in [4] we can prove that if this T does not fulfill the Deddens-Fillmore condition then T does not have property (A) or (B) of Corollary 2 [6]. Thus, by this corollary, T is not para-reflexive.

Let us now show the sufficiency of the Deddens-Fillmore condition. It is clear that we can assume that

$$T = J_0 \oplus J_1 \oplus \left(\bigoplus_{\alpha\in B} J_\alpha\right),$$

where the order n_i of J_i is the maximum occurring in formula (i) above ($i = 0$, 1); thus the order of any J_α ($\alpha \in B$) is not greater than n_1. Now let Z be an operator leaving invariant all para-closed subspaces invariant for T. Then obviously

$$Z = Z_0 \oplus Z_1 \oplus \left(\bigoplus_{\alpha\in B} Z_\alpha\right)$$

and for any $h = h_0 \oplus h_1 \oplus (\bigoplus_{\alpha\in B} h_\alpha)$ there exists a polynomial p_h such that

$$Zh = p_n(T)h, \quad \text{that is,}$$

$$(Z_0 \oplus Z_1 \oplus Z_\alpha)(h_0 \oplus h_1 \oplus h_\alpha) = p_h(J_0 \oplus J_1 \oplus J_\alpha)(h_0 \oplus h_1 \oplus h_\alpha)$$

for every α in B. The above relation shows in particular that $Z_0 \oplus Z_1 \oplus Z_\alpha$ leaves invariant every invariant subspace of $J_0 \oplus J_1 \oplus J_\alpha$. By virtue of the Deddens-Fillmore theorem there exists a unique polynomial q_α of degree $\leqslant n_0$ such that such that

$$Z_0 \oplus Z_1 \oplus Z_\alpha = q_\alpha(J_0 \oplus J_1 \oplus J_\alpha) = q_\alpha(J_0) \oplus q_\alpha(J_1) \oplus q_\alpha(J_\alpha), \tag{2}$$

for every α in B. Thus for α, β in B we have

$$q_\alpha(J_0) = Z_0 = q_\beta(J_0). \tag{3}$$

Since J_0 is of order n_0 and q_α, q_β are of degree $\leqslant n_0$, (3) implies $q_\alpha = q_\beta$. Consequently, there exists a polynomial q of degree $\leqslant n_0$ such that $q_\alpha \equiv q$ for every α in B. From (2) we infer

$$Z = Z_0 \oplus Z_1 \oplus \left(\bigoplus_{\alpha \in B} Z_\alpha \right) = q(J_0) \oplus q(J_1) \oplus \left(\bigoplus_{\alpha \in B} q(J_\alpha) \right) = q(T)$$

which finishes our proof.

THEOREM 4. *An operator T is para-reflexive if and only if either it is nonalgebraic or it is algebraic and the nilpotents corresponding to the points of the spectrum of T satisfy the Deddens-Fillmore condition.*

PROOF. This follows at once from Proposition 2 above, the Dunford-Riesz spectral decomposition of an algebraic operator, and Corollary 1.

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